Can. J. Math., Vol. XL, No. 2, 1988, pp. 257-338

# PROJECTIVE MODULES OVER HIGHER-DIMENSIONAL NON-COMMUTATIVE TORI

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The non-commutative tori provide probably the most accessible interesting examples of non-commutative differentiable manifolds. We can identify an ordinary $n$-torus $T^n$ with its algebra, $C(T^n)$, of continuous complex-valued functions under pointwise multiplication. But $C(T^n)$ is the universal $C^*$-algebra generated by $n$ commuting unitary operators. By definition, [**15**, **16**, **50**], a non-commutative $n$-torus is the universal $C^*$-algebra generated by $n$ unitary operators which, while they need not commute, have as multiplicative commutators various fixed scalar multiples of the identity operator. As Connes has shown [**8**, **10**], these algebras have a natural differentiable structure, defined by a natural ergodic action of $T^n$ as a group of automorphisms. The non-commutative tori behave in many ways like ordinary tori. For instance, it is an almost immediate consequence of the work of Pimsner and Voiculescu [**37**] that the $K$-groups of a non-commutative torus are the same as those of an ordinary torus of the same dimension. (In particular, non-commutative tori are $KK$-equivalent to ordinary tori by Corollary 7.5 of [**52**].) Furthermore, the structure constants of non-commutative tori can be continuously deformed into those for ordinary tori. (This is exploited in [**17**].)

In this paper we study the non-stable behavior of (finitely generated) projective modules over non-commutative tori. These are the appropriate generalization of complex vector bundles over ordinary tori, according to a theorem of Swan [**54**, **45**]. It is well known that for higher-dimensional ordinary tori the non-stable behavior of vector bundles is quite complicated. Our main theme is that, in contrast, as soon as there is any irrationality present, then the non-stable behavior of projective modules over non-commutative tori is quite regular. To make this more precise, let us introduce some notation.

A non-commutative torus is specified by giving the multiplicative commutators for its generators. For our purposes this is most conveniently done by giving a skew bicharacter on $\mathbf{Z}^n$, or better (as first exploited by Elliott [**17**]) by giving a real skew bilinear form, say $\theta$, on $\mathbf{Z}^n$. To each $x \in \mathbf{Z}^n$ we can associate a product, say $u_x$, of the unitary generators, in

Received February 25, 1987. This research was supported in part by National Science Foundation grant DMS-83-08352.

such a way that the non-commutative torus $A_\theta$ corresponding to $\theta$ is the universal $C^*$-algebra generated by the $u_x$'s subject to the relation

$$u_y u_x = \exp(\pi i\theta(x, y))\, u_{x+y}.$$

(For details see Section 4.)

Any non-commutative torus $A_\theta$ has a canonical trace, $\tau$ (generalizing the Lebesgue measure on an ordinary torus), which defines a homomorphism (again denoted by $\tau$) from $K_0(A_\theta)$ to $\mathbf{R}$ (which generalizes the assignment to a vector bundle of its dimension). Furthermore, from the work of Pimsner and Voiculescu [37] it follows rapidly that

$$K_0(A_\theta) \cong Z^{2^{n-1}}.$$

We also recall that, by definition, the positive cone of $K_0(A)$ for any algebra, $A$, consists of the elements which are represented by actual projective $A$-modules (not just differences thereof). So $\tau$ will have positive values on the positive cone of $K_0(A_\theta)$.

We will say that $\theta$ is not rational if its values on the integral lattice $\mathbf{Z}^n \subset \mathbf{R}^n$ are not all rational. Then our main results are as follows:

THEOREM 6.1. *If $\theta$ is not rational, then the positive cone of $K_0(A_\theta)$ consists exactly of the elements of $K_0(A_\theta)$ on which $\tau$ is strictly positive, together with zero.*

THEOREM 7.1 (Cancellation). *If $\theta$ is not rational, then any two projective modules which represent the same element of $K_0(A_\theta)$ are isomorphic. Equivalently, if $U$, $V$ and $W$ are projective $A_\theta$-modules such that $U \oplus W \cong V \oplus W$, then $U \cong V$.*

COROLLARY 7.2 and THEOREM 7.3. *If $\theta$ is not rational, then we have a quite explicit construction of every projective $A_\theta$-module up to isomorphism.*

COROLLARY 7.10. *If $\theta$ is not rational, then the projections in $A_\theta$ itself generate all of $K_0(A_\theta)$.*

These results also have consequences for $K_1(A_\theta)$. Let $UA_\theta$ denote the group of unitary elements in $A_\theta$, and let $U^0A_\theta$ denote the connected component of the identity element of $UA_\theta$.

THEOREM 8.3. *If $\theta$ is not rational, then the natural map from $UA_\theta/U^0A_\theta$ to $K_1(A_\theta)$ is an isomorphism.*

This last result, in turn, has an interesting consequence for the structure of the set of projections in $A_\theta$, namely,

THEOREM 8.13. *If $\theta$ is not rational, then any two projections in $A_\theta$ which represent the same element of $K_0(A_\theta)$ are in the same path component of the set of projections in $A_\theta$.*

Of central importance for the proofs of the above results is a quite explicit method for constructing a large number of projective $A_\theta$-modules, even without any special hypotheses on $\theta$. In Section 1 we describe a general approach to this construction, which is also applicable to some other situations, as indicated in [**50**]. For the $A_\theta$'s, this approach involves the Heisenberg representation of locally compact Abelian groups, and the restrictions of the Heisenberg representation to subgroups. This is discussed in a general way in Section 2, while the special case in which the subgroups are lattices is discussed in Section 3. It is this latter case which actually provides projective $A_\theta$-modules.

Given the myriad projective modules which can be constructed by the method discussed in the first three sections, it is essential to have a way of classifying these modules. The crucial tool which we use for this classification is the generalized Chern character introduced by Connes [**8**], with its associated apparatus of non-commutative differential geometry involving connections and their curvature. Connes' Chern character has already been discussed for the $A_\theta$ by Elliott [**17**], and we will use heavily Elliott's description of the range of Connes' Chern character for the $A_\theta$. Our construction of connections, and the calculation of their corresponding curvatures and Chern characters, is the subject of Sections 4 and 5.

Section 6 is the first section in which we must assume that $\theta$ is not rational. Under this hypothesis, we show that every element of $K_0(A_\theta)$ on which $\tau$ is positive, is represented by a projective module of the kind constructed in the earlier sections. The proof is basically a somewhat lengthy inductive argument on the exterior forms which constitute the range of the Chern character, using in a careful way the non-rationality of $\theta$.

In Section 7 we prove the cancellation theorem and obtain some of its corollaries. The proof involves, in addition to the results of the earlier sections, the theory of topological stable rank which was developed in [**48**] to prove cancellation for irrational rotation algebras (non-commutative 2-tori) in [**49**]. Finally, in Section 8 we discuss the consequences for $K_1(A_\theta)$.

The non-commutative tori, in addition to providing an interesting setting in which to investigate non-commutative differential geometry and algebraic topology, arise naturally in various ways. For example, Poguntke [**38**], building on extensive earlier work, has shown that for any connected Lie group $G$, the unique simple subquotient of $C^*(G)$ corresponding to any primitive ideal of $C^*(G)$ is either the algebra $K$ of compact operators (or a finite dimensional full matrix algebra), or is of the form $K \otimes A_\theta$ where $A_\theta$ is a simple non-commutative torus. He also has obtained an analogous result when $G$ is a (not necessarily connected) compactly generated locally compact two-step nilpotent group. In another direction, Olesen, Pedersen, and Takesaki [**31**] have shown that the non-commutative (and commutative) tori are exactly the $C^*$-algebras which admit an ergodic action of an ordinary torus group $T^n$. In yet another direction, the non-commutative tori provide a useful setting within which to study Schrödinger operators with quasi-periodic potential. Many spectral projections of these Schrödinger operators will correspond to projective modules. For a survey of this matter see [3]. Finally, let us mention that Connes has discovered that non-commutative tori provide a fruitful setting in which to develop Yang-Mills theory [13].

The investigation carried out in this paper can be attempted for other classes of $C^*$-algebras associated with groups. A nice start at doing this has been made for nilpotent discrete groups by Packer [33, 34], and for nilpotent Lie groups by Sheu [53].

I wish to record here my thanks to George A. Elliott for having provided me with a preprint of [17] at an early stage in my investigation of this subject, and to Bruce Blackadar for helpful conversations about aspects of cancellation for projective modules over the $A_\theta$'s.

**1. The general framework.** We discuss in this section a general framework for the construction of projective modules over the twisted group $C^*$-algebras of discrete groups. (Throughout this paper, by "projective module" we will always mean "finitely generated projective module".) Our discussion expands some of the ideas sketched in [50], and has potential application to other situations, as indicated there. But for the present paper, our discussion serves primarily as motivation for the developments in later sections; specific results from this section are not needed later. Thus this section can be read rapidly, but we do use it as an opportunity to introduce some of the notation which will be used throughout the rest of this paper.

Let $D$ be a discrete group. We wish to consider the group $C^*$-algebra of $D$ twisted [60] by a 2-cocycle with values in $T$ (the group of complex numbers of modulus one). While we will eventually need to work with specific cocycles, we find it convenient to put this off as long as possible by using the well-known alternative description [29] in terms of central extensions by $T$. Thus we will assume given a group $E$ containing $T$ as an open central subgroup, with $E/T$ identified with $D$. We will let $\sigma$ denote all of the data giving this central extension of $D$ by $T$. We can then form the group $C^*$-algebra $C^*(E)$, and the reduced algebra $C_r^*(E)$ (see [35]). Let $e$ denote the function on $T$ defined by $e(t) = \exp(2\pi it)$ for $t \in \mathbf{R}$, where we identify $T$ with $\mathbf{R}/\mathbf{Z}$ in the usual way. (We will use this $e$ throughout the paper. Here $\mathbf{R}$ = real numbers, while $\mathbf{Z}$ = integers.) Since $T$ is open in $E$, we can consider $e$ to be a continuous function on $E$ by letting $e$ have value zero off $T$. We choose Haar measure on $E$ so that $T$ has measure one. Then $e$ will be an idempotent in the center of $C_c(E)$ (the algebra of continuous complex-valued functions of compact support on $E$, with convolution), and so will represent a central idempotent in both $C^*(E)$ and $C_r^*(E)$. The algebras we wish to explore are then $eC^*(E)$ and $eC_r^*(E)$, which we denote by $C^*(D, \sigma)$ and $C_r^*(D, \sigma)$ respectively. Our principal objective is to describe a method for constructing projective modules over these algebras.

This method involves embedding $D$ as a cocompact subgroup of a larger (perhaps Lie) group $G$ to which $\sigma$ extends. In view of the description of $\sigma$ which we are presently using, this means that we must consider a group $H$ containing $T$ as a central subgroup, with quotient identified with $G$, together with a cocompact embedding of $E$ in $H$, such that the diagram

$$\begin{tikzcd}
 & & E \arrow[r] \arrow[dd] & D \arrow[dd] \arrow[dr] & \\
1 \arrow[r] & T \arrow[ur] \arrow[dr] & & & 1 \\
 & & H \arrow[r] & G \arrow[ur] &
\end{tikzcd}$$

commutes. We will denote again by $\sigma$ the corresponding extension of $G$ by $T$. If we identify $e$ with the measure on $T$ whose density with respect to the normalized Haar measure on $T$ is $e$, then we can view $e$ as a finite measure on $H$. As such, $e$ can also be viewed as an idempotent in the center of the double centralizer algebras of $C^*(H)$ and $C_r^*(H)$, so that we can form the algebras $eC^*(H)$ and $eC_r^*(H)$. We denote these algebras by $C^*(G, \sigma)$ and $C_r^*(G, \sigma)$ respectively.

Now the space $C_c(H)$, suitably equipped and completed as described in [**42**], forms an imprimitivity (i.e., equivalence) bimodule, $X = \bar{C}_c(H)$, between $C^*(E)$ and the transformation group $C^*$-algebra $C^*(H, H/E)$. I have not noticed the corresponding fact for $C_r^*$ mentioned in the literature, so record it here:

1.1 PROPOSITION. *Let $H$ be any locally compact group and $E$ any closed subgroup. Then a quotient, $X_r$, of the imprimitivity bimodule $X = \bar{C}_c(H)$ between $C^*(H, H/E)$ and $C^*(E)$, provides an imprimitivity bimodule between $C_r^*(H, H/E)$ and $C_r^*(E)$. In fact, $X_r = X/(XI)$ where $I$ is the kernel of the homomorphism from $C^*(E)$ onto $C_r^*(E)$.*

We omit the proof since it is not needed later; but it consists of straight-forward application of the results in Section 3 of [**44**].

We return now to the special $H$ and $E$ considered earlier. It is easily verified that $e$ commutes with everything in sight, so that if we set $Y = eX$ and $Y_r = eX_r$, then $Y$ will be an imprimitivity bimodule between $eC^*(H, H/E)$ and $eC^*(E)$ $(= C^*(D, \sigma))$, while $Y_r$ will be an imprimitivity bimodule between $eC_r^*(H, H/E)$ and $eC_r^*(E)$ $(= C_r^*(D, \sigma))$. The general basis for our construction of projective modules is then:

1.2 PROPOSITION. *Let $A$ and $B$ be $C^*$-algebras, with $A$ having an identity element, and let $Y$ be a $B$-$A$-imprimitivity bimodule. For any projection $p$ in $B$, the right $A$-module $pY$ is projective.*

*Proof.* This follows familiar lines [**46**]. Essentially from the definitions, we can find two finite sequences $(x_i)$ and $(y_i)$, each containing say $m$ elements in $Y$, such that

$$\sum \langle x_i, y_i \rangle_B = p.$$

In fact, it is clear that we can take all $x_i, y_i \in pY$. Then for any $z \in pY$ we have

$$z = pz = \sum \langle x_i, y_i \rangle_B z = \sum x_i \langle y_i, z \rangle_A,$$

so that the $x_i$ form a finite set of generators for $pY$. Furthermore, the mapping of $A^m$ to $pY$ given by

$$(a_i) \mapsto \sum x_i a_i$$

has as right inverse the mapping

$$z \mapsto (\, \langle y_i, z \rangle_A).$$

This expresses $pY$ as a direct summand of $A^m$, so that $pY$ is projective.

Thus we see that one way to construct projective modules over $C^*(D, \sigma)$ (or $C_r^*(D, \sigma)$ ) is to arrange matters so that we can see how to find projections in $eC^*(H, H/E)$ (or $eC_r^*(H, H/E)$ ). Note that $eC^*(H, H/E)$ need not have an identity element.

We now use the assumption that $E$ is cocompact in $H$. This assumption implies that there is a natural homomorphism of $C^*(H)$ into $C^*(H, H/E)$, and so of $eC^*(H)$ into $eC^*(H, H/E)$. Thus one way to find projections in $eC^*(H, H/E)$ is to find projections in $eC^*(H)$. But projections in $eC^*(H)$ correspond, more or less, to square-integrable $\sigma$-representations of $G$. We say "more or less" because, on the one hand, not all square-integrable representations give projections [**19**, **56**], while on the other hand we do not insist that the square-integrable representations be irreducible. Similar considerations apply to $eC_r^*(H)$. Note that $Y$ $(= e\overline{C}_c(H))$ is closely related to the restriction to $D$ of the right regular $\sigma$-representation of $G$.

In order to work effectively with the above generalities, it is very desirable to have matters defined at the level of functions. To begin with, one can hope that $p \in L^1(H)$. Then for $A = eC^*(E)$ (or $eC_r^*(E)$ ) one would try to set, for $f, g \in peL^1(H)$ and $s \in E$ (assuming that $H$ is unimodular),

$$\langle f, g \rangle_A(s) = f^* * g(s) = \int_H \bar{f}(x) g(xs)dx = \langle f * d_s, g \rangle_{L^2(H)}.$$

(Here $d_s$ denotes the "delta-function" at $s$, and we define the inner-product on $L^2(H)$ to be conjugate linear in the first variable.) For this to make sense, it is desirable that the right hand side should, as a function of $s$, be in $L^1(E)$. It is not clear to me how widely this can be expected to hold. It would be interesting to know, for example, how often it holds when $p$ is a minimal projection in the technical sense used in [**2**, **56**].

Anyway, rephrasing matters in terms of $\sigma$-representations, we see that if $\pi$ is an integrable right $\sigma$-representation of $G$ on a Hilbert space $\Xi$, we should seek a dense $*$-subalgebra $A$ of $L^1(D, \sigma)$, and a dense $A$-invariant subspace $V$ of $\Xi$ such that

$$s \mapsto \langle \xi\pi(s), \eta\rangle$$

is in $A$ for all $\xi$, $\eta \in V$. Taking this expression as the $A$-valued inner-product on $V$, we can complete to obtain a right $C^*(D, \sigma)$-module with $C^*(D, \sigma)$-valued inner-product, which may turn out to be a projective module. In the next two sections we will see how this can be accomplished when $D$ is Abelian by using Schwartz spaces for $A$ and $V$.

**2. The Heisenberg equivalence bimodule.** Suppose now that the discrete group $D$ is Abelian. Given a cocycle $\sigma$ on $D$, we wish to embed $D$ in a larger group to which $\sigma$ extends, and for which there exists a square-integrable $\sigma$-representation. One such situation, in which the larger group is also Abelian, is quite familiar, and suffices for our present purposes. Let $M$ be any locally compact Abelian group, let $\hat{M}$ be its dual group, and let $G = M \times \hat{M}$. Then on $G$ we have the canonical bicharacter $\beta$ defined by

$$\beta(\,(m, s), (n, t)\,) = \langle m, t\rangle,$$

where here $\langle\ ,\ \rangle$ denotes the duality between $M$ and $\hat{M}$. (We will also use $\langle\ ,\ \rangle$ to denote the inner-product on $L^2(M)$. The context will make clear which meaning is intended.) Furthermore, $G$ has a canonical square-integrable $\beta$-representation, $\pi$, on $L^2(M)$, the Heisenberg representation [41], defined (using the conventions on page 149 of [58]) by

$$(\pi_{(m,s)}f)(n) = \langle n, s\rangle f(n + m).$$

In view of this, we will refer to $\beta$ as the Heisenberg cocycle on $G$.

The commutation relation among the operators of $\pi$ is given by

$$\pi_x\pi_y = \beta(x, y)\pi_{x+y} = \beta(x, y)\bar{\beta}(y, x)\pi_y\pi_x$$

for $x, y \in G$. It is thus natural to define a skew bicharacter, $\rho$, on $G$ by

$$\rho(x, y) = \beta(x, y)\bar{\beta}(y, x),$$

so that

$$\pi_x\pi_y = \rho(x, y)\pi_y\pi_x.$$

(This $\rho$ will essentially correspond to the $\rho$ in [17].) It is not cohomologous to $\beta$, but rather to $\beta^2$. However, as discussed in [25, 31], $\rho$ determines the cohomology class of $\beta$.

