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What to know for Midterm 2

First of all, this summary is **not** complete! The idea of this summary of what you need to know is to give a list of the most important topics, theorems, test, methods, etc. One way to use this list is to check (after you've studied) if there is any subjects that you're not comfortable with yet. Using this summary can by no means replace the use of the book for studying for you exam.

1 What you still need to know

•Integration.
•Partial fractions.
•l'Hospital's rule.
•A differentiable function is decreasing if its derivative is negative.

2 Sequences

•What is a **sequence**?
•Definition of **Limit** of a sequence, when **convergent**, when **divergent** ?
•If the **Interpolation function** of a sequence converges to L, then the sequence itself converges to the same limit L.
•Limit laws for sequences (page 696).
•Squeeze theorem for sequences.
•Definition of **increasing** and **Decreasing** and **Monotonic**. Note that constant sequences are both increasing and decreasing.
•Definition of **bounded** sequence.
•**Monotonic Sequence Theorem**, every bounded, monotonic sequence is convergent. (This is the basis of the integral test and the comparison test for Series)

3 Series

An (infinite) series is an expression of the form $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$. Its sum is given by the limit of the partial sums $s_n = \sum_{k=1}^{n} a_k$, provided that this limit exists.
There are many techniques to find out whether a series is convergent or divergent. Some of them even give the limit if the series is convergent (telescoping series, geometric series or use of power series). Several assume some condition holds (function is decreasing and positive for integral test, terms are nonnegative for comparison test, etc.) and it is very important to check those conditions. If they are not satisfied, you will have to find another way of proving convergence/divergence.

If the conditions are not satisfied you still don't know anything. For instance, the root test tells you that the series $\sum a_n$ is absolutely convergent (hence convergent) if the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists and $L < 1$. If $L > 1$, then the series is divergent. If $L = 1$, then we don't know anything. For both $a_n = \frac{1}{n}$ and $a_n = \frac{1}{n^2}$ we get $L = 1$, but the series of the former is divergent, while the series of the latter is convergent.

Before you start using any tests or methods, try to make a good guess of convergence or divergence, because this guess will determine what method you will use. If you want to prove a series is divergent with the comparison test, you'll have to find another series that is already divergent of which the positive terms are less (or equal). However, if you want to prove that a series is convergent, you will have to find another series that is convergent of which the terms are at larger (or equal). Therefore, spend a little more time on the guess that you make, convergent or divergent, so that you don't waste your precious time trying to prove the wrong thing.

In general, if the terms of the series go to zero **fast enough**, the series will be convergent. Note that to prove convergence, it is **not enough** to test that the terms go to zero. Fast enough is for instance something like $\sum \frac{n^2+2}{n^4-\pi^4}$. The guess is that this goes like $\sum n^2/n^4 = \sum 1/n^2$ (highest degree in numerator and denominator), which is convergent by p-series, so we can try to use the limit comparison test with $b_n = 1/n^2$. Another series for which the terms go to zero fast enough is for instance $\sum \frac{c^n}{n!}$, use the ratio test. The terms in $\sum \frac{1}{n}$ also go to zero, but not fast enough to converge. This **harmonic series** is divergent.

Section 11.7 has a good summary of all the techniques. We'll also present a list of important methods and tests here. Note that very often you will need to use more then one method on a single exercise.

•Definition of **series**, when convergent, divergent?
•Geometric series, if $|r| < 1$, then $\sum_{m=0}^{\infty} ar^m = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. How do you find a and r? If $|r| < 1$, this not only gives convergence, but even a limit. If $|r| \geq 1$, then the series is divergent.
•Test of divergence (converse does **not** hold!).
•**If** two series $s = \sum a_n$ and $t = \sum b_n$ are convergent, then so are the series $\sum(a_n \pm b_n)$, and $\sum cb_n$ with the obvious limits $s \pm t$ and ct.
•Telescoping series. If a series is of the form $\sum_{n=1}^{\infty}(g(n) - g(n+1))$, then the partial sums s_n are given by $s_n = g(1) - g(n+1)$, so the series converges to $\lim_{n\to\infty}(g(1) - g(n+1))$ provided this limit exists.
•Integral test. Conditions are that $f(x)$ is a continuous, positive, decreasing function for x big enough, say for $x \geq T$ for some $T > 0$. Then the series $\sum a_n$ with $a_n = f(n)$ is convergent if and only if the improper integral $\int_T^{\infty} f(x)\,dx$ is convergent.

