Trigonometric Identities and Integration
Math 1B, Spring 2003, Section 213

This is not a summary of the book, certainly not of the lectures. Some important aspects of Math 1B are put together here, but this is not enough to learn your mid-terms and/or final with. Also remember that it is impossible to give a strategy to find integrals without thinking. Creativity will allways play a big role in finding the right method to use. The more you practice, the more creative you get.

This review is written much denser then the book, so if you don't understand something in here completely, you can go back to the book to review that part. If you do understand all this, great! But don't get excited yet, because there is much more to know...

Trigonometry Identities

Rewriting to other functions

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$

$$\cos x = \cot x \sin x = \frac{\cot x}{\csc x}$$
$$\sin x = \tan x \cos x = \frac{\tan x}{\sec x}$$
$$\cos^2 x + \sin^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Changing the angles

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin^2 x = (1 - \cos 2x)/2$$
$$\cos^2 x = (1 + \cos 2x)/2$$

$$\sin x \cos y = (\sin(x-y) + \sin(x+y))/2$$
$$\sin x \sin y = (\cos(x-y) - \cos(x+y))/2$$
$$\cos x \cos y = (\cos(x-y) + \cos(x+y))/2$$

Differentiation

$$\frac{d}{dx} \sin x = \cos x$$
$$\frac{d}{dx} \cos x = -\sin x$$
$$\frac{d}{dx} \tan x = \sec^2 x$$
$$\frac{d}{dx} \sec x = \sec x \tan x$$
$$\frac{d}{dx} \cot x = -\csc^2 x$$
$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

Some basic integrals

For all f and g with $\frac{d}{dx} f = g$ we have $\int g\, dx = f + C$, so from the derivatives above we already find several basic integrals. Here are a few more.

$$\int \tan x\, dx = \log|\sec x| + C$$
$$\int \cot x\, dx = -\log|\csc x| + C$$
$$\int \sec x\, dx = \log|\tan x + \sec x| + C$$
$$\int \csc x\, dx = \log|\cot x - \csc x| + C$$

Hyperbolics

For complete discussion on hyperbolic functions including their graphs read section 3.9.

Rewriting to other functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \coth x = \frac{\cosh x}{\sinh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \cosh^2 x - \sinh^2 x = 1$$
$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$
$$\coth^2 x - \operatorname{csch}^2 x = 1$$

Using these identities, one easily finds more of them analoguous to the trigonometry identities.

Inverses of hyperbolic functions

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$
$$\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$$
$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Note that the domains for these functions are $\mathbb{R}$, $[1, \infty)$ and $(-1, 1)$ respectively.

Changing "angles"

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Again, using these identities one can find more of them analoguous to the trigonometric identities.

Differentiation

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$$
$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \qquad \frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \qquad \frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$$

Some basic integrals

Again, for all f and g with $\frac{d}{dx} f = g$ we have $\int g\, dx = f + C$, so from the derivatives above we already find several basic integrals. Here is a couple more.

$$\int \operatorname{sech} x\, dx = \tan^{-1} |\sinh x| + C$$
$$\int \tanh x\, dx = \log \cosh x + C$$
$$\int \operatorname{csch} x\, dx = \log |\tan \frac{1}{2} x| + C$$
$$\int \coth x\, dx = \log |\sinh x| + C$$

Methods of integration

Useful for substitution

$$d \sin x = \cos x\, dx$$
$$-d \cos x = \sin x\, dx$$
$$d \tan x = \sec^2 x\, dx$$
$$d \sec x = \sec x \tan x\, dx$$
$$-d \cot x = \csc^2 x\, dx$$
$$-d \csc x = \csc x \cot x\, dx$$
$$\frac{1}{n}\, dx^n = x^{n-1}\, dx \quad n \neq 0$$

Which substitution

First a warning. If you make a substitution $x = f(u)$, then make sure you replace every occurence of x by $f(u)$ and dx by $f'(u)\, du$. If you start with an integral with one variable x and you end up with an integral with two variables x and u, then something went wrong...

Substitutions are supposed to make integrals easier. If that doesn't happen, you may need to look for another substitution or even another method of integration. However, sometimes more than one substitutions are needed, so don't give up too soon. Allways be on the look-out for substitutions before you use more sophisticated (and time consuming) methods.

- Sometimes we need to find the antiderivative of a function that contains $\sin x$ and $\cos x$. We might use a factor $\cos x$ to write $\cos x\,dx = d\sin x$, then rewrite all occurrences of $\cos x$ in terms of $\sin x$, using the identity $\cos^2 x + \sin^2 x = 1$. Then a substitution $u = \sin x$ could make the integral easier.

  Similar methods hold for functions expressed in $\sec x$ and $\tan x$ because both their derivatives can be expressed in $\sec x$ and $\tan x$ and we have the identity $\sec^2 x - \tan^2 x = 1$. This is all very well explained in section 7.2 of the book. That section also contains several good examples and strategies.

  Can you find similar strategies for finding antiderivatives of functions of the form $\csc^m x \cot^n x$ or $\sinh^m x \cosh^n x$?