As suggested in the first section, we will in later sections prefer to work with right modules. However, left representations are more familiar, and for this reason we will in this section work primarily with left modules. Towards the end of the next section a convenient way of reinterpreting our formulas in terms of right modules will emerge.

We will be concerned with embedding discrete groups in $G$. But in this section discreteness is not important. Thus until further notice we will let $D$ be any locally compact Abelian group, usually viewed as a closed subgroup of $G$. To show the generality of our situation, we point out that if $\gamma$ is any (continuous) bicharacter on $D$, then an embedding of $D$ into an appropriate $G$ can always be found such that $\gamma$ is the restriction to $D$ of the Heisenberg cocycle on $G$. To see this, let $\phi$ denote the homomorphism of $D$ into $\hat{D}$ defined by

$$\langle x, \phi(y) \rangle = \gamma(x, y).$$

Let $G = D \times \hat{D}$ with its Heisenberg cocycle $\beta$, and let $\psi$ denote the embedding of $D$ into $G$ defined by

$$\psi(x) = (x, \phi(x)).$$

Then

$$\beta(\psi(x), \psi(y)) = \langle x, \phi(y) \rangle = \gamma(x, y)$$

as desired. (Notice also that if $D$ is discrete, then $\psi(D)$ is a lattice in $G$.) From now on we will not use $\gamma$, but rather will use $\beta$ to denote both the Heisenberg cocycle and its restriction to $D$. Similarly we will denote the restrictions to $D$ of $\rho$ and $\pi$ again by $\rho$ and $\pi$, and we will denote $C^*(D, \beta)$ by $B$ (or $B_\beta$). We recall from **[60]** that for $\Phi, \Psi \in L^1(D, \beta)$ the operations in $B_\beta$ are defined by

$$(\Phi\Psi)(x) = \int_D \Phi(y)\Psi(x - y)\beta(y, x - y)dy$$

$$\Phi^*(x) = \beta(x, x)\bar{\Phi}(-x).$$

It will be important for us to know that the representation $\pi$ of $B_\beta$ on $L^2(M)$ is faithful. In order to show this, and for other purposes, we need to consider the dual action of the dual group of $D$ on $B_\beta$. Now the dual group, $\hat{G}$, of $G$ can be identified with $\hat{M} \times M$ in the evident way. And from this it is easily seen that every character of $G$ is of the form

$$x \mapsto \rho(x, y)$$

for some $y \in G$. This establishes a specific isomorphism between $G$ and $\hat{G}$. Now every character of $D$ extends (not uniquely) to a character of $G$, so that every character of $D$ will be of the form

$$w \mapsto \rho(w, y)$$

for some $y \in G$, where $w \in D$. Here $y$ is not unique, and the indicated homomorphism from $G$ to $\hat{D}$ has as kernel exactly the subgroup

$$D^{\perp} = \{y \in G: \quad \rho(w, y) = 1 \text{ for all } w \in D\}.$$

This subgroup $D^{\perp}$ will play a major role shortly.

Let $\alpha$ denote the dual action of $\hat{D}$ on $B_\beta$. This action is defined on $C_c(D, \beta)$ by

$$(\alpha_t \Phi)(w) = \langle w, t\rangle \Phi(w)$$

for $\Phi \in C_c(D, \beta)$, $t \in \hat{D}$ and $w \in D$ (so that $\beta$ is not explicitly involved). This action gives an action of $\hat{D}$ on the primitive ideal space of $B_\beta$, and P. Green has shown in Proposition 34 of **[18]** (see also the last sentence of its proof) that:

2.1 Proposition. *The dual action of $\hat{D}$ on the primitive ideal space of $B_\beta$ is transitive. In particular, there are no proper $\hat{D}$-invariant ideals in $B_\beta$.*

Consider now the representation $\pi$ of $B_\beta$ on $L^2(M)$. To show that this representation is faithful, it suffices to show, according to Proposition 2.1, that the kernel of $\pi$ is $\hat{D}$-invariant. But the integrated form of the relation

$$\pi_x \pi_w \pi_x^* = \rho(x, w)\pi_w$$

for $x \in G$ and $w \in D$, is easily seen to be

$$\pi_x \pi(\Phi)\pi_x^* = \pi(\alpha_x \Phi)$$

for $\Phi \in B_\beta$, where by $\alpha_x$ we denote the dual automorphism of $B_\beta$ corresponding to the character $w \mapsto \rho(x, w)$ of $D$. It follows that if $\Phi$ is in the kernel of $\pi$, then so is $\alpha_x(\Phi)$ for any $x \in G$. Since every character of $D$ comes from an $x \in G$, as seen above, the kernel of $\pi$ must be a $\hat{D}$-invariant ideal of $B_\beta$. Consequently we obtain, as desired:

2.2 Proposition. *The representation $\pi$ of $B_\beta$ on $L^2(M)$ is faithful.*

Following the method described in the first section, we wish to construct from the above situation a left $B_\beta$-module $V$ with $B_\beta$-valued inner-product. For this we need suitable spaces of functions. In the present context this means that we need a space of functions on $M$ which behaves well under both the Fourier transform and restriction to subgroups. As suggested by Weil **[58]**, the appropriate space is the space $S(M)$ of Schwartz functions on $M$, as defined by Bruhat [7]. When $M$ is "elementary" in the sense of no. 11 of **[58]**, that is, when $M$ is a Lie group of the form $\mathbf{R}^p \times \mathbf{Z}^q \times T^m \times F$ where $F$ is a finite group, then $S(M)$ is defined as usual as the space of infinitely differentiable functions which, together with all their derivatives, vanish at infinity more rapidly than any polynomial grows (where "polynomial" only refers to the $\mathbf{R}^p \times Z^q$ part of $M$). Since in later sections we will only need to consider the case in which $M$ is elementary, no difficulty will occur if the reader assumes throughout this section also, that all groups considered are elementary.

The crucial property which we obtain by using Schwartz functions is:

2.3 LEMMA. *If $f$, $g \in S(M)$, then the function on $G$ defined by*

$$x \mapsto \langle f, \pi_x g \rangle$$

*is in $S(G)$.*

*Proof.* In this section the inner-product in $L^2(M)$ is taken to be conjugate linear in the second variable, since we work with left representations. For $m \in M$ and $s \in \hat{M}$, we have

$$\langle f, \pi_{(m,s)} g \rangle = \int_M f(n) \langle n, s \rangle^{-} \bar{g}(n + m) dn.$$

Now for fixed $m$ this is just the Fourier transform of the function

$$n \mapsto f(n) \bar{g}(n + m).$$

But for $f$, $g \in S(M)$ the function $f(n)\bar{g}(n + m)$ on $M \times M$ is easily seen to be in $S(M \times M)$, and it is easily seen that the operation of taking Fourier transforms in the second variable is an isomorphism of $S(M \times M)$ onto $S(M \times \hat{M}) = S(G)$, as indicated at the top of page 159 of [**58**].

It is also easily seen that for any $f \in S(G)$ the restriction of $f$ to $D$ will be in $S(D)$. (A somewhat more general fact is indicated at the top of page 167 of [**58**].) In particular, we obtain the following crucial fact:

2.4 COROLLARY. *For $f$, $g \in S(M)$ the function on $D$ defined by*

$$w \mapsto \langle f, \pi_w g \rangle$$

*is in $S(D)$.*

2.5 *Notation.* For $f, g \in S(M)$ we let $\langle f, g \rangle_B$ denote the function in $S(D)$ defined by

$$\langle f, g \rangle_B(w) = \langle f, \pi_w g \rangle$$

for $w \in D$.

We wish to show that this $S(D)$-valued inner-product is compatible with the action of $S(D)$ on $S(M)$. But first we must make sure that this action is well-defined.

2.6 LEMMA. *Let $D$ be a closed subgroup of $G$, let $f \in S(M)$, and let $\Phi \in S(D)$. Then $\pi(\Phi)f$ is in $S(M)$.*

*Proof.* We indicate the proof for the case of elementary groups, and leave to the reader its extension to the general case. (But see Sections 6, 7 and 9 of [7].) For $M$, and so $D$, elementary, we will see that we do not need to assume that $\Phi$ is differentiable, but only that $\Phi$ vanishes rapidly at infinity. We must show that if $P$ is any polynomial on $M$ and if $\Delta$ is any differential operator with constant coefficients on $M$, then $P\Delta(\pi(\Phi)f)$ is a bounded function. (See page 158 of [**58**].) Now, to begin with,

$$(\pi(\Phi)f)(m) = \int_D \Phi(w)\langle m, w''\rangle f(m + w')dw,$$

where $w = (w', w'') \in M \times \hat{M}$. From this it is fairly evident that $\pi(\Phi)f$ is infinitely differentiable.

Next, any differential operator with constant coefficients is a linear combination of products of the operators $\partial/\partial m_j$, where for $M = \mathbf{R}^k \times \mathbf{Z}^p \times T^q \times F$ the $m_j$ are coordinates in $\mathbf{R}^k$ or $T^q$ (and, for the $T^q$ variables, functions are viewed as defined on $\mathbf{R}^q$ but constant on cosets of $\mathbf{Z}^q \subset \mathbf{R}^q$). To compute, we must first clarify that our convention concerning the identification of $\mathbf{R}$ with $\hat{\mathbf{R}}$ is that

$$\langle r, s\rangle = e(rs) = \exp(2\pi i rs) \quad \text{for } r, s \in \mathbf{R}.$$

Then

$$(\partial(\pi(\Phi)f)/\partial m_j)(m) = \int_D \Phi(w)2\pi i w_j''\langle m, w''\rangle f(m + w')dw$$
$$+ \int_D \Phi(w)\langle m, w''\rangle(\partial f/\partial m_j)(m + w')dw.$$

But $\Phi(w)2\pi i w_j''$ is again in $S(D)$ and $\partial f/\partial m_j$ is again in $S(M)$. It follows that any $\Delta(\pi(\Phi)f)$ is a finite sum of terms of the form $\pi(\Phi)f$ for other $\Phi$'s and $f$'s. Thus it suffices to show that for any $\Phi \in S(D)$ and any $f \in S(M)$ the function $P\pi(\Phi)f$ is bounded, for any polynomial $P$ on $M$.

But any polynomial $P$ is a sum of products of coordinates $m_j$, where now these are coordinates in $\mathbf{R}^k$ or $\mathbf{Z}^p$. Then

$$\begin{aligned} m_j(\pi(\Phi)f)(m) &= \int_D m_j\Phi(w)\langle m, w''\rangle f(m + w')dw \\ &= \int_D \Phi(w)\langle m, w''\rangle(m_j + w_j')f(m + w')dw \\ &- \int_D w_j'\Phi(w)\langle m, w''\rangle f(m + w')dw. \end{aligned}$$

Now $n \mapsto n_j f(n)$ is in $S(M)$ for $n \in M$, and $w \mapsto w_j'\Phi(w)$ is in $S(D)$, so it follows that for any polynomial $P$ the function $P\pi(\Phi)f$ is a finite sum of terms of the form $\pi(\Phi)f$ for other $\Phi$'s and $f$'s. Thus it suffices to show that any $\pi(\Phi)f$ is a bounded function. But this follows from simple estimates using the fact that $\Phi \in L^1(D)$.

2.7 *Notation.* For $f \in S(M)$ and $\Phi \in S(D)$ we will denote $\pi(\Phi)f$ by $\Phi f$.

2.8 Proposition. *For $f$, $g \in S(M)$ and $\Phi \in S(D)$ we have*

$$\langle \Phi f, g\rangle_B = \Phi\langle f, g\rangle_B.$$

*Proof.* For $w \in D$ we have

$$\begin{aligned}\langle \Phi f, g\rangle_B(w) &= \langle \pi(\Phi)f, \pi_w g\rangle \\ &= \int_D \Phi(v)\langle \pi_v f, \pi_w g\rangle dv = \int_D \Phi(v)\langle f, \pi_v^* \pi_w g\rangle dv \\ &= \int_D \Phi(v)\langle f, \pi_{w-v} g\rangle \beta(v, w-v) dv \\ &= (\Phi * \langle f, g\rangle_B)(w),\end{aligned}$$

since $\pi_v \pi_{w-v} = \beta(v, w-v)\pi_w$.

We now continue to show that with the operations defined above, $S(M)$ is, in effect, an $S(D)$-rigged space, in the terminology of [**42**]. For this we need:

2.9 Proposition. *With operations defined as above, we have*

(1) $\langle f, g\rangle_B^* = \langle g, f\rangle_B$ *for* $f, g \in S(M)$.
(2) *The linear span of the range of* $\langle\ ,\ \rangle_B$ *is dense in* $B_\beta = C^*(D, \beta)$.

*Proof.* For (1) we have

$$\begin{aligned}(\langle f, g\rangle_B^*)(w) &= \beta(w, w)(\langle f, g\rangle_B(-w))^- \\ &= \beta(w, w)\langle \pi_{-w} g, f\rangle = \beta(w, w)\langle g, (\pi_w)^* f\rangle \\ &= \beta(w, w)\langle g, \beta(w, w)\pi_w f\rangle = \langle g, f\rangle_B(w),\end{aligned}$$

as desired.

For (2), we notice first that from part (1) and from Proposition 2.8, the linear span of the range of $\langle\ ,\ \rangle_B$ is an ideal in $S(D)$. Let us denote the norm closure of this linear span by $I$, so that $I$ is an ideal in $B_\beta$. We will show that $I$ is $\hat{D}$-invariant, so that by Proposition 2.1 it must be all of $B_\beta$. Now for any $x \in G$ and $f, g \in S(M)$, we have

$$\begin{aligned}(\alpha_x(\langle f, g\rangle_B))(w) &= \rho(x, w)\langle f, \pi_w g\rangle \\ &= \rho(x, w)\langle \pi_x f, \pi_x \pi_w g\rangle = \langle \pi_x f, \pi_w \pi_x g\rangle \\ &= \langle \pi_x f, \pi_x g\rangle_B(w).\end{aligned}$$

It follows that $I$ is $\hat{D}$-invariant, as desired.

The remaining fact which we need in order to know that $S(M)$ is an $S(D)$-rigged space is that $\langle f, f\rangle_B$ is a positive element of the $C^*$-algebra $B_\beta$ for any $f \in S(M)$. It will be convenient to defer the proof of this fact until we have established some facts about the commutant of the action of $B_\beta$ on $L^2(M)$.

Now it is clear from the commutation relation for the $\pi_x$'s that the set of $x$'s in $G$ such that $\pi_x$ commutes with $\pi_w$ for all $w \in D$ is exactly $D^\perp$. It follows that all operators in the range of the representation of $C^*(D^\perp, \beta)$

on $L^2(M)$ will commute with those from $B_\beta$. Anticipating our later preference for right modules, we here prefer to use, instead of $C^*(D^\perp, \beta)$, its opposite algebra, viewed as acting on the right on $L^2(M)$. It is clear from the commutation relation for the $\pi_x$'s that this opposite algebra is $C^*(D^\perp, \tilde{\beta})$ where we define

$$\tilde{\beta}(x, y) = \beta(y, x).$$

Now $\tilde{\beta}$ is cohomologous to $\bar{\beta}$, since $\tilde{\beta}\beta$ is symmetric and so a coboundary (by Lemma 7.2 of [25]). It turns out that various formulas are simpler if we use $\bar{\beta}$ instead of $\tilde{\beta}$. We can arrange to do this by replacing $\pi_z$ by $\pi_z^*$. Accordingly, for $\Omega \in S(D^\perp)$ and $f \in S(M)$ we define $f\Omega$ by

$$f\Omega = \int_{D^\perp} (\pi_z^* f)\Omega(z)dz.$$

Then, as in Lemma 2.6, one checks that $f\Omega \in S(M)$. It is easily seen that we obtain thus a right $*$-representation of

$$S(D^\perp) \subset L^1(D^\perp, \bar{\beta}).$$

We let $A$ (or $A_{\bar{\beta}}$) denote the enveloping $C^*$-algebra of $L^1(D^\perp, \bar{\beta})$. Then, exactly as in Proposition 2.2, the above action extends to a faithful right action of $A$ on $L^2(M)$. We now define an inner-product, $\langle\ ,\ \rangle_A$, on $S(M)$ with values in $S(D^\perp) \subseteq A$, by

$$\langle f, g\rangle_A(z) = \langle \pi_z g, f\rangle$$

for $f, g \in S(M)$, $z \in D^\perp$. As in Corollary 2.4, one checks that $\langle f, g\rangle_A \in S(D^\perp)$. Moreover, as in Proposition 2.8, one has

$$\langle f, g\Omega\rangle_A = \langle f, g\rangle_A\Omega,$$

while, as in Proposition 2.9, we find that

$$\langle f, g\rangle_A^* = \langle g, f\rangle_A,$$

and that the linear span of the range of $\langle\ ,\ \rangle_A$ is dense in $A$. Thus we have verified all of the requirements for $S(M)$ to be a right-rigged $S(D^\perp)$-space except the positivity of the inner-product, whose proof we again defer.

But what we really need is that $S(M)$ provide an equivalence bimodule (i.e., imprimitivity bimodule in the terminology of [42]) between $B$ and $A$. For this we need to verify that for any $f, g, h \in S(M)$ we have

$$\langle f, g\rangle_B h = f\langle g, h\rangle_A.$$

This is not straightforward, since one side involves an integral over $D$ while the other involves an integral over $D^\perp$. In fact, noticing this, we see that the equation cannot be true unless we have normalized the Haar measures on $D$ and $D^\perp$ properly. We set about doing this by first fixing arbitrary Haar measures on $M$ and $D$. We then choose the Plancherel Haar measure on $\hat{M}$, and the corresponding product Haar measure on $G$. We remark that this latter is a canonical choice of Haar measure on $G$ (for its decomposition as $M \times \hat{M}$) in the sense that it is independent of the choice of Haar measure on $M$. In fact it is the unique Haar measure on $G$ which is "self-Plancherel" for the identification of $G$ with $\hat{G}$ determined by $\rho$. Now under this identification, $D^{\perp}$ is the annihilator of $D$. It follows that $D^{\perp}$ can be identified as the dual group of $G/D$. On $G/D$ we choose as usual the Haar measure such that for $F \in C_c(G)$ we have

$$\int_G F(x)dx = \int_{G/D}\left(\int_D F(x + w)dw\right)d\dot{x}.$$

Then on $D^{\perp}$ we choose the corresponding Plancherel Haar measure, that is, the Haar measure such that for $f, g \in C_c(G/D)$ we have

$$\int_{G/D} f(\dot{x})\bar{g}(\dot{x})d\dot{x} = \int_{D^{\perp}} \hat{f}(z)\bar{\hat{g}}(z)dz,$$

where $\hat{f}$ is the Fourier transform of $f$ defined via $\rho$, that is,

$$\hat{f}(z) = \int_{G/D} \rho(z, \dot{x})f(\dot{x})d\dot{x}$$

for $z \in D^{\perp}$. We are now in a situation to which we will be able to apply the Poisson summation formula, in the general form found on page 153 of [**28**]. Specifically, if $F \in S(G)$, then

$$\int_D F(w)dw = \int_{D^{\perp}} \hat{F}(z)dz$$

where $\hat{F}$ is defined by

$$\hat{F}(y) = \int_G \rho(y, x)F(x)dx.$$

(It was Paul Chernoff, ardent fan of the Poisson summation formula, who suggested its use to simplify my original arguments.)