•Remainder estimate for the integral test. Suppose that f satisfies the conditions of the integral test and $a_n = f(n)$. If $\sum a_n$ converges by the integral test, $s = \sum_{n=1}^{\infty} a_n$ and s_n is the partial sum $s_n = \sum_{k=1}^{n} a_k$, then for the error $R_n = s - s_n$ we have

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Note that $R_n \geq 0$, because all the terms a_n are non-negative. Therefore, if you want to find an n, such that $R_n < 0.001$, it is enough to find an n such that $\int_n^{\infty} f(x)\,dx < 0.001$.
•The p-series.
•The comparison test, conditions are that $\sum a_n$ and $\sum b_n$ are series with **positive** terms for n large enough. This condition is important!
•The limit comparison test. conditions are that $\sum a_n$ and $\sum b_n$ are series with **positive** terms for n large enough. The series are either both divergent or both convergent if $\lim \frac{a_n}{b_n} = c$ and $c > 0$. For the extended limit comparison test, see exercises 40 and 41 on page 726. Example: $\sum a_n$ with

$$a_n = \frac{n^2+2}{n^4-\pi^4}$$

For n large enough the terms are positive. Numerator has degree 2, denominator has degree 4, so we compare with $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$. We get

$$\frac{a_n}{b_n} = \frac{n^2+2}{n^4-\pi^4} \cdot \frac{n^2}{1} = \frac{1+\frac{2}{n^2}}{1-\frac{\pi^4}{n^4}},$$

so $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1+0}{1-0} = 1$ and $1 > 0$. Now $\sum b_n$ is convergent by p-series, so by limit comparison test, also $\sum a_n$ is convergent.
•Alternating series test. Consider $\sum (-1)^n b_n$ with b_n positive. If the sequence $\{b_n\}$ is decreasing for n large enough and $\lim b_n = 0$, then the series is convergent. To test that $\{b_n\}$ is decreasing, you prove that $b_{n+1} \leq b_n$ for n large enough, or you prove that an interpolation function $f(x)$ is decreasing by showing that its derivative is negative for x large enough.
•Alternating Series Estimation Theorem. Suppose $s = \sum (-1)^n b_n$ is the sum of a convergent alternating series for which the sequence $\{b_n\}$ is decreasing and $\lim b_n = 0$. If $s_n = \sum_{k=0}^{n} (-1)^k b_k$ is the partial sum, then for the error $R_n = s - s_n$ we get $|R_n| \leq b_{n+1}$. Therefore, if you want to find an n such that $|R_n| < 0.001$, it is enough to find an n with $b_{n+1} < 0.001$.
•Definition of **absolutely convergent** and **conditionally convergent**.
•Theorem: If a series $\sum a_n$ is absolutely convergent , then it is convergent.
•The ratio test. Suppose that the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists. If $L < 1$, then the series $\sum a_n$ is absolutely convergent, so convergent. If $L > 1$, then the series is divergent. If $L = 1$, then we don't know anything.

•The root test. Suppose that the limit $\lim \sqrt[n]{|a_n|} = L$ exists. If $L < 1$ then $\sum a_n$ is absolutely convergent, so convergent. If $L > 1$ then the series $\sum a_n$ is divergent. If $L = 1$, then we don't know anything.
•Section 11.7 is good to read...

4 Power series

•Definition of **power series** with center a.
•What is **radius of convergence**?
•Interval of convergence is symmetric around center a of power series, if you exclude the endpoints.
•Can you find power series with interval of convergence of the form $(-\infty, \infty)$ or $\{a\}$ or (r, s) or $[r, s)$ or $(r, s]$ or $[r, s]$?
•If $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, then the radius of convergence of $\sum c_n(x - a)^n$ is $R = 1/c$.
•Suppose that a power series $\sum c_n(x - a)^n$ with radius of convergence R converges to a function $f(x)$ on its interval of convergence.
Then the function f is differentiable and the power series we get by differentiating $\sum c_n(x - a)^n$ term by term converges to f' and has the same radius of convergence R.
Furthermore, the power series we get by taking antiderivative term by term, converges to an antiderivative of f.
Read Theorem 2 in section 11.9 and do lots of exercises. Using this theorem you can find representation as power series of many functions, starting with the ones you already know, such as $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.
•**If** $f(x)$ is represented by a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ for $a-R < x < a+R$ with $R > 0$, then the c_n are given by

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

•For any function f that is infinitely differentiable at a, the Taylor series of f centered at a is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

The remark above says that if any power series converges to f on some interval, then it is the Taylor series of f.
•**Warning!** It is possible that two different functions have the same Taylor series. For instance the zero-function and the function g given by $g(0) = 0$ and $g(x) = e^{-1/x^2}$ for $x \neq 0$ both have a Taylor series with all coefficients equal to zero. So there are functions g of which the Taylor series converges, but not to the function g itself.
•You should know the power series expansion of the following functions, e^x, $\sin x$, $\cos x$, $\frac{1}{1-x}$, $\ln(1 - x)$, $\tan^{-1}(x)$, $(1 + x)^k$ for all real numbers k. All are centered around 0. You should also know their interval of convergence. You can find them in section 11.10 and 11.11.

5 Limits

This is partly material, that comes from before MAth 1B. You should know how to work with them, how to use l'Hospitals Rule and several standard limits.
•$x^{1/x} \to 1$ for $x \to \infty$.
•$c^{1/x} \to 1$ for $x \to \infty$ for any constant $c > 0$.
•$(1 + \frac{1}{x})^x \to e$ for $x \to \infty$.
•$\frac{\sin x}{x} \to 1$ for $x \to 0$.
•$\frac{x^n}{n!} \to 0$ for $n \to \infty$ for every real number x.