- Sometimes it is possible to find a substitution such that you end up with a rational function. The book calls this a rationalizing substitution. A certain amount of luck and creativity is needed to find such substitutions, but they can make life much easier sometimes. For instance, if we substitute $x = u^{12}$ in the following integral, then $dx = 12u^{11}\,du$ and we find

$$\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}} = \int \frac{u^{11}}{u^4 + u^3}\,du = \int \frac{u^8}{u+1}\,du.$$

  Now the substitution $v = u + 1$ makes this an easy integral.

- If the function you need to integrate is several levels deep, i.e., of the form $f(g(h(x)))$, then it might be useful to substitute $u = h(x)$. Be careful! Here the new variable u is expressed in the old variable x. This is different from a substitution of the form $x = j(v)$, where we can easily find $dx = j'(v)\,dv$. For the substitution $u = h(x)$ it can sometimes be very hard to find out what dx is.

  In finding the integral

$$\int \frac{1}{e^{2x} - e^x - 6}\,dx$$

  the deepest part is e^x, so we substitute $u = e^x$. To find dx we can either write $du = e^x\,dx = u\,dx$, so $dx = \frac{1}{u}\,du$, or we write $x = \log u$, to find $dx = \frac{1}{u}\,du$. We are then left with the integral of a rational function again.

- The expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$.

  Because of the identity $a^2 + a^2\tan^2\theta = \sec^2\theta$, the substitution $x = \tan\theta$ in $\sqrt{a^2 + x^2}$ gives $\sqrt{\sec^2\theta} = |\sec\theta|$. If θ runs through an interval of length π, then $x = \tan\theta$ runs through exactly all real numbers. If we choose more specifically the interval $-\pi/2 < \theta < \pi/2$, then $\sec\theta = 1/\cos\theta > 0$ as $\cos\theta > 0$. Therefore we can then write $\sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$. Similarly for the other two, we make the following substitutions.

expression	substitution	interval	identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 < \theta < \pi/2$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

  Be careful again! You see that these substitutions can be made if you have something of the form $\pm x^2 \pm a^2$, a constant term and a quadratic term, but NO linear term. If you do have a linear term, first get rid of it by **completing the square**. For example, if we have

$$\int \frac{1}{\sqrt{x - \frac{1}{2}x^2}}\,dx,$$

Then first we write $x - \frac{1}{2}x^2 = -\frac{1}{2}(x^2 - 2x) = -\frac{1}{2}\left((x-1)^2 - 1\right) = \frac{1}{2}\left(1 - (x-1)^2\right)$ to get

$$\int \frac{1}{\sqrt{x - \frac{1}{2}x^2}}\,dx = \int \frac{dx}{\sqrt{\frac{1}{2}}\sqrt{1-(x-1)^2}} = \int \frac{\sqrt{2}\,dx}{\sqrt{1-(x-1)^2}}.$$

Now substitute $u = x - 1$ and only then do a trig-substitution.

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

One obvious reason to use integration by parts is that sometimes the integral $\int v\,du$ is easier than $\int u\,dv$. In that case, using the formula of integration by parts we can expess the harder integral $\int u\,dv$ in terms of the function uv and the easier integral $\int v\,du$. To use the formula of integration by parts on an integral $\int f\,dx$, you need to write $f(x)$ as a product $f(x) = u(x) \cdot v'(x)$, such that $f\,dx = uv'\,dx = u\,dv$. This means you have to make choices for u and dv. It is impossible to say which choice you have to make in general, but there are some typical examples.

We can rewrite the formula of integration by parts as

$$\int uv'\,dx = uv - \int vu'\,dx.$$

This means that if we compare the "difficult" integral $\int uv'\,dx$ to the easier $\int vu'\,dx$, then the function u is replaced by u' and v' by v. Therefore, one typical case of integration by parts is to write (if possible) a function f as $f = uv'$ for two other functions u and v where u' is an "easier" function than u and/or v is "easier" than v'. Some typical examples.

- Functions as $\sin x$, $\cos x$ and e^x are just as easy as their derivates. Take for instance $\sin x$ and suppose we have a function f that we can write as $f = u \sin x$ for a function u with "easier" derivative u' than the function u itself. Then we take that u together with $dv = \sin x\,dx$. Take for instance
$$\int x^2 \sin x\,dx.$$
Here we take $u = x^2$ and $dv = \sin x\,dx$, because then $v = -\cos x$ is not more difficult than $v' = \sin x$, but $u' = 2x$ is easier than $u = x^2$. To find the integral we can do integration by parts again. Work this out yourself.

- Inverses of trigonometric functions are fairly difficult compared to their derivates. It is therefore usually fruitful to try to choose such a function as u and the rest as dv. Take for instance
$$\int x \tan^{-1} x\,dx.$$
Here we take $u = \tan^{-1} x$ and $dv = v'\,dx = x\,dx$. Even though we get $v = \frac{1}{2}x^2$, which is more complicated than $v' = x$, we win because $u' = 1/(1+x^2)$ is much easier than $u = \tan^{-1} x$. The integral that we are left with is that of a rational function, whence much easier than the one we started with. Again, work this out yourself.