We now need the well-known orthogonality relation for the Heisenberg representation, and, in particular, the fact that for our specific choice of Haar measures the formal dimension is 1.

2.10 Lemma. *If $f, g, h, k \in S(M)$, then*

$$\int_G \langle f, \pi_x g\rangle\langle h, \pi_x k\rangle^{-} dx = \langle f, h\rangle\langle g, k\rangle^{-}.$$

*Proof.* For $m \in M$ let $L_m f$ be defined by

$$(L_m f)(n) = f(n + m).$$

With this notation, and with $x = (m, s)$ etc., the left hand side becomes

$$\int_M \int_{\hat{M}} \int_M f(n)\langle n, s\rangle^{-}\bar{g}(n + m)dn$$

$$\times \int_M \bar{h}(p)\langle p, s\rangle k(p + m)dpdsdm$$

$$= \int_M \int_{\hat{M}} (fL_m\bar{g})^{\wedge}(s)(hL_m\bar{k})^{\wedge -}(s)dsdm$$

where $^\wedge$ here denotes the ordinary Fourier transform. Applying the ordinary Plancherel theorem, we obtain

$$\int_M \int_M (fL_m\bar{g})(n)(hL_m\bar{k})^-(n)dndm$$

$$= \int_M f(n)\bar{h}(n) \int_M \bar{g}(n + m)k(n + m)dmdn$$

$$= \langle f, h\rangle\langle g, k\rangle^-.$$

The next result is the key to the relation between the inner-products with values in $A$ and $B$ respectively.

2.11 Proposition. *Let $f, g, h, k \in S(M)$, let $D$ be a closed subgroup of $G$, and let the Haar measure on $D^\perp$ be normalized in terms of that on $D$ as discussed above. Then*

$$\int_D \langle f, \pi_w g\rangle\langle h, \pi_w k\rangle^- dw = \int_{D^\perp} \langle f, \pi_z h\rangle\langle g, \pi_z k\rangle^- dz.$$

*Proof.* Define $F$ on $G$ by

$$F(x) = \langle f, \pi_x g\rangle\langle h, \pi_x k\rangle^-,$$

so that $F \in S(G)$ by Lemma 2.3. Let $\hat{F}$ be the Fourier transform of $F$ using $\rho$. Then

$$\hat{F}(y) = \int_G \langle f, \pi_x g\rangle\langle h, \pi_x k\rangle^- \rho(y, x)dx$$

$$= \int_G \langle \pi_y f, \pi_y\pi_x g\rangle\langle h, \pi_x k\rangle^- \rho(y, x)dx$$

$$= \int_G \langle \pi_y f, \pi_x\pi_y g\rangle\langle h, \pi_x k\rangle^- dx$$

$$= \langle \pi_y f, h\rangle\langle \pi_y g, k\rangle^-,$$

where the last step uses Lemma 2.10. If we now apply to $F$ the generalized Poisson summation formula in the form given above, we obtain

$$\int_D \langle f, \pi_w g\rangle\langle h, \pi_w k\rangle^- dw$$

$$= \int_{D^\perp} \langle \pi_z f, h\rangle\langle \pi_z g, k\rangle^- dz$$

$$= \int_{D^\perp} \langle f, \beta(z, z)\pi_{-z}h\rangle\langle g, \beta(z, z)\pi_{-z}k\rangle^- dz$$

$$= \int_{D^\perp} \langle f, \pi_z h\rangle\langle g, \pi_z k\rangle^- dz.$$

2.12 Proposition. *If $f$, $g$, $h \in S(M)$, then*

$$\langle f, g\rangle_B h = f\langle g, h\rangle_A.$$

*Proof.* Since $S(M)$ is dense in $L^2(M)$, it suffices to show that the inner-products of both sides with any $k \in S(M)$ are equal. But

$$\langle\, \langle f, g\rangle_B h, k\rangle = \left\langle \int_D \langle f, g\rangle_B(w)\pi_w h dw, k\right\rangle$$
$$= \int_D \langle f, \pi_w g\rangle\langle k, \pi_w h\rangle^- dw.$$

Applying Proposition 2.11 with $h$ and $k$ interchanged, we obtain

$$= \int_{D^\perp} \langle f, \pi_z k\rangle\langle g, \pi_z h\rangle^- dz$$

$$= \left\langle \int_{D^\perp} (\pi_z^* f)\langle g, h\rangle_A(z) dz, k\right\rangle$$

$$= \langle f\langle g, h\rangle_A, k\rangle$$

as desired.

By using the same techniques, we can now show the positivity of the inner-products.

2.13 Proposition. *If $f \in S(M)$, then $\langle f, f\rangle_A$ and $\langle f, f\rangle_B$ are positive elements of the $C^*$-algebras $A$ and $B$ respectively.*

*Proof.* According to Proposition 2.2 the representations of $A$ and $B$ on $L^2(M)$ are faithful, so it suffices to show that $\langle f, f\rangle_A$ and $\langle f, f\rangle_B$ are positive as operators on $L^2(M)$. Since $S(M)$ is dense in $L^2(M)$, it suffices to verify positivity on $S(M)$. But for $h \in S(M)$ we have

$$\langle\, \langle f, f\rangle_B h, h\rangle = \left\langle \int_D \langle f, f\rangle_B(w)\pi_w h, h\right\rangle$$

$$= \int_D \langle f, \pi_w f\rangle\langle h, \pi_w h\rangle^- dw$$

$$= \int_{D^\perp} \langle f, \pi_z h\rangle\langle f, \pi_z h\rangle^- dz \geqq 0,$$

where we have used Proposition 2.11 in the last step. The proof of positivity for $\langle f, f\rangle_A$ is similar.

We have now verified all of the conditions for $S(M)$ to provide an equivalence bimodule (i.e., imprimitivity bimodule as defined in 6.10 of [**42**]) except for the relation with the operator norms of $A$ and $B$. For this we must show:

2.14 PROPOSITION. *For $f \in S(M)$, $\Phi \in B$, and $\Omega \in A$, we have*

$$\langle \Phi f, \Phi f\rangle_A \leqq \|\Phi\|^2 \langle f, f\rangle_A,$$

*and*

$$\langle f\Omega, f\Omega\rangle_B \leqq \|\Omega\|^2 \langle f, f\rangle_B.$$

*Proof.* The key fact used in this proof is that the representations of $A$ and $B$ on $L^2(M)$ are faithful (Proposition 2.2), so that it suffices to verify the above relations for the corresponding operators on $L^2(M)$. Now for any $h \in S(M)$ we have

$$\begin{aligned}\langle h\langle \Phi f, \Phi f\rangle_A, h\rangle &= \langle\, \langle h, \Phi f\rangle_B \Phi f, h\rangle \\ &= \langle \Phi f, \langle \Phi f, h\rangle_B h\rangle = \langle \Phi f, \Phi f\langle h, h\rangle_A\rangle \\ &= \langle \Phi(f\langle h, h\rangle_A{}^{1/2}), \Phi(f\langle h, h\rangle_A{}^{1/2})\,\rangle \\ &\leqq \|\Phi\|^2\langle f, f\langle h, h\rangle_A\rangle = \|\Phi\|^2\langle h\langle f, f\rangle_A, h\rangle.\end{aligned}$$

The desired inequality follows from the density of $S(M)$ in $L^2(M)$. A similar calculation works for the other case.

We can now define a norm on $S(M)$ by letting the norm of $h \in S(M)$ be $\|\,\langle h, h\rangle_A\|^{1/2}$, or equivalently, by Proposition 3.1 of [**44**], $\|\,\langle h, h\rangle_B\|^{1/2}$. Then the completion of $S(M)$ will be a $B$-$A$-equivalence bimodule, where $B$ and $A$ are now the completed $C^*$-algebras. We summarize all of the above as follows:

2.15 THEOREM. *Let $M$ be a locally compact Abelian group, let $G = M \times \hat{M}$, and let $\beta$ be the cocycle for the Heisenberg projective representation of $G$ on $L^2(M)$. Let $D$ be any closed subgroup of $G$, and let $D^\perp$ be the annihilator of $D$ with respect to the skew cocycle coming from $\beta$, with Haar measure normalized as discussed earlier. Let $B$ be the group algebra of $D$ twisted by the restriction of $\beta$ to $D$, and let $A$ be the group algebra of $D^\perp$ twisted by the restriction of $\bar{\beta}$ to $D^\perp$. Then the Schwartz space $S(M)$, suitably completed, and with the operations defined earlier, becomes a $B$-$A$-equivalence bimodule.*

We now digress briefly to consider the situation for the corresponding von Neumann algebras. If, for the moment, we let $B$ and $A$ denote the algebras $S(D, \beta)$ and $S(D^\perp, \bar{\beta})$, with their actions on $S(M)$, and if we equip $S(M)$ with the ordinary inner-product from $L^2(M)$, then it is easily seen from the results above that $S(M)$ becomes a Hilbert $B$-$A$-birigged space, as defined in 1.1 of [**43**]. Then Theorem 1.9 of [**43**] is immediately applicable, and so we conclude that $B$ and $A$ generate each other's commutant, as algebras of operators on $L^2(M)$. But it is easily seen that these commutants are also generated by the corresponding projective representations of $D$ and $D^\perp$. Thus we obtain:

2.16 THEOREM. *Let $M$, $G$, $\beta$, $D$ and $D^\perp$ be as in Theorem* 2.15, *and let $\pi$ be the Heisenberg projective representation of $G$ on $L^2(M)$. Let $\bar{B}$ and $\bar{A}$ be the von Neumann algebras generated by $\pi(D)$ and $\pi(D^\perp)$ respectively. Then $\bar{B}$ and $\bar{A}$ are each other's commutant.*

This result is essentially contained in Proposition III.4.4 of [**14**], where it is obtained by quite different methods.

We close this section by reinterpreting some of its main results in a way which is quite suggestive of further developments, although we will not specifically need this reinterpretation in later sections of this paper.

Let $K$ denote the algebra of compact operators on $L^2(M)$. Conjugation by the Heisenberg representation gives an ordinary action of $G$ as a group of automorphisms of $K$, which for the moment we denote by $\alpha$. We claim that $\alpha$ is integrable in a quite strong sense. There are various definitions of what is meant by an integrable action (see [**35**] ). But the following version does not seem to have been considered before.

2.17 *Definition.* Let $G$ be any locally compact group, let $K$ be any $C^*$-algebra, with $M(K)$ its multiplier algebra, and let $\alpha$ be an action of $G$ on $K$. Then we say that $\alpha$ is *strongly subgroup integrable* if $K$ has a dense $\alpha$-invariant $*$-subalgebra, $K_0$, such that for any closed subgroup $D$ of $G$, and any $a \in K_0$ there is a $b \in M(K)$ such that for all $c \in K_0$ the function $w \mapsto \alpha_w(a)c$ on $D$ is integrable, and

$$bc = \int_D \alpha_w(a)cdw.$$

By taking adjoints one sees immediately that $w \mapsto c\alpha_w(a)$ also is integrable. By considering integrals over the net of compact subsets of $D$ it is easily seen that also

$$cb = \int_D \alpha_w(a)cdw.$$

It is natural to write symbolically

$$b = \int_D \alpha_w(a)dw.$$

2.18 THEOREM. *Let $M$ be a locally compact Abelian group, let $G = M \times \hat{M}$, and let $\alpha$ be the action of $G$ on $K = K(L^2(M))$ obtained by conjugating by the Heisenberg projective representation of $G$. Then $\alpha$ is strongly integrable.*

*Proof.* As the dense subalgebra $K_0$ we take the algebra of finite linear combinations of rank one operators $\langle f, g\rangle_K$ for $f, g \in S(M)$ (where by definition $\langle f, g\rangle_K h = \langle h, g\rangle f$). Then it suffices to consider $a = \langle f, g\rangle_K$ and $c = \langle h, k\rangle_K$ for $f, g, h, k \in S(M)$. If $D$ is any closed subgroup of $G$, and $w \in D$, then

$$\begin{aligned}\alpha_w(a)c &= \pi_w\langle f, g\rangle_K \pi_w^*\langle h, k\rangle_K \\ &= \langle \pi_w f, \pi_w g\rangle_K \langle h, k\rangle_K \\ &= \langle h, \pi_w g\rangle\langle \pi_w f, k\rangle_K,\end{aligned}$$

which is integrable over $D$ since $w \mapsto \langle h, \pi_w g\rangle$ is in $S(D)$ by Corollary 2.4. Furthermore,

$$\begin{aligned}\int_D \alpha_w(a)c\,dw &= \left\langle \int_D \langle h, \pi_w g\rangle \pi_w f\,dw,\ k\right\rangle_K \\ &= \langle\, \langle h, g\rangle_B f, k\rangle_K,\end{aligned}$$

which by Proposition 2.12

$$= \langle h\langle g, f\rangle_A, k\rangle_K = \langle g, f\rangle_A\langle h, k\rangle_K.$$

Thus, symbolically,

$$\int_D \alpha_w(\,\langle f, g\rangle_K)dw = \langle g, f\rangle_A.$$

In analogy with the situation for a free wandering action on a locally compact space $X$, for which the corresponding action on $C_\infty(X)$ is strongly integrable and the algebra generated by all the $\int_D \alpha_w(a)dw$ is just $C_\infty(X/D)$, we can, for any strongly integrable action $\alpha$ of a group $D$ on a $C^*$-algebra $K$, suggestively write $K/D$ for the $C^*$-subalgebra of $M(K)$ generated by all the $\int_D \alpha_w(a)dw$. Then the proof of Theorem 2.18 makes clear that, in that setting, $K/D = A$.

Suppose now that $M$ is the real line, $\mathbf{R}$, so that $G = \mathbf{R}^2$. We can view the action of $G$ on $K(L^2(R)\,)$ to be a "quantized plane". (In fact, for a suitable insertion of a "Planck's constant" $h$ parametrizing the infinite-dimensional irreducible representations of the Heisenberg group, the algebra $K(L^2(R)\,)$ will, as $h$ goes to 0, become $C_\infty(R^2)$, with the action of $G$ becoming translation.) We will see in the next section that if $D$ is chosen to be a lattice in $\mathbf{R}^2$ (so $\cong \mathbf{Z}^2$), then $A = C^*(D^\perp, \bar{\beta})$ will be a non-commutative 2-torus, and in fact an irrational rotation algebra if $D$ is irrationally skewed to the usual decomposition of $\mathbf{R}^2$. Thus we have

$$\text{(quantized plane)}/(D = \mathbf{Z}^2) = \text{(non-commutative torus)}.$$

Given the usual relation between an ordinary plane, a lattice it contains, and an ordinary torus, it is reasonable to consider the "quantized plane" $K$ to be the "simply-connected covering space" of the non-commutative 2-torus, and the action of $D$ on $K$ to be the "fundamental group" of the non-commutative 2-torus.

It is clear that a similar situation will prevail in higher dimensions, but we will not discuss it here, nor will we explicitly use this point of view later, although it can be helpful to keep it in mind.

**3. The case of lattices.** We suppose now that $D$ is a discrete (Abelian) group. Then $C^*(D, \beta)$ is a $C^*$-algebra with identity element. We suppose further that $D$ is a lattice in $G$ $(= M \times \hat{M})$, that is, a discrete subgroup such that $G/D$ is compact. Then the dual group for $G/D$ will be discrete. But we saw in the last section that this dual group can be identified with $D^\perp$. Thus $D^\perp$ must be a discrete subgroup of $G$. In the same way, the dual of $G/D^\perp$ can be identified with $D$, which is discrete. It follows that $G/D^\perp$ is compact. We have thus shown:

3.1 LEMMA. *If $D$ is a lattice in $G$, then so is $D^\perp$.*

In particular, we see that $C^*(D^\perp, \bar{\beta})$ will have an identity element. But it is well-known that if $A$ and $B$ are both $C^*$-algebras with identity elements and if $V$ is a $B$-$A$-equivalence bimodule, then $V$ is a projective right $A$-module, and a projective left $B$-module. (The arguments are contained, for example, in the proof of Proposition 2.1 of [**46**].) Furthermore, $B$ will be the full endomorphism ring of the $A$-module $V$. Thus we have shown:

3.2 PROPOSITION. *Let $D$ be a lattice in $G$ $(= M \times \hat{M})$, let $A = C^*(D^\perp, \bar{\beta})$, and let $V$ denote the right $A$-module obtained by completing $S(M)$ as described earlier. Then $V$ is a projective $A$-module whose full endomorphism ring is $C^*(D, \beta)$, acting as described earlier.*

Thus $V$ represents an element, $[V]$, of $K_0(A)$. Since $D^\perp$ is discrete, $A$ has a canonical finite normalized trace, $\tau$ (or $\tau_A$), coming from evaluating elements of $S(D^\perp)$ at the identity element of $D^\perp$. Then $\tau$ defines a group homomorphism from $K_0(A)$ into $\mathbf{R}$, which we also denote by $\tau$. We can then ask how to compute $\tau([V])$. To answer this, we recall from Proposition 2.2 of [**46**] that corresponding to $\tau_A$ there will be a canonical finite (non-normalized) trace $\tau'_B$ on $B$ such that

$$\tau'_B(\langle f, g\rangle_B) = \tau_A(\langle g, f\rangle_A)$$

for all $f, g \in V$. In terms of this we have:

3.3 PROPOSITION. *Let $A$ and $B$ be $C^*$-algebras with identity element, and let $V$ be a $B$-$A$-equivalence bimodule, so that $V$ represents an element of $K_0(A)$. Let $\tau_A$ be a finite normalized trace on $A$ and let $\tau'_B$ be the corresponding (non-normalized) trace on $B$. Then*

$$\tau_A([V]) = \tau'_B(1_B).$$

*Proof.* As in the proof of Proposition 2.1 of [**46**], we can find elements $v_1, \ldots, v_n$ of $V$ such that

$$\sum \langle v_i, v_i \rangle_B = 1_B,$$

and thus so that $\{ \langle v_i, v_j \rangle_A \}$ is a projection, $P$, in $M_n(A)$, the algebra of $n \times n$ matrices over $A$. Then it is easily checked, much as in the proof of Proposition 1.2, that the mapping

$$v \mapsto ( \langle v_i, v \rangle_A )$$

of $V$ into $A^n$ is an isomorphism of $V$ onto $P(A^n)$, with inverse given (on $P(A^n)$ ) by

$$(a_i) \mapsto \sum v_i a_i.$$

By the definition of how $\tau_A$ extends to $K_0(A)$, we have

$$\begin{aligned} \tau_A([V]) &= \tau_A(P) = \sum \tau_A( \langle v_i, v_i \rangle_A ) \\ &= \sum \tau'_B( \langle v_i, v_i \rangle_B ) = \tau'_B(\sum \langle v_i, v_i \rangle_B ) = \tau'_B(1_B). \end{aligned}$$

Thus for the specific modules constructed above, we must determine $\tau'_B$ and then $\tau'_B(1_B)$. We let $\tau_B$ denote the normalized trace on $B$.

To proceed, we must first determine, for the present special case in which $D$ is a lattice, how the normalizations of Haar measure made before Lemma 2.10 specialize. Since $D$ is discrete, it is natural to begin by choosing counting measure on $D$ as its Haar measure. On $M$ we can take any Haar measure, as long as we then take the corresponding Plancherel measure on $\hat{M}$. Then from the formula used to define the Haar measure on $G/D$, it is easily seen that this Haar measure, which now must be finite, must give $G/D$ a total volume equal to the volume in $G$ of a fundamental domain for $D$. Let us denote this volume of a fundamental domain by $|G/D|$. Then the corresponding Plancherel Haar measure on $D^\perp$, which must be a multiple of counting measure on $D^\perp$, must give each point of $D^\perp$ a mass of $|G/D|^{-1}$. Since we will shortly be focusing our attention on $D^\perp$ rather than $D$, so that we can work with right modules, we wish to express $|G/D|^{-1}$ in terms of $D^\perp$. Now the Poisson summation formula, as used above, is entirely symmetric in $D$ and $D^\perp$. Examination of its proof then shows that the above Haar measure on $D^\perp$ must be such that if we define the Haar measure on $G/D^\perp$ to be the Plancherel Haar measure for the Haar (counting) measure on $D$ (so that $G/D^\perp$ has total mass 1), then for suitable functions $F$ on $G$

$$\int_G F(x) = \int_{G/D^\perp} \int_{D^\perp} F(x + z) dz d\dot{x}.$$

If we now let $F$ be the characteristic function of a precompact measurable fundamental domain for $D^\perp$ in $G$, we obtain

$$|G/D^\perp| = \int_{G/D^\perp} |G/D|^{-1} = |G/D|^{-1}.$$

Thus the Plancherel measure on $D^\perp$ gives each point the mass $|G/D^\perp|$.

Again because we will be emphasizing $D^\perp$, we prefer just to use counting measure on $D^\perp$. It is clear from the Poisson summation formula that we can do this if on $D$ we choose the Haar measure assigning mass $|G/D^\perp|^{-1} = |G/D|$ to each point. We make this assumption from now on.

With the conventions just made, it is clear that the normalized trace, $\tau_A$, on $A$ comes just from evaluation at 0. Thus for $f, g \in S(M)$,

$$\tau_A(\langle f, g\rangle_A) = \langle f, g\rangle_A(0) = \langle g, f\rangle.$$

We must find the corresponding (non-normalized) trace on $B$. But if we let $\tau'_B$ just be evaluation of functions at 0 we obtain

$$\tau'_B(\langle g, f\rangle_B) = \langle g, f\rangle_B(0) = \langle g, f\rangle.$$

Thus this is the corresponding trace on $B$. It is non-normalized because, with the choice of Haar measure just made on $D$, the identity element of $B$ is easily seen to be the function with value $|G/D^\perp|$ at 0 and value 0 elsewhere. Thus

$$\tau'_B(1_B) = |G/D^\perp|.$$

In view of Proposition 3.3, we have obtained:

3.4 THEOREM. *Let $D$ be a lattice in $G$ $(= M \times \hat{M})$, let $A = C^*(D^\perp, \bar{\beta})$, and let $V$ be the finitely generated projective right $A$-module obtained by completing $S(M)$ as described earlier. Let $\tau_A$ be the canonical normalized trace on $A$ coming from evaluating functions at* 0, *and view $\tau_A$ as defining the corresponding functional on $K_0(A)$. Then*

$$\tau_A([V]) = |G/D^\perp|,$$

*where $|G/D^\perp|$ is the volume of a fundamental domain for $D^\perp$.*

We now digress to consider the von Neumann algebra aspects of the situation, along the lines begun in Theorem 2.16. Because $D$ and $D^\perp$ are now lattices, so that $A$ and $B$ have finite faithful traces, the corresponding von Neumann algebras, $\bar{A}$ and $\bar{B}$, will be finite von Neumann algebras, and we can ask for the coupling function between them, much as was done in **[47]**. We first clarify the situation by noting that, for the same reasons as given in the proof of Proposition 2.3 of **[47]**, the von Neumann algebra generated by $S(D, \beta)$ acting on $L^2(M)$ is naturally isomorphic to the left regular von Neumann algebra generated by $S(D, \beta)$ with respect to its finite trace. We will thus let $\bar{B}$ denote each of these algebras interchangeably. A similar comment applies to $\bar{A}$. We wish to invoke Theorem 2.6 of **[47]**, and so we must determine the center-valued traces on $\bar{A}$ and $\bar{B}$. Since $\bar{A}$ and $\bar{B}$ are each other's commutant by Theorem 2.16, their common center is $\bar{A} \cap \bar{B}$. We claim that this common center is generated by $S(D_0)$ where $D_0 = D \cap D^\perp$. Certainly $S(D_0)$ is contained in the center, and the center-valued trace will act as the identity operator on

$S(D_0)$. But suppose $w$ is in $D$ but not in $D_0$, so that there is a $y \in D$ with $\rho(w, y) \neq 1$. For any $z \in D$ let $u_z$ denote the unitary in $\bar{B}$ corresponding to the $\delta$-function at $z$. Then, if we let ${}^{\natural B}$ denote the center-valued trace on $\bar{B}$, we will have

$$(u_w)^{\natural B} = (u_y u_w u_y{}^{-1})^{\natural B} = \rho(w, y)(u_w)^{\natural B},$$

from which it follows that

$$u_w^{\natural B} = 0.$$

Since ${}^{\natural B}$ is norm continuous, it follows that ${}^{\natural B}$ on $S(D, \beta)$ consists just of restricting functions to $D_0$. In the same way, the center-valued trace ${}^{\natural A}$ on $\bar{A}$ will on $S(D^{\perp}, \bar{\beta})$ consist just of restricting functions to $D_0$. Then for $f, g \in S(M)$ we have for $y \in D_0$

$$(\langle f, g\rangle_A)^{\natural A}(y) = \langle \pi_y g, f\rangle,$$

while

$$(\langle g, f\rangle_B)^{\natural B}(y) = \langle g, \pi_y f\rangle.$$

These are almost the same, but we must remember that we are using different Haar measures on $D$ and $D^{\perp}$, and so we must compare the above as operators on $L^2(M)$. But for $h \in L^2(M)$ we have

$$(\langle f, g\rangle_B)^{\natural B} h = |G/D| \sum_{D_0} \langle g, \pi_y f\rangle(\pi_y h),$$

while

$$\begin{aligned} h(\langle g, f\rangle_A)^{\natural A} &= \sum_{D_0} \pi_y^* h\langle \pi_y g, f\rangle \\ &= \sum \pi_y^* h\langle g, \pi_y^* f\rangle \\ &= \sum \beta(y, y)\pi_{-y} h\langle g, \beta(y, y)\pi_{-y} f\rangle \\ &= \sum \pi_y h\langle g, \pi_y f\rangle. \end{aligned}$$

Thus as operators we have

$$(\langle f, g\rangle_B)^{\natural B} = |G/D|\,(\langle g, f\rangle_A)^{\natural A}.$$

Then from Theorem 2.6 of [**47**] we immediately obtain:

3.5 Theorem. *Let $D$ be a lattice in $G$ $(= M \times \hat{M})$, and let $\bar{A}$ and $\bar{B}$ be the finite von Neumann algebras on $L^2(M)$ generated by $S(D^{\perp}, \bar{\beta})$ and $S(D, \beta)$ respectively. Then the coupling function for $\bar{A}$ and $\bar{B}$ is the scalar operator $|G/D|^{-1}$ $(= |G/D^{\perp}|)$.*

We now return to our discussion of the $C^*$-algebras, and we specialize to the case in which $D = \mathbf{Z}^n$. Let us first consider the nature of an Abelian group $M$ such that $D$ can be embedded as a lattice in $M \times \hat{M} = G$. Since $D$

is finitely generated, it will be contained in an open compactly generated subgroup, $H$, of $G$. But since $G/D$ is compact, the same will be true of $G/H$. But $G/H$ is discrete, and so must be finite. It follows that $G$ is compactly generated. Then by Theorem 9.8 of [**22**] $G$ is of the form $\mathbf{R}^a \times \mathbf{Z}^b \times K$ where $K$ is compact. But $G$ is clearly self-dual, and so must equally be of the form $\mathbf{R}^a \times T^b \times \hat{K}$, where $\hat{K}$ is discrete. Since $G$ is compactly generated, so must $\hat{K}$ be, so that $\hat{K}$ is of the form $\mathbf{Z}^c \times F$ where $F$ is a finite Abelian group. Thus $G$ is of the form $\mathbf{R}^a \times T^b \times \mathbf{Z}^c \times F$, that is, an "elementary" group as defined in [**58**]. Since $M$ is a summand of $G$, it too must be of this form, say

$$M \cong \mathbf{R}^p \times \mathbf{Z}^q \times T^m \times F.$$

Then, of course,

$$G \cong \mathbf{R}^{2p} \times \mathbf{Z}^{q+m} \times T^{q+m} \times F \times \hat{F}.$$

But for $\mathbf{Z}^n$ to be a lattice in $G$, it is easily seen from Theorem 9.12 of [**22**] that one must have $n = 2p + q + m$. We have thus shown:

3.6 Proposition. *If $M$ is an Abelian locally compact group such that $\mathbf{Z}^n$ embeds as a lattice in $M \times \hat{M}$, then $M$ is of the form*

$$M = \mathbf{R}^p \times \mathbf{Z}^q \times T^m \times F,$$

*where $2p + q + m = n$, and $F$ is a finite Abelian group.*

We now begin to shift attention to $D^\perp$ and right modules. It is clear that the above considerations apply equally well if we are instead insisting that $D^\perp = \mathbf{Z}^n$. Of course, $D$ $(= D^{\perp\perp})$ will be a lattice in $G$, and so, in view of the form which $G$ must have, $D$ must be of the form $\mathbf{Z}^n \times F_0$ for some finite group $F_0$. The module $V$ of Proposition 3.2 will be a right module over $A = C^*(\mathbf{Z}^n, \bar{\beta})$, whose full endomorphism ring will be $B = C^*(D, \beta)$ acting on the left.

We recall that the dual group, $T^n$, of $\mathbf{Z}^n$ has the natural dual action $\alpha$ on $A$ given, for $f \in L^1(\mathbf{Z}^n, \beta)$, $x \in \mathbf{Z}^n$ and $t \in T^n$, by

$$(\alpha_t(f))(x) = \langle x, t \rangle f(x).$$

As in the first lemma of Section 13 of [**10**], the space of $C^\infty$-vectors for this action will be exactly $S(\mathbf{Z}^n)$. In the same way, the dual action of $\hat{D}$ on $C^*(D, \beta)$ will have $S(D)$ as its space of $C^\infty$-vectors (much as in the proof of Theorem 4.1 of [**6**]). We recall that these Schwartz spaces, as algebras, are closed under the holomorphic functional calculus, for the reasons given in the section on smoothing in the appendix to [**9**]. In particular, any element of $S(D, \beta)$ which is invertible in $C^*(D, \beta)$ will be invertible in $S(D, \beta)$. (That is, the inverse of an invertible $C^\infty$-element is $C^\infty$.)

We need now to place the module $S(M)$ in the framework described by Connes in Lemma 1 of [**8**]. What we need, both here and especially later is:

3.7 PROPOSITION. *Let $A$ be a $C^*$-algebra with identity element, let $V$ be a projective right $A$-module with $A$-valued inner-product, and let $B = \mathrm{End}_A(V)$. So $B$ is a $C^*$-algebra, and $V$ has a corresponding $B$-valued inner-product. Let $A_0$ and $B_0$ be dense $*$-subalgebras of $A$ and $B$ respectively containing the identity elements, and let $V_0$ be a dense subspace of $V$ which is closed under the actions of $A_0$ and $B_0$, and such that the restrictions to $V_0$ of the inner-products have values in $A_0$ and $B_0$ respectively. If $B_0$ has the property that any element of $B_0$ which is invertible in $B$ has its inverse in $B_0$, then $V_0$ is a projective right $A_0$-module. In addition, the mapping from $V_0 \otimes_{A_0} A$ to $V$ defined by $v \otimes a \mapsto va$ is an isomorphism of right $A$-modules.*

*Proof.* Since $V$ is finitely generated and projective, there is a finite collection, $v_1, \ldots, v_n, y_1, \ldots, y_n$ of elements of $V$ such that

$$\sum \langle v_i, y_i \rangle_B = 1_B.$$

Since $V_0$ is dense, we can approximate the $v_i$'s and $y_i$'s closely enough that the corresponding sum of inner-products, which is an element of $B_0$, will be invertible in $B$. By hypotheses its inverse is in $B_0$, and so, multiplying the sum by the inverse, we find that $1_B$ is expressed as the sum of inner-products of elements of $V_0$. It follows easily, much as in the proof of Proposition 1.2 (or the proposition in [**45**]) that $V_0$ is a projective $A$-module.

Let now the $v_i$'s and $y_i$'s be as above except in $V_0$. The indicated map from $V_0 \otimes_{A_0} A$ is surjective because for any $v \in V$

$$v = 1_B v = \sum \langle v_i, y_i \rangle_B v = \sum v_i \langle y_i, v \rangle_A.$$

But this map is also injective, for if $\sum z_j a_j = 0$ for certain $z_j \in V_0$ and $a_j \in A$, then

$$\begin{aligned} \sum z_j \otimes a_j &= \sum \sum \langle v_i, y_i \rangle_B z_j \otimes a_j \\ &= \sum \sum v_i \langle y_i, z_j \rangle_A \otimes a_j = \sum \sum v_i \otimes \langle y_i, z_j \rangle_A a_j \\ &= \sum v_i \otimes \langle y_i, \sum z_j a_j \rangle = 0. \end{aligned}$$

3.8 COROLLARY. *With $D$, $M$ and $V$ as before, $S(M)$ is a projective right $A_0$-module, where $A_0 = S(\mathbf{Z}^n, \bar{\beta})$. Furthermore*

$$V \cong S(M) \otimes_{A_0} A \quad \textit{where } A = C^*(\mathbf{Z}^n, \bar{\beta}).$$

For the cancellation theorem in Section 7 we need to have an upper bound on the topological stable rank (as defined in [**48**]) of the endomorphism ring of $V$. We obtain this from:

3.9 PROPOSITION. *Let $D$ be any group of the form $\mathbf{Z}^n \times F$ for some finite Abelian group $F$, and let $\beta$ be any bicharacter on $D$. Then the topological stable rank of $C^*(D, \beta)$ is no larger than $n + 1$.*

*Proof.* We shall show that $C^*(D, \beta)$ can be built up from $C^*(F, \beta)$ by successive crossed products by $\mathbf{Z}$. (This is mentioned at the end of 1.7 of [**17**], at least for $D$ torsion-free.) Once this is shown, the proposition follows immediately by induction on $n$, using Theorem 7.1 of [**48**]. Note that $D$ is finite if $n = 0$, so that $C^*(D, \beta)$ is finite dimensional, and its topological stable rank is 1 by Section 3 of [**48**]. Thus to complete the proof (as well as for later use) we only need:

3.10 Proposition. *Let $D$ be any group of form $\mathbf{Z}^n \times F$ for some finite group $F$, and let $\beta$ be any bicharacter on $D$. Let $D'$ denote the subgroup of $D$ generated by the first $n - 1$ generators of $\mathbf{Z}^n$ together with $F$, and denote the restriction of $\beta$ to $D'$ still by $\beta$. Let $\alpha$ denote the automorphism of $C^*(D', \beta)$ obtained from conjugation by the last generator of $\mathbf{Z}^n$, and let $\alpha$ also denote the corresponding action of $Z$ on $C^*(D', \beta)$. Then*

$$C^*(D, \beta) \cong C^*(D', \beta) \times_\alpha Z.$$

*Proof.* Let $D$ be embedded, as discussed early in Section 2, as a closed subgroup of a $G = M \times \hat{M}$ such that the restriction to $D$ of the Heisenberg cocycle on $G$ coincides with $\beta$. Then $D'$ is also so embedded. It follows from Proposition 2.2 that both $C^*(D, \beta)$ and $C^*(D', \beta)$ are faithfully represented on $L^2(M)$, so that $C^*(D', \beta)$ can be viewed as a subalgebra of $C^*(D, \beta)$. Let $u$ be the unitary in $C^*(D, \beta)$ corresponding to the last generator of $\mathbf{Z}^n$. Then from the commutation relations it is clear that $u$ normalizes $C^*(D', \beta)$, so that $\alpha$ is well-defined. It is also clear that $u$ and $C^*(D', \beta)$ generate $C^*(D, \beta)$. Thus there is an evident homomorphism, $\eta$, of the crossed product onto $C^*(D, \beta)$. We must show that $\eta$ is injective. From the discussion above it is clear that $\eta$ is injective on the subalgebra $C^*(D', \beta)$. Let $I$ denote the kernel of $\eta$. Now we have seen earlier that we have the dual action of $\hat{D}$ on $C^*(D, \beta)$. But $\hat{D} \cong T^n \times \hat{F}$, and in particular we can single out the subgroup $T$ of $\hat{D}$ which sees only the last copy of $\mathbf{Z}$ in $\mathbf{Z}^n \times F = D$. But then it is evident that $\eta$ is equivariant for this action of $T$ on $C^*(D, \beta)$ and for the dual action of $T$ on the crossed-product. It follows that the kernel $I$ must be invariant under this dual action of $T$. But then by averaging a positive element of $I$ over $T$ using the dual action, it follows that if $I$ is not trivial, it must contain non-zero elements which are invariant under the dual action. But it is well-known (see Proposition 7.8.9 of [**35**] ) that such elements must belong to $C^*(D', \beta)$, on which we have seen that $\eta$ is faithful. Thus $I$ is trivial and $\eta$ is an isomorphism.