- Sometimes the trivial factorization $f = f \cdot 1$ can be used, especially if f contains logarithms. Take
$$\int (\log x)^3\,dx.$$
Here we can choose $u = (\log x)^3$ and $dv = 1\,dx$. We find $v = x$ and again $v = x$ is a more complicated function than $v' = 1$. However, this sacrifice will pay back, because using the

chain-rule we find $u' = 3(\log x)^2 \cdot \frac{1}{x}$. This means that u' is again less complicated than u because the exponent of the $\log x$ is now 2 instead of 3. Furthermore, the factor $\frac{1}{x}$ cancels to the $x = v$, so we find

$$\int (\log x)^3 \, dx = x(\log x)^3 - 3\int (\log x)^2 \, dx.$$

Check that the formula above is true and find the integral $\int (\log x)^n \, dx$ for general n.

We have seen that in integration by parts we sometimes choose u and v where v is not at all easier than v'. This can be a sacrifice that has to be made because u' is much much easier than u. It can also be done to make some factors cancel as in the last example. Anyway, it must be clear now, that some creativity is needed to make the right choice. Also, sometimes several steps of integration are needed, so apart from creativity, also endurance will pay off.

We have just seen how integration by parts can make integrals easier. Sometimes we can do integration by parts twice (or even more times) to get back to the integral we started with. For instance, suppose we have the integral

$$\int e^x \cos x \, dx$$

Choose $u_1 = e^x$ and $dv_1 = \cos x \, dx$, then $v_1 = \sin x$ and $du_1 = e^x \, dx$ and we get

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx. \tag{1}$$

We will again do integration by parts, so again, we will have to make a choice. However, we just ended with $u_1 v_1 - \int v_1 \, du_1$, in which $\int v_1 \, du_1 = \int \sin x \, de^x$. Therefore, if we now chose $u_2 = \sin x = v_1$ and $dv_2 = de^x = du_1$, then we would get back to where we were, ending with the trivially true identity

$$\int u_1 \, dv_1 = u_1 v_1 - \int v_1 \, du_1 = u_1 v_1 - \left(v_1 u_1 - \int u_1 \, dv_1 \right).$$

That doesn't get us anywhere, so instead we choose $u_2 = e^x = u_1'$ and $dv_2 = \sin x \, dx = v_1 \, dx$ to get $v_2 = -\cos x$ and hence

$$\int e^x \sin x \, dx = -e^x \cos x - \int -e^x \cos x \, dx.$$

Together with (1) this gives

$$\int e^x \cos x \, dx = e^x(\sin x + \cos x) - \int e^x \cos x \, dx.$$

Bringing the integral on the right to the left and dividing by 2 gives

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x).$$

Another good example of the same trick is Example 8 on page 481 of the book. It shows how to find the integral $\int \sec^3 x \, dx$.

Partial fractions

For the integration of rational functions there is a very effective method, using partial fractions. It is based on the fact that we know how to do integrals of the form

$$\int \frac{A}{(x+B)^n}\,dx$$
$$\int \frac{C}{(x^2+D^2)^m}\,dx$$
$$\int \frac{Ex}{(x^2+D^2)^m}\,dx$$

The first can be done with a linear substitution $u = x + B$, the second with a trig substitution and the third with the substitution $u = x^2 + D^2$.

Any rational function can be written as the sum of a polynomial and a proper rational function. To do this, you use long division. For instance

$$\frac{x^3+3x^2+5x+4}{x^2+2x+3} = x+1+\frac{1}{x^2+2x+3}$$

The next step is to break up the proper rational function into partial fractions. To do this, you factorize the denominator of the proper partial fraction into irreducible polynomials. Over the real numbers, that means into linear factors and quadratic polynomials without real roots. For instance we could get something like

$$f = \frac{x^3-4x^2+2x-6}{(x+1)(x-1)^3(x^2+1)^2}.$$

Since this is proper and all factors of the denomiator are irreducible, this can be written as

$$\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}.$$

Note that there are 3 irreducible factors in the denominator of f with different exponents.

- The exponent of $x+1$ is 1, so we have only one term $\frac{A}{x+1}$. It's numerator is constant because $x+1$ is linear.
- The exponent of $x-1$ is 3, so we get 3 terms, $\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$. Again, the numerators are constant, because $x-1$ is also linear.
- The exponent of x^2+1 is 2, so we get 2 terms $\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}$. The numerators are linear, because x^2+1 is quadratic.

What could cost you many points on exams

Many many points on exams get lost because of stupid mistakes. Sometimes these mistakes make an exercise so different, that also no partial credit can be given anymore. Therefore be aware, and **never** use the following non-equalities as equalities.

$$\frac{1}{a+b} \neq \frac{1}{a}+\frac{1}{b}$$
$$(a+b)^2 \neq a^2+b^2$$
$$\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$$
$$g^a \cdot g^b \neq g^{ab}$$