Actually one can often do much better than Proposition 3.9. For example, by using in part ideas of Bruce Blackadar, I have been able to show that when $\beta$ is not rational (that is, its range is not entirely contained in the roots of unity), the topological stable range of $C^*(D, \beta)$ is no bigger than 2, and is equal to 2 if $C^*(D, \beta)$ is not simple. But a remarkable argument of Riedel [**40**, 1] shows that when $C^*(D, \beta)$ is simple, its topological stable rank can sometimes be 1, although exactly how often this happens remains mysterious. In another direction, if $\beta$ is trivial, then $C^*(D, \beta)$ is isomorphic to the tensor product of $C(T^n)$ with a finite-dimensional commutative algebra. It follows from Theorem 2.8 of [**48**] that in this case the topological stable rank is $[n/2] + 1$, where [ ] denotes "integer part of". Presumably the other cases where $\beta$ is rational fall somewhere in-between, especially in view of Theorem 6.1 of [**48**], but I have not investigated this matter. At any rate, Proposition 3.9 is quite adequate for our present purposes.

In much of this section we have been working in a setting where cocycles need not be cohomologous to skew bicharacters; whereas in the next section we will restrict attention to a setting where skew bicharacters are sufficient, and in which formulas will be considerably simpler if one uses skew bicharacters rather than general cocycles or bicharacters. We conclude this section by reviewing briefly how to pass between cohomologous cocycles [**60**]. Since for this $D$ need not be commutative, we use multiplicative notation.

Let $\beta$ and $\sigma$ be cocycles on $D$ which are cohomologous, so that there is a function, $\eta$, from $D$ to $T$ such that

$$\sigma(x, y) = \bar{\eta}(x)\bar{\eta}(y)\eta(xy)\beta(x, y)$$

for $x, y \in D$. For any $x \in D$ let $u_x$ as before denote the unitary corresponding to the delta-function at $x$, and let $A_\beta$, $= C^*(D, \beta)$, be the enveloping $C^*$-algebra of the algebra $C_c(D)$ for which the product is given by

$$u_x u_y = \beta(x, y)u_{xy},$$

and similarly for $A_\sigma$. (The involution is given by

$$(u_x)^* = \beta(x, x)u_{x^{-1}},$$

and similarly for $\sigma$.) Then we can define an isomorphism, $\phi$, from $A_\beta$ to $A_\sigma$ by setting

$$\phi(f)(x) = \eta(x)f(x)$$

for $f \in C_c(D)$ and $x \in D$, and extending to the completions. The inverse of $\phi$ is then determined as above by $\bar{\eta}$.

Suppose now that $V$ is a right $A_\beta$-module with inner-product $\langle\ ,\ \rangle_\beta$ with values in $A_\beta$, which is the completion of a right $L^1(D)$-module $V_0$ for which

$$\langle V_0, V_0\rangle_\beta \subset L^1(D).$$

Then we can make $V$ into a right $A_\sigma$-module with inner-product $\langle\ ,\ \rangle_\sigma$ by setting

$$v \overset{\sigma}{\cdot} f = v(\bar{\eta} f)$$

$$\langle v, w \rangle_\sigma(x) = \eta(x) \langle v, w \rangle_\beta(x)$$

for $v, w \in V_0$, $x \in D$, $f \in L^1(D)$. This process will preserve the property of being projective.

**4. Elementary projective modules and their Chern characters.** Let $D = \mathbf{Z}^n$, which we will let play the role of the $D^\perp$ of earlier sections. Let $\gamma$ be any bicharacter on $D$, and consider all the various embeddings of $D$ as a lattice in $M \times \hat{M}$ for various $M$'s such that $\gamma = \bar{\beta}$ where $\beta$ is the corresponding Heisenberg cocycle, together with the various finite direct sums of the corresponding projective right modules over $C^*(D, \gamma)$. We obtain in this way a rather bewildering variety of projective modules. We thus need some method for classifying the modules so obtained. The trace on $K_0$ is usually not adequate. However the Chern character introduced by Connes [8], and already discussed by Elliott [17] for the algebras $C^*(D, \gamma)$, turns out to be the ideal classification tool. We begin to develop its use in this section. But we will consider here only $M$'s of the special form $\mathbf{R}^p \times \mathbf{Z}^q$ with $2p + q = n$. (We will call the corresponding projective modules *elementary* projective modules.) Since in this case $\hat{M} \cong \mathbf{R}^p \times T^q$, it turns out to be unnecessary to consider the more general form $\mathbf{R}^p \times \mathbf{Z}^k \times T^m$. We will defer until Section 5 treatment of the case in which $M$ contains a finite subgroup.

We will not need the more general version of the Chern character which Connes has developed in [11, 12]. The version in [8], based on actions of Lie groups, suffices. The action which we will employ is the dual action $\alpha$ of $T^n$ on $C^*(D, \gamma)$. Following [8], the corresponding Chern character will then have its values in the cohomology group $H^*_R(T^n)$. But since $T^n$ is commutative, $H^*_R(T_n)$ can be identified with the exterior algebra $\wedge L^*$, where $L$ denotes the Lie algebra of $T^n$ and $L^*$ denotes its dual vector-space.

We identify $L$ with $\mathbf{R}^n$ in the evident way, and denote its standard basis by $E_1, \ldots, E_n$. We then denote the dual basis for $L^*$ by $\bar{E}_1, \ldots, \bar{E}_n$. The standard basis and its dual determine the orientations and volume elements on $L$ and $L^*$ which we will use.

We will identify $D$ with the lattice in $L^*$ generated by the dual basis $\{\bar{E}_i\}$ of $L^*$. This is very convenient, because if $x \in D$ and if $u_x$ denotes the corresponding unitary in $C^*(D, \gamma)$, then the derivation on $C^*(D, \gamma)$ defined by any $X \in L$ by means of $\alpha$ is given on $u_x$ by

$$X(u_x) = 2\pi i \langle X, x \rangle u_x,$$

where here $\langle\ ,\ \rangle$ denotes the pairing between $L$ and $L^*$. To see this, recall that the one-parameter group $\alpha^X$ in $T^n$ defined by $X$ acts on $u_x$ by

$$\alpha_t^X(u_x) = e(\langle tX, x\rangle)u_x$$

for $t \in \mathbf{R}$, where, as before, $e$ is the function from $\mathbf{R}$ to $T$ defined by

$$e(t) = \exp(2\pi it).$$

One then just differentiates this formula. (Notice that $\gamma$ does not appear in the formula for the derivation corresponding to $X$.)

Actually, we will never explicitly need the dual action of $T^n$ in our formulas for the Chern character. Rather, in making calculations it will suffice to view the Chern character as measuring the interaction between a Lie algebra of derivations of an algebra and the projective modules over the algebra. See **[32]** for the case of commutative algebras. But, of course, the dual action of $T^n$ is needed to ensure that everything works well once one completes to obtain the corresponding $C^*$-algebra.

Since $D$ ($\cong \mathbf{Z}^n$) is free, any bicharacter $\gamma$ on $D$ can be lifted (not uniquely) to a bicharacter into the covering group of $T$, that is, into $\mathbf{R}$. This can then be extended to a bilinear form on $L^* \supset D$, which we denote for the moment by $J$. Thus for $x, y \in D$ we have

$$\gamma(x, y) = e(J(x, y)).$$

Let $\theta$ be twice the negative of the antisymmetric part of $J$, that is, for $x, y \in D$,

$$\theta(x, y) = -(J(x, y) - J(y, x)).$$

If $J$ and $-\theta/2$ are viewed as $\mathbf{R}$-valued cocycles on $D$, then they are cohomologous, because $J + \theta/2$ will be the symmetric part of $J$, which is easily seen to be the coboundary of the $\mathbf{R}$-valued function

$$x \mapsto J(x, x)/2$$

on $D$. This implies that if we let $\sigma$ denote the skew 2-cocycle on $D$ defined by

$$\sigma(x, y) = \bar{e}(\theta(x, y)/2),$$

then $\gamma$ and $\sigma$ are cohomologous. Since various formulas will be simpler if we use skew bicharacters, we will work primarily with $\sigma$ and $\theta$.

Our notation is chosen so that our $\theta$ is the negative of the $\theta$ used by Elliott. This choice is necessary in order for Elliott's other formulas to be correct. We will discuss this matter further at the end of this section. We note that aside from this, our earlier $\rho$ becomes the same as Elliott's, while our $\gamma$ is Elliott's $\alpha$. In particular,

$$\rho(x, y) = \bar{e}(\theta(x, y))$$

and

$$u_x u_y = \rho(x, y)u_y u_x$$

for $x, y \in D$. Note that $\sigma^2 = \rho$.

Elliott **[17]** shows that the Chern character is very conveniently expressed in terms of $\theta$. For this reason we will find it notationally simpler to denote $C^*(D, \sigma)$ by $A_\theta$ instead of the $A_\rho$ which Elliott uses, even though many different $\theta$'s give the same $\rho$ and so isomorphic algebras. In the same way we will write $S_\theta$ for $S(D, \sigma)$. Since $\theta$ is a skew bilinear form on $L^*$, we can view it as an element of $\wedge^2 L$. It will be very useful to do this.

As indicated earlier, in this section we will take $M$ to be of the form $\mathbf{R}^p \times \mathbf{Z}^q$, so that $G$ will be of the form $\mathbf{R}^{2p} \times \mathbf{Z}^q \times T^q$. Since $D$ is free, any homomorphism from $D$ into $G$ lifts to a homomorphism into the covering group $\mathbf{R}^{2p} \times \mathbf{Z}^q \times \mathbf{R}^q$ of $G$, and so into $\mathbf{R}^{2p} \times \mathbf{R}^{2q}$, where we take $\mathbf{Z}^q \subset \mathbf{R}^q$. But such a homomorphism will then extend to a linear map from $L^*$ into $\mathbf{R}^{2p} \times \mathbf{R}^{2q}$. It will be most convenient to view matters at this level. To help in understanding various formulas, it will be useful to distinguish between $\mathbf{R}$ and its dual vector space, which we denote by $\mathbf{R}^*$. Then $\mathbf{R}^{*m}$ will denote the dual of $\mathbf{R}^m$. For any $m$ we will let $e_1, \ldots, e_m$ denote the standard basis for $\mathbf{R}^m$, and then $\bar{e}_1, \ldots, \bar{e}_m$ will denote the dual basis for $\mathbf{R}^{*m}$. In view of the fact that $\mathbf{R}^{2p} \times \mathbf{R}^{2q}$ comes from $M \times \hat{M}$, and that we are considering mappings from $L^*$, it will be useful to view $\mathbf{R}^{2p+2q}$ more specifically as $\mathbf{R}^{p+q} \times \mathbf{R}^{*p+q}$, and to denote it by $H^*$. However, we will order the basis for $H^*$ by $e_1, \bar{e}_1, e_2, \bar{e}_2, \ldots,$ with corresponding orientation and volume element. We will frequently view $H^*$ as

$$H^* = \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{*p} \times \mathbf{R}^{*q},$$

and denote its dual vector space by $H$, writing

$$H = \mathbf{R}^{*p} \times \mathbf{R}^{*q} \times \mathbf{R}^p \times \mathbf{R}^q.$$

The dual basis for $H$ will be denoted $\bar{e}_1, e_1, \bar{e}_2, e_2, \ldots$ in that order. This coincidence of notation should not cause difficulties, and has certain advantages. (We could, in fact, identify $H$ with $H^*$.)

It should be clear that the Heisenberg cocycle $\beta$ on $M \times \hat{M}$ comes from the usual pairing of $\mathbf{R}^{p+q}$ with $\mathbf{R}^{*p+q}$. Specifically, if $x = (m, \hat{s})$ and $y = (n, \hat{t})$, where $m, n \in \mathbf{R}^p \times \mathbf{Z}^q$ and where $\hat{s}$ and $\hat{t}$ are the images in $\mathbf{R}^{*p} \times T^q$ of $s$ and $t$ in $\mathbf{R}^{*p} \times \mathbf{R}^{*q}$, then

$$\beta(x, y) = e(\langle m, t \rangle).$$

Just as we prefer to work with $\sigma$ and $\theta$, we prefer here to work with the skew bicharacter $\beta\beta^*$, which is given by

$$\beta\beta^*(x, y) = e(\langle m, t \rangle - \langle n, s \rangle).$$

The alternating bilinear form inside parentheses on the right, defined on $H^*$, is easily seen to be the one given by the standard 2-form

$$\omega = \bar{e}_1 \wedge e_1 + \bar{e}_2 \wedge e_2 + \ldots + \bar{e}_{p+q} \wedge e_{p+q}$$

in $\bigwedge^2 H$. If the Heisenberg cocycle $\beta$ on $G$ is to pull back to $\bar{\gamma}$ on $D$, it is clear that $\omega$ should pull back to $-\theta$.

We have seen that the embedding of $D$ into $G$ is determined by a linear mapping of $L^*$ into $H^*$, and we now see that this linear mapping must pull $\omega$ back into $-\theta$. But it must also result in embedding $D$ as a lattice in $H^*$.

4.1 *Definition*. By an *embedding map* we mean a linear map $T$ from $L^*$ to $H^*$ such that:

(1) $T(D) \subset \mathbf{R}^p \times \mathbf{Z}^q \times \mathbf{R}^{*p} \times \mathbf{R}^{*q}$.

(2) $T(D)$ is a lattice in $\mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{*p} \times (\mathbf{R}^{*q}/\mathbf{Z}^q)$.

(3) The form $\omega$ on $H^*$ is pulled back by $T$ to the form $-\theta$ on $L^*$, that is, if $T^*$ denotes the adjoint of $T$, viewed as a map from $H$ to $L$, then

$$(\textstyle\bigwedge^2 T^*)(\omega) = -\theta.$$

The integer $p$ will be called the *height* of $T$.

Let $\tilde{H}^*$ denote $\mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{*p}$, and let $\tilde{T}$ denote $T$ composed with the evident projection of $H^*$ onto $\tilde{H}^*$ (along $\bar{e}_{p+1}, \ldots, \bar{e}_{p+q}$). Since $\mathbf{R}^{*q}/\mathbf{Z}^q$ is compact, and $\tilde{H}^*$ has the same dimension as $L^*$, it is evident that condition (2) above is equivalent to

(2′) $\tilde{T}$ is invertible (from $L^*$ to $\tilde{H}^*$).

We remark that if $\theta$ is sufficiently degenerate (e.g. $\theta \equiv 0$) then, because of condition (3), there may be few embedding maps. But we will see in the next section that this does not matter (because $\theta$ is not unique, and can be chosen to be non-degenerate).

Let $S(M)$ be, as before, the Schwartz space on $M = \mathbf{R}^p \times \mathbf{Z}^q$. Given an embedding map $T$, we wish to equip $S(M)$, as before, with the structure of a right $S_\theta$-rigged module. To do this we must take account of our change to skew bicharacters, and of the change from $\beta$ to $\bar{\beta}$ which occurs when using right modules. So, let $T$ be an embedding map, and let $T = (T', T'')$ be the decomposition of $T$ into its components going into $\mathbf{R}^{p+q}$ and $\mathbf{R}^{*p+q}$ respectively. Then we will actually embed $D$ into $M \times \hat{M}$ by composing

$$x \mapsto (T'(x), -T''(x))$$

with the mapping to $M \times \hat{M}$. The corresponding linear map from $L^*$ to $H^*$ then pulls $\omega$ back to $+\theta$. If we let $\gamma$ be the bicharacter on $D$ such that $\bar{\gamma}$ is the corresponding pull-back of the Heisenberg bicharacter $\beta$, so that

$$\gamma(x, y) = e(\langle T'(x), T''(y) \rangle)$$

for $x, y \in D$, it follows that $\gamma\gamma^* = \rho$ as desired. Thus the right action on $S(M)$ as defined in Section 2 will be an action of $S(D, \gamma)$. According to the formula found somewhat after the proof of Proposition 2.9, this action is given, for $f \in S(M)$, $x \in D$, and $m \in M$, by

$$\begin{aligned}(fu_x)(m) &= (\pi_x^* f)(m) = \bar{\gamma}(x, x)(\pi_{-x} f)(m) \\ &= e(\langle m - T'(x), T''(x)\rangle) f(m - T'(x)).\end{aligned}$$

This action extends to $S(D, \gamma)$ in the evident way. We wish to change this to an action of $S(D, \sigma)$, using the method described at the end of Section 3. To do this, define the function $\eta$ on $D$ by

$$\eta(x) = \bar{e}(\langle T'(x), T''(x)\rangle/2).$$

Then a straight-forward calculation shows that

$$\bar{\eta}(x)\bar{\eta}(y)\eta(x + y)\gamma(x, y) = \sigma(x, y).$$

This is exactly the coboundary formula found near the end of Section 3, except that $\beta$ has been replaced by $\gamma$. We can thus use the discussion there to find the formulas for the corresponding right action of $S(D, \sigma)$. Specifically, the new action is defined, for $f \in S(M)$ and $x \in D$, by

$$\begin{aligned}(f \overset{\sigma}{\cdot} u_x)(m) &= (f(\bar{\eta}(x)u_x))(m) \\ &= \bar{\eta}(x)e(\langle m - T'(x), T''(x)\rangle) f(m - T'(x)) \\ &= e(\langle m - T'(x)/2, T''(x)\rangle) f(m - T'(x)).\end{aligned}$$

This action extends to $S(D, \sigma)$ in the evident way. We will not need $\gamma$ from now on, so we will denote this new action just by $fu_x$.

We have arrived at the formula for this action by a fairly long path, so it is worth commenting here that for the next part of our development we do not explicitly need the earlier steps, as it is easy to verify directly that under the above action $S(M)$ becomes a right $S(D, \sigma)$-module. We could derive the formula for the corresponding inner-product with values in $S(D, \sigma)$, but we will not have explicit need for it either. But we do need later to keep in mind the earlier steps, as they ensure that $S(M)$ will be a projective module that relates well to the $C^*$-completions (via the inner-product), and they describe its endomorphism algebra. We summarize much of the development in this section so far by:

4.2 *Definition*. Let $\theta \in \wedge^2 L$. Define a skew cocycle, $\sigma$, on the standard lattice, $D$, in $L^*$ by

$$\sigma(x, y) = \bar{e}(\theta(x, y)/2).$$

Denote by $S_\theta$ the *-algebra $S(D, \sigma)$. For any embedding map $T$ of $L^*$ into $H^*$, define a right action of $S_\theta$ on $S(M)$, where $M = \mathbf{R}^p \times \mathbf{Z}^q$, by

$$(fu_x)(m) = e(\langle m - T'(x)/2, T''(x)\rangle) f(m - T'(x)),$$

where $T = (T', T'')$ for $H^* = \mathbf{R}^{p+q} \times \mathbf{R}^{*p+q}$. Then $S(M)$ becomes a projective right $S_\theta$-module, which we denote by $V^T$. We will also let $V^T$

denote the completion of $S(M)$ as a projective right $A_\theta$-module, where $A_\theta = C^*(D, \sigma)$. The modules of the form $V^T$, over either $S_\theta$ or $A_\theta$, will be called *elementary* modules.

We remark that among the elementary modules are the free modules. These will correspond to embedding maps of height zero, that is, where $p = 0$ and $q = n$. This can be checked directly, and should be clear by the end of the proof of Theorem 4.5.

Let us now determine the image under the canonical normalized trace, $\tau$, on $A_\theta$ of the element of $K_0(A_\theta)$ which is represented by $V^T$. We recall from Sections 2 and 3 that we must normalize Haar measures appropriately. Specifically, on $G = M \times \hat{M}$ we must take Plancherel Haar measures. For the present situation we can do this by taking counting measure on $\mathbf{Z}^q$, the Haar measure of mass 1 on $(\mathbf{R}/\mathbf{Z})^q$, and Lebesgue measure on $\mathbf{R}^{2p}$. (This is a product of Plancherel measures because our pairing of $\mathbf{R}^p$ with $\mathbf{R}^{*p}$ is in terms of $e(t) = \exp(2\pi i t)$, which has the factor $2\pi$ built in.) We notice that the corresponding measure on $H^*$ is just Lebesgue measure, which is just the measure coming from the volume element associated with the standard basis. If we let $\det(\tilde{T})$ denote the determinant of $\tilde{T}$, defined to be the factor by which $\tilde{T}$ changes the volume element for the standard basis of $L^*$ to that for $\tilde{H}^*$, then it is clear that for the above normalizations the covolume of $D$ in $G$ is just $|\det(\tilde{T})|$. In view of Theorem 3.4, we obtain:

4.3 PROPOSITION. *Let $T$ be an embedding map, with $V^T$ the corresponding projective module. Let $\tau$ denote the canonical normalized trace on $A_\theta$, viewed as a functional on $K_0(A_\theta)$. Then*

$$\tau([V^T]) = |\det(\tilde{T})|.$$

For future purposes, we recall that this means that if $I$ denotes the identity operator on $V^T$, and if $\tau'$ denotes the non-normalized trace on $\mathrm{End}_A(V^T)$ corresponding to $\tau$, as described just after Proposition 3.2, then

$$\tau'(I) = |\det(\tilde{T})|.$$

While the trace is a complete isomorphism invariant for projective modules over the irrational rotation $C^*$-algebras [**49**], in general (and already for the rational rotation $C^*$-algebras), it is not even faithful on $K_0$. However, as pointed out by Elliott [**17**], the Chern character of Connes is always a complete invariant for the elements of $K_0(A_\theta)$. Thus it is crucial for us to calculate the Chern character of the $V^T$'s. To do this, we must define on $V^T$ a connection, $\nabla$, with respect to the action of the Lie algebra $L$ on $A_\theta$. As domain for this connection we take

$$S(M) = S(\mathbf{R}^p \times \mathbf{Z}^q).$$

Thus we are looking for a mapping, $\nabla$, of $L$ into the linear maps on $S(M)$ such that for any $X \in L$, $f \in S(M)$ and $x \in D$,

$$\nabla_X(fu_x) = (\nabla_X(f))u_x + f(X(u_x)).$$

(We use here the density in $S_\theta$ of the span of the $u_x$'s, and the continuity of all our operations.) It is natural to seek $\nabla$ as a linear combination of three types of operators on $S(M)$, defined for $(r, a) \in \mathbf{R}^p \times \mathbf{Z}^q$, $s \in \mathbf{R}^{*p}$, $t \in \mathbf{R}^{*q}$, $u \in \mathbf{R}^p$, and $f \in S(M)$ by

$$(Q_s^1 f)(r, a) = 2\pi i\langle r, s\rangle f(r, a),$$
$$(Q_t^2 f)(r, a) = 2\pi i\langle a, t\rangle f(r, a),$$
$$(Q_u^3 f)(r, a) = \sum u_j(\partial f/\partial r_j)(r, a).$$

Straight-forward calculations show that the commutation relations among these types of operators are:

$$[Q_u^3, Q_s^1] = 2\pi i\langle u, s\rangle I$$
$$[Q_s^1, Q_t^2] = 0 = [Q_t^2, Q_u^3],$$

where $I$ denotes the identity operator on $S(M)$. Notice also that for any fixed $j$, the various $Q^j$ all commute among themselves for different parameter values. We also need the commutation relations of the $Q^j$ with the operators corresponding to elements of $D$. Specifically, for $x \in D$ and for $u_x$ the corresponding element of $S_\theta$, let $W_x$ denote the operator on $S(M)$ consisting of right multiplication by $u_x$. That is,

$$\begin{aligned}(W_x f)(r, a) &= (fu_x)(r, a)\\ &= e(\langle (r, a) - T'(x)/2, T''(x)\rangle)f((r, a) - T'(x)).\end{aligned}$$

To conveniently express the commutation relations, we must let $T = (T_1, T_2, T_3, T_4)$ denote the decomposition of $T$ into its four components in

$$H^* = \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{*p} \times \mathbf{R}^{*q}.$$

Then straight-forward calculations show that

$$[Q_s^1, W_x] = 2\pi i\langle T_1(x), s\rangle W_x,$$
$$[Q_t^2, W_x] = 2\pi i\langle T_2(x), t\rangle W_x,$$
$$[Q_u^3, W_x] = 2\pi i\langle u, T_3(x)\rangle W_x.$$

Now for $z \in \tilde{H} = \mathbf{R}^{*p} \times \mathbf{R}^{*q} \times \mathbf{R}^p$, with $z = (s, t, u)$, let us define an operator $Q_z$ on $S(M)$ by

$$Q_z = Q_s^1 + Q_t^2 + Q_u^3.$$

From the commutation relations above we immediately see that for $x \in D$ we have

$$[Q_z, W_x] = 2\pi i\langle z, \tilde{T}(x)\rangle W_x,$$

where now $\langle\ ,\ \rangle$ denotes the evident pairing between $\tilde{H}$ and $\tilde{H}^*$.

If we rewrite the defining equation for a connection in terms of the commutator with $W_x$, we obtain

$$[\nabla_X, W_x]f = fX(u_x).$$

But the right hand side, as seen earlier, is

$$2\pi i\langle X, x\rangle W_x f = 2\pi i\langle (\tilde{T}^{-1})^*(X), \tilde{T}(x)\rangle W_x f.$$

Comparing this with the commutator of $Q_z$ with $W_x$, we see that we must define $\nabla$ by

$$\nabla_X = Q_{(\tilde{T}^{-1})^*(X)}.$$

To calculate the Chern character of $V^T$, we must find the curvature, $\Omega$, of $\nabla$. To ease the notation, we let $S = (\tilde{T}^{-1})^*$, which goes from $L$ to $\tilde{H} = \mathbf{R}^{*p} \times \mathbf{R}^{*q} \times \mathbf{R}^p$, and we let $S = (S_1, S_2, S_3)$ be the corresponding decomposition of $S$. Then, since $L$ is commutative, $\Omega$ is defined for $X, Y \in L$ by

$$\begin{aligned}\Omega(X, Y) &= [\nabla_X, \nabla_Y] = [Q_{S(X)}, Q_{S(Y)}]\\ &= [Q^1_{S_1(X)}, Q^3_{S_3(Y)}] + [Q^3_{S_3(X)}, Q^1_{S_1(Y)}]\\ &= 2\pi i(\langle S_3(X), S_1(Y)\rangle - \langle S_3(Y), S_1(X)\rangle)I\\ &= 2\pi i \sum_{j=1}^{p} (\langle S(X), \bar{e}_j\rangle\langle S(Y), e_j\rangle\\ &\qquad\qquad - \langle S(Y), \bar{e}_j\rangle\langle S(X), e_j\rangle)I\\ &= 2\pi i \sum_{j=1}^{p} (\langle X, \tilde{T}^{-1}(\bar{e}_j)\rangle\langle Y, \tilde{T}^{-1}(e_j)\rangle\\ &\qquad\qquad - \langle Y, \tilde{T}^{-1}(\bar{e}_j)\rangle\langle X, \tilde{T}^{-1}(e_j)\rangle)I.\end{aligned}$$

This suggests that we define $\bar{Y}_j \in L^*$ by

$$\bar{Y}_j = \begin{cases}\tilde{T}^{-1}(\bar{e}_j) & \text{for } 1 \leqq j \leqq p\\ \tilde{T}^{-1}(e_{j-p}) & \text{for } p + 1 \leqq j \leqq n\ (= 2p + q),\end{cases}$$

so that $\{\bar{Y}_j\}$ is a basis for $L^*$. In terms of this notation we see that

$$\Omega(X, Y) = 2\pi i\left\langle X \wedge Y, \sum_{j=1}^{p} \bar{Y}_j \wedge \bar{Y}_{j+p}\right\rangle I.$$

That is,

$$\Omega = 2\pi i\left(\sum_{j=1}^{p} \bar{Y}_j \wedge \bar{Y}_{j+p}\right)I.$$

(In case $p = 0$, sums such as this, here and later, are to be taken to have value 0. Here this corresponds to the fact that on free modules nice connections should have zero curvature.) Shortly we will transform this formula for the curvature so as to make explicit the role of $\theta$. But recall that the Chern character is defined in terms of the various exterior powers, $\Omega^k$, of $\Omega$. The above formula is convenient for computing these exterior powers, so we carry out this computation first. Note that for different $j$'s the terms $\bar{Y}_j \wedge \bar{Y}_{j+1}$ commute among themselves (for the exterior product), and that the square of any such term is 0. Then if we let $P(k)$ denote the collection of subsets of $\{1, \ldots, p\}$ of cardinality $k$, we easily see that

$$\Omega^k = (2\pi i)^k k! \left(\Sigma\left\{\prod_{j\in M} \bar{Y}_j \wedge \bar{Y}_{j+p} : M \in P(k)\right\}\right) I.$$

Let $\{Y_j\}$ denote the basis for $L$ which is dual to the basis $\{\bar{Y}_j\}$ for $L^*$. Then for arbitrary elements $X_1, \ldots, X_{2k}$ of $L$ and any $M \in P(k)$ we have

$$\left\langle X_1 \wedge \ldots \wedge X_{2k}, \prod_{j\in M} \bar{Y}_j \wedge \bar{Y}_{j+p}\right\rangle$$

$$= \left\langle X_1 \wedge \ldots \wedge X_{2k} \wedge \left(\prod_{j\notin M} Y_j \wedge Y_{j+p}\right), \prod_{j=1}^{p} \bar{Y}_j \wedge \bar{Y}_{j+p}\right\rangle.$$

We now set

$$\mu^T = d \prod_{j=1}^{p} \bar{Y}_j \wedge \bar{Y}_{j+p}$$

for $d = |\det(\tilde{T})|$ (where if $p = 0$ we set $\mu^T = d$). We will find that $\mu^T$ and $\theta$ together determine the Chern character. Notice that $\mu^T$ is a $2p$-form, where $p$ is the height of $T$. Anyway, in terms of this new notation the above expression becomes

$$d^{-1}\left\langle X_1 \wedge \ldots \wedge X_{2k} \wedge \left(\prod_{j\notin M} Y_j \wedge Y_{j+p}\right), \mu^T\right\rangle.$$

Thus

$$\Omega^k(X_1 \wedge \ldots \wedge X_{2k}) = (2\pi i)^k k! d^{-1}\Big\langle X_1 \wedge \ldots \wedge X_{2k} \wedge \Sigma\left\{\prod_{j\notin M} Y_j \wedge Y_{j+p} : M \in P(k)\right\}, \mu^T\Big\rangle I.$$

We now transform this to make explicit the role of $\theta$. Recall that $\{\bar{Y}_j\}$ was defined to be the image under $\tilde{T}^{-1}$ of the basis

$$\bar{e}_1, \ldots, \bar{e}_p, e_1, \ldots, e_{p+q}$$

for $\widetilde{H}^*$. Then the dual basis $\{Y_j\}$ will be the image under $\widetilde{T}^*$ of the dual basis

$$e_1, \ldots, e_p, \bar{e}_1, \ldots, \bar{e}_{p+q}$$

for $\widetilde{H}$. By condition 3 in the definition of an embedding map, $T^*$ carries $\omega$ to $-\theta$. In terms of the basis $\{Y_j\}$, this says that

$$\begin{aligned}\theta &= -(\wedge^2 T^*)(\omega) = -\sum_{j=1}^{p+q} T^*(\bar{e}_j) \wedge T^*(e_j) \\ &= -\sum_{j=1}^{p} Y_{j+p} \wedge Y_j - \sum_{j=1}^{q} Y_{2p+j} \wedge Z_j\end{aligned}$$

where $Z_j = T^*(e_{p+j})$ for $j = 1, \ldots, n$. That is,

$$\theta = \sum_{j=1}^{p} Y_j \wedge Y_{j+p} + \sum_{j=1}^{q} Z_j \wedge Y_{2p+j}.$$

Let $C^p$ denote anything with terms which involve at least one $Y_j$ for $2p + 1 \leqq j \leqq n$. Then

$$\begin{aligned}\theta^{p-k} &= (p-k)! \sum\left\{\prod_{j\in N} Y_j \wedge Y_{p+j} : N \in P(p-k)\right\} + C^p \\ &= (p-k)! \sum\left\{\prod_{j\notin M} Y_j \wedge Y_{p+j} : M \in P(k)\right\} + C^p.\end{aligned}$$

From the definition of $\mu^T$ we see that $\langle C^p, \mu^T\rangle = 0$, so that from the earlier formula for $\Omega^k$ we obtain

$$\begin{aligned}&\Omega^k(X_1 \wedge \ldots \wedge X_{2k}) \\ &= (2\pi i)^k k!((p-k)!)^{-1} d^{-1}\langle X_1 \wedge \ldots \wedge X_{2k} \wedge \theta^{k-j}, \mu^T\rangle I.\end{aligned}$$

In particular, we find that for $X, Y \in L$ we have

$$\Omega(X, Y) = 2\pi i((p-1)!)^{-1} d^{-1}\langle X \wedge Y \wedge \theta^{p-1}, \mu^T\rangle I.$$

Now Connes [**8**] defines the Chern character, ch, to be

$$\mathrm{ch}_k(X_1 \wedge \ldots \wedge X_{2k}) = (2\pi i)^{-k}(k!)^{-1}\tau'(\Omega^k(X_1 \wedge \ldots \wedge X_{2k})),$$

where $\tau'$ is the non-normalized trace on $\mathrm{End}(V^T)$ corresponding to $\tau$. Recalling that

$$\tau'(I) = |\det(\widetilde{T})| = d$$

by the comment just after Proposition 4.3, we obtain from above

$$\mathrm{ch}_k(X_1 \wedge \ldots \wedge X_{2k}) = \langle X_1 \wedge \ldots \wedge X_{2k} \wedge \theta^{p-k}, \mu^T\rangle/(p-k)!$$

for $k \geqq 1$. By definition

$$\begin{aligned}\mathrm{ch}_0 &= \tau'(I) = |\det \tilde{T}| = d \\ &= d\langle Y_1 \wedge Y_{p+1} \wedge \ldots \wedge Y_p \wedge Y_{2p}, \bar{Y}_1 \wedge \bar{Y}_{p+1} \wedge \ldots \wedge \bar{Y}_p \wedge Y_{2p}\rangle \\ &= \langle \theta^p, \mu^T\rangle / p!.\end{aligned}$$

Thus the formula for $\mathrm{ch}_k$ given above for $k \geqq 1$ is also valid for $k = 0$. Notice, in particular, that $\langle \theta^p, \mu^T\rangle > 0$, and also that $\mathrm{ch}_p \neq 0$, while $\mathrm{ch}_k = 0$ for $k > p$, where $p$ is the height of $T$. If we let $\lrcorner$ denote contraction, as done in [5], and if $\exp(\theta)$ is defined using the usual power series with the exterior product as done in [17], then it is evident that the Chern character can be written succinctly as

$$\mathrm{ch}(V^T) = \exp(\theta) \lrcorner \mu^T.$$

Let us now investigate the nature of $\mu^T$. Note first that since $\tilde{T}(D) \subseteq \mathbf{R}^p \times \mathbf{Z}^q \times \mathbf{R}^{*p}$, we have

$$\langle \bar{e}_j, \tilde{T}(x) \rangle \in \mathbf{Z} \quad \text{for } p + 1 \leqq j \leqq p + q,$$

for any $x \in D$. But each $\bar{E}_i$ is in $D$, and

$$\langle \bar{e}_j, \tilde{T}(\bar{E}_i) \rangle = \langle \tilde{T}^*(\bar{e}_j), \bar{E}_i\rangle = \langle Y_{p+j}, \bar{E}_i\rangle.$$

It follows that $Y_j$ is integral for $2p + 1 \leqq j \leqq n$. Let $\epsilon$ denote the sign of $\det(\tilde{T})$. Now because $\{\bar{Y}_j\}$ is the image under $\tilde{T}^{-1}$ of the standard basis for $\tilde{H}^*$, and because of the orientation of that basis, we have

$$\begin{aligned}\epsilon d^{-1} = \det(\tilde{T}^{-1}) = \langle & E_1 \wedge \ldots \wedge E_n, \\ & \bar{Y}_{p+1} \wedge \bar{Y}_1 \wedge \ldots \wedge \bar{Y}_{2p} \wedge \bar{Y}_p \wedge \bar{Y}_{2p+1} \wedge \ldots \wedge \bar{Y}_n\rangle.\end{aligned}$$

Thus

$$\begin{aligned}&\mu^T \wedge \bar{Y}_{2p+1} \wedge \ldots \wedge \bar{Y}_n \\ &= d(-1)^p \bar{Y}_{p+1} \wedge \bar{Y}_1 \wedge \ldots \wedge \bar{Y}_{2p} \wedge \bar{Y}_p \wedge \bar{Y}_{2p+1} \wedge \ldots \wedge \bar{Y}_n \\ &= \epsilon(-1)^p \bar{E}_1 \wedge \ldots \wedge \bar{E}_n.\end{aligned}$$

Then if $N$ is any subset of $\{1, \ldots, n\}$ of cardinality $2p$, and if $E_N$ is the corresponding basis element for $\wedge^{2p} L$ coming from the basis $\{E_i\}$ of $L$, we have

$$\begin{aligned}\langle E_N, \mu^T\rangle &= \langle E_N \wedge Y_{2p+1} \wedge \ldots \wedge Y_n, \mu^t \wedge \bar{Y}_{2p+1} \wedge \ldots \wedge \bar{Y}_n\rangle \\ &= \epsilon(-1)^p \langle E_N \wedge Y_{2p+1} \wedge \ldots \wedge Y_n, \bar{E}_1 \wedge \ldots \wedge \bar{E}_n\rangle,\end{aligned}$$

which is an integer because $Y_{2p+1}, \ldots, Y_n$ are integral as seen above. It follows that $\mu^T$ is integral. Now by definition $\mu^T$ is decomposable over $\mathbf{R}$. We show that, in fact, $\mu^T$ is decomposable over $\mathbf{Z}$, that is:

4.4 LEMMA. *There is an integral basis $\{\bar{F}_j\}$ for $D \subset L^*$, and an integer $m$, such that*

$$\mu^T = m\bar{F}_1 \wedge \ldots \wedge \bar{F}_{2p}.$$

*Proof.* Let $C$ denote the subgroup of $\mathbf{Z}^n \subset L$ spanned by $Y_{2p+1}, \ldots, Y_n$. By Theorem 5 on page 393 of [27], there is a basis $\{F_j\}$ for $\mathbf{Z}^n$ and there are integers $k_1, \ldots, k_q$ such that $k_1 F_{2p+1}, \ldots, k_q F_n$ is a basis for $C$. In particular,

$$Y_{2p+1} \wedge \ldots \wedge Y_n = cF_{2p+1} \wedge \ldots \wedge F_n$$

for some integer $c$. Let $\{\bar{F}_j\}$ denote the dual basis for $L^*$. For any subset $N$ of cardinality $2p$ in $\{1, \ldots, n\}$ let $\bar{F}_N$ denote the corresponding basis element for $\wedge^{2p} L^*$. Then there are real numbers $a_N$ such that

$$\mu^T = \sum a_N \bar{F}_N$$

as $N$ ranges over all possible such subsets. For such $N$, let $F_N$ denote the corresponding basis element for $\wedge^{2p} L$. Then

$$\begin{aligned}\langle F_N, \mu^T\rangle &= \langle F_N \wedge Y_{2p+1} \wedge \ldots \wedge Y_n, \mu^T \wedge \bar{Y}_{2p+1} \wedge \ldots \wedge \bar{Y}_n\rangle \\ &= c\langle F_N \wedge F_{2p+1} \wedge \ldots \wedge F_n, \mu^T \wedge \bar{Y}_{2p+1} \wedge \ldots \wedge \bar{Y}_n\rangle,\end{aligned}$$

from which it is clear that $\langle F_N, \mu^T\rangle = 0$ unless $N = \{1, \ldots, 2p\}$. It follows that all the $a_N = 0$ unless this condition is met, so that

$$\mu^T = m\bar{F}_1 \wedge \ldots \wedge \bar{F}_{2p}$$

for some number $m$. But $m \in \mathbf{Z}$ by the integrality of $\mu_p$.

We have thus shown all but the existence part of:

4.5 THEOREM. *Let $\theta \in \wedge^2 L$ be given. For each embedding map $T$ of height $p$ there is a decomposable element $\mu^T$ of $\wedge^{2p}\mathbf{Z}^n \subset \wedge^{2p} L^*$ such that the Chern character of the projective $A_\theta$-module $V^T$ is given by*

$$\mathrm{ch}(V^T) = \exp(\theta) \lrcorner \mu^T,$$

*and the curvature, for an appropriate connection, is given by*

$$\Omega(X, Y) = 2\pi i((p-1)!)^{-1}|\det(\tilde{T})|^{-1}\langle X \wedge Y \wedge \theta^{p-1}, \mu^T\rangle I.$$

*In particular, $\langle \theta^p, \mu^T\rangle > 0$.*

*Conversely, for any decomposable $\mu$ in $\wedge^{2p}\mathbf{Z}^n$ such that $\langle \theta^p, \mu\rangle > 0$, there exists an embedding map $T$ of height $p$ such that for some positive integer $m$ the sum of $m$ copies of $V^T$ has Chern character and curvature as given above with $\mu = m\mu^T$.*

*Proof of existence.* By assumption there is an oriented basis $\{\bar{F}_j\}$ for $\mathbf{Z}^n \subset L^*$ such that

$$\mu = m\epsilon \bar{F}_1 \wedge \bar{F}_{p+1} \wedge \ldots \wedge \bar{F}_p \wedge \bar{F}_{2p}$$

where $m$ is a positive integer and $\epsilon = \pm 1$. Let $\upsilon = \mu/m$, so that $\upsilon$ also satisfies the hypotheses of the theorem. We will produce an embedding map $T$ such that $\mu^T = \upsilon$. Then $m$ copies of $V^T$ will have the desired Chern character and curvature. Let $\{F_j\}$ denote the dual basis for $L$, and let $W$ denote the linear span of $F_1, \ldots, F_{2p}$. Since $\langle \theta^p, \upsilon \rangle \neq 0$, we can find a new basis $\{Y_j\}$ for $W$, not necessarily integral, such that if we extend it to a basis for $L$ by $Y_{2p+j} = F_{2p+j}$ for $j = 1, \ldots, q$, then

$$\theta = \sum_{j=1}^{p} Y_j \wedge Y_{p+j} + \sum_{j=1}^{q} Z_j \wedge Y_{2p+j}$$

for certain $Z_j \in L$. Define $T^*: H \to L$ by

$$T^*(e_j) = \begin{cases} Y_j & \text{for } 1 \leqq j \leqq p \\ Z_{j-p} & \text{for } p+1 \leqq j \leqq p+q \end{cases}$$

$$T^*(\bar{e}_j) = Y_{p+j} \quad \text{for } 1 \leqq j \leqq p+q.$$

Then

$$\begin{aligned} (\wedge^2 T^*)(\omega) &= \sum_{j=1}^{p+q} T^*(\bar{e}_j) \wedge T^*(e_j) \\ &= \sum_{j=1}^{p} Y_{p+j} \wedge Y_j + \sum_{j=1}^{q} Y_{2p+j} \wedge Z_j \\ &= -\theta. \end{aligned}$$

Let $T$ be the adjoint of $T^*$, so $T$ goes from $L^*$ to $H^*$. The above calculation shows that $T$ satisfies condition 3 in Definition 4.1 of an embedding map. Let $\{\bar{Y}_j\}$ be the dual basis to $\{Y_j\}$ for $L^*$. Note that $\bar{Y}_{2p+j} = \bar{F}_{2p+j}$ for $1 \leqq j \leqq q$, while the span of $\bar{Y}_1, \ldots, \bar{Y}_{2p}$ is the same as the span of $\bar{F}_1, \ldots, \bar{F}_{2p}$. Then for fixed $k$

$$\langle Y_j, \bar{Y}_k \rangle = \begin{cases} \langle T^*(e_j), \bar{Y}_k \rangle = \langle e_j, T(\bar{Y}_k) \rangle & \text{for } 1 \leqq j \leqq p \\ \langle T^*(\bar{e}_{j-p}), \bar{Y}_k \rangle = \langle \bar{e}_{j-p}, T(\bar{Y}_k) \rangle & \text{for } p+1 \leqq j \leqq n. \end{cases}$$

From this it is clear that

$$\bar{Y}_k = \begin{cases} \tilde{T}^{-1}(\bar{e}_k) & \text{for } 1 \leqq k \leqq p \\ \tilde{T}^{-1}(e_{k-p}) & \text{for } p+1 \leqq k \leqq n. \end{cases}$$

In particular,

$$\tilde{T}(\bar{F}_{2p+j}) = e_{p+j} \quad \text{for } 1 \leqq j \leqq q,$$

while $\tilde{T}(\bar{F}_j)$ for $1 \leqq j \leqq 2p$ is contained in the span of $\tilde{T}(\bar{Y}_1), \ldots, \tilde{T}(\bar{Y}_{2p})$, which is the span of $\bar{e}_1, e_1, \ldots, \bar{e}_p, e_p$. It follows that $T$ satisfies condition 1 in the definition of an embedding map. It is also clear from the above that $T$ satisfies condition 2′, and hence condition 2. Thus $T$ is an embedding map for $\theta$. As before, with $d = |\det(\tilde{T})|$, we have

$$\mu^T = d \sum_{j=1}^{p} \bar{Y}_j \wedge \bar{Y}_{j+p}.$$

But by the definition of $\det(\tilde{T})$ we have

$$\det(\tilde{T})\ \bar{Y}_1 \wedge \bar{Y}_{p+1} \wedge \ldots \wedge \bar{Y}_p \wedge \bar{Y}_{2p} \wedge \ldots \wedge \bar{Y}_n$$
$$= \bar{F}_1 \wedge \ldots \wedge \bar{F}_n,$$

so that, remembering that $\bar{Y}_j = \bar{F}_j$ for $j \geqq 2p + 1$, we obtain

$$\det(\tilde{T}) \prod_{j=1}^{p} \bar{Y}_j \wedge \bar{Y}_{j+p} = \bar{F}_1 \wedge \ldots \wedge \bar{F}_{2p}.$$

Thus $\mu^T$ agrees with $\upsilon$ up to a sign. But $\langle \theta^p, \mu^T \rangle > 0$, since up to a positive constant this is $\mathrm{ch}_0(V^T)$, while $\langle \theta^p, \upsilon \rangle > 0$ by assumption. Hence the sign must be positive, and $\mu^T = \upsilon$. Thus $V^T$ has curvature and Chern character as stated in the theorem, except using $\upsilon$ instead of $\mu$. Since the Chern character is additive on direct sums of modules, the sum of $m$ copies of $V^T$ will have the desired curvature and Chern character for $\mu$.

Actually, if $2p < n$, then we can alter the above definition of $T$ by setting

$$T^*(\bar{e}_{p+1}) = m^{-1} Z_1, \quad T^*(e_{p+1}) = m Y_{2p+1},$$

and then we will still have $\wedge^2 T^*(\omega) = -\theta$, but the determinant of $\tilde{T}$ will be multiplied by $m$, so that $\mu^T = \mu$. However if $2p = n$, there does not seem to be enough room for such a maneuver.

We remark that if $\mu \in \wedge^0 L^*$, so that $p = 0$, then it is easily seen that the $V^T$ constructed above is free of rank $\mu$.

Let us discuss now the reason for defining $\theta$ by

$$\rho(x, y) = \bar{e}(\theta(x, y)),$$

rather than by the $e(\theta(x, y))$ which Elliott uses in [**17**]. Elliott's formulas for the Chern character are not quite correct as stated, but need to have $\theta$ replaced everywhere by $-\theta$, or to have changed the basic commutation relation. (See his comment in the second paragraph of [**16**].) The source of this problem occurs in the middle of page 180, where Elliott appeals to Connes' calculation on page 601 of [**8**]. The problem is that Connes' calculation is off by a sign. To be more precise, if for $\lambda = e(\theta_0)$ with $\theta_0 \in (0, 1)$ one uses the commutation relation $U_1 U_2 = \lambda U_2 U_1$, as does Connes on page 601, then $\mathrm{ch}_1$ for the module with trace $\theta_0$ is $-1$; while if one uses the relation $U_1 U_2 = \bar{\lambda} U_2 U_1$, as does Connes on page 602, then $\mathrm{ch}_1$ for this module is $+1$. As Elliott explains on page 178 of [**17**], the Chern character, unlike the trace on $K_0$, is not intrinsic to the $C^*$-algebra, but depends on the formulation of the dual group action. In particular, it is easily seen to depend on the orientation of the Lie algebra. The difference in commutation relations above can be viewed as an implicit change in this orientation, in that the second version can be rewritten as $U_2U_1 = \lambda U_1U_2$.

**5. Tensor products with finite-dimensional representations.** In this section we, in effect, generalize the results of the previous section to the case in which, as in Proposition 3.6, the group $M$ has a finite group as a factor. But we do this by considering a slightly more general case, which involves tensoring projective modules by the spaces of finite dimensional cocycle representations.

Suppose that, much as in Proposition 3.6, $M = N \times F$ where $N \cong \mathbf{R}^p \times \mathbf{Z}^q$ and $F$ is a finite commutative group. Because $F$ is finite, it is clear that $S(M) = S(N) \otimes S(F)$, where $S(F)$ is just the space of complex-valued functions on $F$, and the tensor product is just an algebraic one. The Heisenberg cocycle on $M \times \hat{M}$ will clearly be the product of that from $N$ and that from $F$, and the Heisenberg representation will decompose correspondingly. If $T$ is a homomorphism of $D = \mathbf{Z}^n$ into $M \times \hat{M}$, then it is clear that $T$ is an embedding with cocompact range if and only if its projection into $N \times \hat{N}$ is. In this case the completion of $S(N)$ will form an elementary projective module for the pull-back of the Heisenberg cocycle from $N$, while $S(F)$ will be, under the projection of $T$ into $F \times \hat{F}$, just the vector space of a finite dimensional cocycle representation of $D$. (We avoid the terminology "projective representation" for evident reasons.) The cocycle for $M$ is just the product of those for $N$ and $F$, exactly as happens when forming the inner tensor product of cocycle (i.e., "multiplier") unitary representations [29]. This suggests that instead of concerning ourselves with the effects of all the possible finite groups $F$ and all the possible homomorphisms of $D$ into $F \times \hat{F}$ which can be used, we simply consider the process of tensoring with all possible finite dimensional cocycle representations of $D$. We proceed to explore this process in this section. A hint of the existence of such a process can be found in the construction near the top of page 602 of [8]. In anticipation that the process may be useful for other groups, we will for a while consider arbitrary discrete groups.

I should mention at this point that the statement of the second proposition of my announcement [50] concerning this tensoring process is not quite correct, in that it ignores what happens with the norms involved. But we will see that it is correct at the level of the various dense subalgebras we use. To handle the norms, we proceed by defining a suitable bimodule by which we can "induce" projective modules.

Let $D$ be any discrete group. Given a cocycle $\sigma$ on $D$, let $A_\sigma = C^*(D, \sigma)$, and let $C_\sigma$ be the dense $*$-subalgebra $C_c(D, \sigma)$ of $A_\sigma$. Let $\gamma$ be another cocycle on $D$, and let $\Xi$ be the Hilbert space of a finite dimensional right unitary $\gamma$-representation of $D$. Define a right cocycle action of $D$ on $C_\sigma \otimes \Xi$ by

$$(f \otimes \xi)x = (fx) \otimes (\xi x), \quad x \in D.$$

Note that we use module notation for the action of $D$ on $\Xi$ and $C_\sigma$. Caution must be exercised here because cocycles are involved, so that, for example, $(\xi x)y = \gamma(x, y)\xi(xy)$ instead of $= \xi(xy)$. But if this is kept in mind, it is easily seen that the above right action of $D$ has cocycle $\sigma\gamma$. We need to extend this action to an action of $A_{\sigma\gamma}$ on a suitable completion. For this we need an inner-product with values in $C_{\sigma\gamma}$. Since we are using right actions, it is convenient to choose the ordinary inner-product on $\Xi$ to be linear in the second variable. We recall that the inner-product on $C_\sigma$ with values in $C_\sigma$ is defined by

$$\langle f, g\rangle_\sigma(x) = (f^* * g)(x).$$

Then on $C_\sigma \otimes \Xi$, as a right $C_{\sigma\gamma}$-module for the action defined above, we define an inner-product with values in $C_{\sigma\gamma}$ by

$$\langle f \otimes \xi, g \otimes \eta\rangle_{\sigma\gamma}(x) = \langle f, g\rangle_\sigma(x)\langle \xi x, \eta\rangle.$$

We defer momentarily verification of its properties, and notice instead that the left action of $C_\sigma$ on $C_\sigma \otimes \Xi$ coming from the action on the first factor is "unitary" for this inner-product, that is, for $y \in D$

$$\begin{aligned}\langle y(f \otimes \xi), y(g \otimes \eta)\rangle_{\sigma\gamma}(x) &= \langle u_y f, u_y g\rangle_\sigma(x)\langle \xi x, \eta\rangle \\ &= \langle f, g\rangle_\sigma(x)\langle \xi x, \eta\rangle \\ &= \langle f \otimes \xi, g \otimes \eta\rangle_{\sigma\gamma}(x).\end{aligned}$$

This means that when we form the completion of $C_\sigma \otimes \Xi$, the left action of $C_\sigma$ will extend to an action of $A_\sigma$.

We now argue along the same lines that one uses when showing that left-regular representations of groups absorb all other representations under inner tensor products. Let $\Xi_0$ denote $\Xi$ but with the trivial representation of the group $D$. Then $C_{\sigma\gamma} \otimes \Xi_0$ is a right $C_{\sigma\gamma}$-module by action on the first factor, and has an evident $C_{\sigma\gamma}$-valued inner-product. Define a bijection, $J$, from $C_{\sigma\gamma} \otimes \Xi_0$ onto $C_\sigma \otimes \Xi$ by

$$J(u_x \otimes \xi) = u_x \otimes \xi x.$$

It is easily checked that $J$ is a $C_{\sigma\gamma}$-module homomorphism. We verify that it preserves the inner-products. For $x, y, z \in D$ and $\xi, \eta \in \Xi$ we have

$$\begin{aligned}\langle J(u_x \otimes \xi), J(u_y \otimes \eta)\rangle_{\sigma\gamma}(z) &= \langle u_x \otimes \xi x, u_y \otimes \eta y\rangle_{\sigma\gamma}(z) \\ &= (u_x^* \underset{\sigma}{*} u_y)(z)\langle (\xi x)z, \eta y\rangle,\end{aligned}$$

which is non-zero only when $z = x^{-1}y$, so we can substitute this expression to obtain

$$= (u_x^* \underset{\sigma}{*} u_y)(z)\langle (\xi x)(x^{-1}y), \eta y\rangle$$

$$= (u_x^* \underset{\sigma}{*} u_y)(z)\bar{\gamma}(x, x^{-1}y)\langle \xi y, \eta y\rangle$$

$$= (u_x^* \underset{\sigma\gamma}{*} u_y)(z)\langle \xi, \eta\rangle$$

$$= \langle u_x \otimes \xi, u_y \otimes \eta\rangle_{\sigma\gamma}(z),$$

where the last inner-product is that on $C_{\sigma\gamma} \otimes \Xi_0$. Since it is clear that the inner-product on $C_{\sigma\gamma} \otimes \Xi_0$ is indeed an inner-product, it follows that that on $C_\sigma \otimes \Xi$ is one also. It is now also clear that the completion, $P^\Xi$, of $C_\sigma \otimes \Xi$ for this inner-product is isomorphic to $(A_{\sigma\gamma})^m$, where $m$ is the dimension of $\Xi$, and thus is a projective (free) right $A_{\sigma\gamma}$-module. Since we had checked earlier that the left action of $D$ on $C_\sigma \otimes \Xi$ is "unitary" for the inner-product, it is clear that this left action extends to an action on $P^\Xi$, and so gives a $*$-homomorphism of $A_\sigma$ into the $C^*$-algebra $\mathrm{End}_{A_{\sigma\gamma}}(P^\Xi)$. (We remark that we could have defined $P^\Xi$ as just the completion of $C_{\sigma\gamma} \otimes \Xi_0$, that is as $(A_{\sigma\gamma})^m$, but then the left action of $A_\sigma$ would have had a somewhat more complicated, unmotivated, expression.) Now $\mathrm{End}_{A_{\sigma\gamma}}(P^\Xi)$ is isomorphic to $M_m(A_{\sigma\gamma})$, the algebra of $m \times m$ matrices over $A_{\sigma\gamma}$. Thus one has an isomorphism of $K_0(A_{\sigma\gamma})$ with $K_0$ of this endomorphism algebra, which is order-preserving but in general does not preserve the order unit. The homomorphism of $A_\sigma$ into the endomorphism algebra is clearly unital, and so defines a corresponding homomorphism of $K_0$-groups which is order-preserving and preserves order-units, but need not be an isomorphism. Composing, we obtain an order-preserving homomorphism from $K_0(A_\sigma)$ into $K_0(A_{\sigma\gamma})$. At the level of projective modules this homomorphism just comes by "inducing". That is, given a projective right $A_\sigma$-module $V$, we let

$${}^\Xi V = V \otimes_{A_\sigma} P^\Xi$$

(where this is the purely algebraic tensor product). As usual, ${}^\Xi V$ is seen to be projective by first noticing that this is clear if $V$ is (finitely generated) free, and then using the fact that tensor products preserve direct sums. We summarize the above by:

5.1 PROPOSITION. *Let $D$ be a discrete group, let $\sigma$ be a cocycle on $D$, and let $A_\sigma = C^*(D, \sigma)$. Let $\Xi$ be the Hilbert space for a finite-dimensional unitary right $\gamma$-representation of $D$, where $\gamma$ is a cocycle on $D$. Then $\Xi$ determines a functor from the category of projective $A_\sigma$-modules to the category of projective $A_{\sigma\gamma}$-modules. This functor consists of tensoring with the $A_\sigma$- $A_{\sigma\gamma}$-bimodule $P^\Xi$ which is the completion of $C_c(D, \sigma) \otimes \Xi$, with the left action of $A_\sigma$ coming from the evident action of $D$ on the first factor, with the right action defined by $(f \otimes \xi)x = (fx) \otimes (\xi x)$, and with $A_{\sigma\gamma}$-valued inner-product defined by*

$$\langle f \otimes \xi, g \otimes \eta \rangle_{\sigma\gamma}(x) = \langle f, g \rangle_{\sigma}(x) \langle \xi x, \eta \rangle.$$

*A similar situation prevails for the reduced $C^*$-algebras.*

*Proof.* The only assertion we have not yet verified is the one which concerns the reduced $C^*$-algebras. For this we need the following facts which should have been made explicit in [**42**], and whose proofs are routine.

5.2 LEMMA. *Let $B$ be a $C^*$-algebra and let $X$ be a right $B$-module with definite $B$-valued inner-product. Let $L(X)$ be the pre-$C^*$-algebra of "bounded" operators on $X$. Let $Y$ be the Hilbert space of a faithful representation of $B$. Then the norm of any element of $L(X)$ is the same as its norm as an operator on the Hilbert space obtained by inducing $Y$ via $X$, i.e., on $X \otimes_B Y$ completed in the usual way.*

5.3 COROLLARY. *With $X$ and $B$ as above, let $p$ be a faithful state of $B$. Then the norm of an element of $L(X)$ is the same as its norm as an operator on the Hilbert space obtained by completing $X$ for the ordinary inner-product defined by*

$$\langle x, x' \rangle_p = p(\langle x, x' \rangle_B).$$

We continue the proof of Proposition 5.1. The reduced algebra $C_r^*(D, \sigma\gamma)$ comes from the tracial state on $C_c(D, \sigma\gamma)$ consisting of evaluating functions at the identity element, $e$, of $D$. Since the corresponding representation is faithful for $C_r^*(D, \sigma)$, we can apply Corollary 5.3. But the corresponding ordinary inner-product on $C_c(D, \sigma) \otimes \Xi$ is given by

$$\langle f \otimes \xi, g \otimes \eta \rangle = \langle f \otimes \xi, g \otimes \eta \rangle_{\sigma\gamma}(e) = \langle f, g \rangle_{\sigma}(e) \langle \xi, \eta \rangle.$$

The representation of $C_c(D, \sigma)$ on the left is thus equivalent to $m$ copies of the left regular representation, where $m$ is the dimension of $\Xi$, and so does give a representation of $C_r^*(D, \sigma)$ on the completion of $C_c(D, \sigma) \otimes \Xi$. The rest of the proof works as for the full $C^*$-algebras.

In order to compute Chern characters we really need the above set-up at the level of Schwartz spaces, but, of course, our problem is that we do not know how to define $S(D)$ for an arbitrary discrete group. Thus we specialize now to the case in which $D = \mathbf{Z}^n$ (where the full and reduced $C^*$-algebras coincide). We will let $S_\sigma = S(D, \sigma)$ and similarly for $\sigma\gamma$. Let $Q^\Xi = S_\sigma \otimes \Xi$. Then, with exactly the same formulas as before, $Q^\Xi$ becomes a left-$S_\sigma$ right-$S_{\sigma\gamma}$-bimodule with $S_{\sigma\gamma}$-valued inner-product, which as a right $S_{\sigma\gamma}$-module is isomorphic to $(S_{\sigma\gamma})^m$. The only detail which needs a moment's thought is that the range of the inner-product lies in $S_{\sigma\gamma}$, but this follows immediately from the fact that the pointwise product of a Schwartz function on $D$ by a bounded function is again a Schwartz function.

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Thus $Q^{\Xi}$ defines a functor from the projective right modules over $S_\sigma$ to those over $S_{\sigma\gamma}$. Since we have not yet completed, this functor takes an especially simple form (as is also true when working with the $C_\sigma$ and $C_{\sigma\gamma}$ above), namely

$$^{\Xi}V = V \otimes_{S_\sigma} Q^{\Xi} = V \otimes_{S_\sigma} (S_\sigma \otimes \Xi) \cong V \otimes \Xi,$$

with the right action defined by

$$(v \otimes \xi)x = (vx) \otimes (\xi x)$$

and with the $S_{\sigma\gamma}$-valued inner-product defined by

$$\langle v \otimes \xi, w \otimes \eta\rangle_{\sigma\gamma}(x) = \langle v, w\rangle_\sigma(x)\langle \xi x, \eta\rangle$$

(the point being that the right-hand side is not so readily understood when working with the completion).

We let the Lie algebra $L$ of $T^n$ act as a Lie algebra of derivations on both $S_\sigma$ and $S_{\sigma\gamma}$ as in the previous section.

5.4 PROPOSITION. *Let $V$ be a projective right $S_\sigma$-module, and let $\nabla$ be a connection on $V$ for the action of $L$ on $S_\sigma$. Let $\Xi$ be the Hilbert space of a finite-dimensional unitary right $\gamma$-representation of $D$. Define $\tilde{\nabla}$ on the right $S_{\sigma\gamma}$-module $V \otimes \Xi$ by*

$$\tilde{\nabla}(v \otimes \xi) = (\nabla v) \otimes \xi.$$

*Then $\tilde{\nabla}$ is a connection. Let $\Omega$ be the curvature of $\nabla$. Then the curvature, $\tilde{\Omega}$, of $\tilde{\nabla}$ is given by*

$$\tilde{\Omega}(X, Y) = \Omega(X, Y) \otimes I_{\Xi} \in \mathrm{End}_{S_{\sigma\gamma}}(V \otimes \Xi),$$

*where $I_{\Xi}$ is the identity operator on $\Xi$.*

*Proof.* For $X \in L$, $v \otimes \xi \in V \otimes \Xi$, and $x \in D$ we have

$$\begin{aligned}
\tilde{\nabla}_X((v \otimes \xi)u_x) &= \tilde{\nabla}_X((vx) \otimes (\xi x)) = (\nabla_X(vx)) \otimes \xi x \\
&= ((\nabla_X(v))u_x + v(X(u_x))) \otimes \xi x \\
&= (\nabla_X(v) \otimes \xi)u_x + 2\pi i\langle x, X\rangle(vu_x) \otimes \xi x \\
&= (\tilde{\nabla}_X(v \otimes \xi))u_x + 2\pi i\langle x, X\rangle(v \otimes \xi)u_x \\
&= (\tilde{\nabla}_X(v \otimes \xi))u_x + (v \otimes \xi)(X(u_x)),
\end{aligned}$$

so that $\tilde{\nabla}$ is indeed a connection. (Here, our notation is ambiguous as to when $u_x$ is in $S_\sigma$ or $S_{\sigma\gamma}$.) Note that we have used strongly the special form of the action of $L$ on $S_\sigma$ and $S_{\sigma\gamma}$. The asserted form of $\tilde{\Omega}$ follows from a straight-forward calculation.

The Chern characters of $V$ and $V \otimes \Xi$ are defined in terms of the canonical normalized traces $\tau^\sigma$ and $\tau^{\sigma\gamma}$ on $S_\sigma$ and $S_{\sigma\gamma}$ respectively, and of the corresponding traces on the endomorphism algebras. We must determine how these traces are related. Assume, as before, that $V$ is equipped with an $S_\sigma$-valued inner-product. Then for $v, w \in V$ and $\xi, \eta \in \Xi$ we have

$$\tau^{\sigma\gamma}(\langle v \otimes \xi, w \otimes \eta\rangle_{\sigma\gamma}) = \langle v \otimes \xi, w \otimes \eta\rangle_{\sigma\gamma}(e)$$
$$= \langle v, w\rangle_\sigma(e)\langle \xi e, \eta\rangle = \tau^\sigma(\langle v, w\rangle_\sigma)\langle \xi, \eta\rangle.$$

Let $E_\sigma = \mathrm{End}_{S_\sigma}(V)$, which is where $\Omega$ takes its values, and let

$$E_{\sigma\gamma} = \mathrm{End}_{S_{\sigma\gamma}}(V \otimes \Xi).$$

Then on $V$ we have the $E_\sigma$-valued inner-product defined by

$$\langle v, w\rangle_{E_\sigma}(v') = v\langle w, v'\rangle_\sigma$$

for $v' \in V$. In terms of this inner-product there is, as discussed before Proposition 3.3, the canonically associated (unnormalized) trace on $E_\sigma$, which we again denote by $\tau^\sigma$, defined by

$$\tau^\sigma(\langle v, w\rangle_{E_\sigma}) = \tau^\sigma(\langle w, v\rangle_\sigma).$$

(So we let the context determine the intended domain of $\tau^\sigma$, instead of decorating it with $V$.) In the same way there is the canonically associated trace $\tau^{\sigma\gamma}$ on $E_{\sigma\gamma}$.

5.5 Lemma. *Let $V$ and $\Xi$ be as above. Let $E_\sigma$ and $E_{\sigma\gamma}$ be the endomorphism algebras of $V$ and $V \otimes \Xi$, with canonical traces, $\tau^\sigma$ and $\tau^{\sigma\gamma}$ as above. Let $T \in E_\sigma$, so that $T \otimes I_\Xi \in E_{\sigma\gamma}$. Then*

$$\tau^{\sigma\gamma}(T \otimes I_\Xi) = \tau^\sigma(T)\dim(\Xi).$$

*Proof*. It suffices to verify this for $T$ of the form $\langle v, w\rangle_{E_\sigma}$, since these span $E_\sigma$. Let $\{\xi_i\}$ be an orthonormal basis for $\Xi$. Then for any $\eta \in \Xi$ and $v' \in V$ we have

$$\sum \langle v \otimes \xi_i, w \otimes \xi_i\rangle_{E_{\sigma\gamma}}(v' \otimes \eta)$$
$$= \sum (v \otimes \xi_i)\langle w \otimes \xi_i, v' \otimes \eta\rangle_{\sigma\gamma}$$
$$= \sum_i \sum_x ((v \otimes \xi_i)x)\langle w \otimes \xi_i, v' \otimes \eta\rangle_{\sigma\gamma}(x)$$
$$= \sum_x \left(vx \otimes \sum_i (\xi_i x)\langle \xi_i x, \eta\rangle\right)\langle w, v'\rangle_\sigma(x)$$
$$= v\langle w, v'\rangle_\sigma \otimes \eta = (\langle v, w\rangle_{E_\sigma} \otimes I_\Xi)(v' \otimes \eta).$$

That is,

$$\sum \langle v \otimes \xi_i, w \otimes \xi_i\rangle_{E_{\sigma\gamma}} = \langle v, w\rangle_{E_\sigma} \otimes I_\Xi.$$

Then

$$\begin{aligned}\tau^{\sigma\gamma}(\langle v, w\rangle_\sigma \otimes I_\Xi) &= \tau^{\sigma\gamma}(\textstyle\sum \langle v \otimes \xi_i, w \otimes \xi_i\rangle_{E_{\sigma\gamma}}) \\ &= \textstyle\sum \tau^\sigma(\langle v, w\rangle_\sigma)\langle \xi_i, \xi_i\rangle \\ &= \tau^\sigma(\langle v, w\rangle_\sigma)\dim(\Xi),\end{aligned}$$

where we used here one of the calculations made in the paragraph before the lemma.

We can now obtain the main result of this section:

5.6 THEOREM. *Let $V$ be a projective right $S_\sigma$-module, and let $\Xi$ be the Hilbert space of a finite-dimensional right unitary $\gamma$-representation of $D$, so that $V \otimes \Xi$ is a projective $S_{\sigma\gamma}$-module. Then the Chern characters* $\mathrm{ch}^V$ *and* $\mathrm{ch}^{V\otimes\Xi}$ *of $V$ and $V \otimes \Xi$ respectively are related by*

$$\mathrm{ch}^{V\otimes\Xi} = (\dim \Xi)\mathrm{ch}^V.$$

*Proof.* Note that this makes sense even though the modules are over different algebras, since these algebras are acted on by the same Lie algebra, $L$, and the Chern characters are just formal sums of alternating multi-linear forms on $L$. To compute the Chern characters, we equip $V$ with an $S_\sigma$-valued inner-product and a connection, $\nabla$, for the action of $L$ on $S_\sigma$. As indicated in [**8**], the Chern character of $V$ will be independent of these choices. We equip $V \otimes \Xi$ with the corresponding $S_{\sigma\gamma}$-valued inner-product and the connection $\tilde{\nabla}$ defined above. Let $\Omega$ and $\tilde{\Omega}$ be the curvatures of $\nabla$ and $\tilde{\nabla}$, which have values in the endomorphism algebras $E_\sigma$ and $E_{\sigma\gamma}$ of $V$ and $V \otimes \Xi$ respectively. We let $\tau^\sigma$ and $\tau^{\sigma\gamma}$ be the traces on $E_\sigma$ and $E_{\sigma\gamma}$ as described above.

The Chern characters are defined in terms of exterior powers of $\Omega$ and $\tilde{\Omega}$ respectively, so we must see how these exterior powers are related. According to Proposition 5.4 we have

$$\tilde{\Omega}(X, Y) = \Omega(X, Y) \otimes I_\Xi,$$

that is, $\tilde{\Omega}$ is the composition of $\Omega$ with the homomorphism from $E_\sigma$ into $E_{\sigma\gamma}$ which takes a $T$ in $E_\sigma$ to $T \otimes I_\Xi$. But it is then easily checked that exterior powers will be related by the same composition, that is,

$$\begin{aligned}&(\tilde{\Omega} \wedge \ldots \wedge \tilde{\Omega})(X_1 \wedge \ldots \wedge X_{2k}) \\ &= ((\Omega \wedge \ldots \wedge \Omega)(X_1 \wedge \ldots \wedge X_{2k})) \otimes I_\Xi.\end{aligned}$$

When we take traces, using Lemma 5.5, we obtain

$$\begin{aligned}&\tau^{\sigma\gamma}((\tilde{\Omega} \wedge \ldots \wedge \tilde{\Omega})(X_1 \wedge \ldots \wedge X_{2k})) \\ &= (\dim \Xi)\tau^\sigma((\Omega \wedge \ldots \wedge \Omega)(X_1 \wedge \ldots \wedge X_{2k})).\end{aligned}$$

Putting in the required factors of $(2\pi i)^{-k}/k!$, we obtain the desired result.

Actually, we want to construct modules over a fixed algebra $S_\sigma$, and so we now change our point of view slightly and assume that $V$ is an $S_{\sigma\gamma}$-module while $\Xi$ is the space of a $\bar{\gamma}$-representation of $D$, so that it is $V \otimes \Xi$ which is an $S_\sigma$-module. We also change back to notation in terms of skew bilinear forms. Thus let

$$\sigma(x, y) = \bar{e}(\theta(x, y)/2)$$

as in the previous section, and let

$$\gamma(x, y) = \bar{e}(\psi(x, y)/2)$$

for some rational $\psi$ in $\bigwedge^2 L$, so that

$$(\sigma\gamma)(x, y) = \bar{e}((\theta + \psi)(x, y)12).$$

Putting everything together, we see that if $T$ is an embedding map for $\theta + \psi$, then $V^T \otimes \Xi$ will be an $S_\theta$-module whose Chern character is

$$(\dim \Xi) \exp(\theta + \psi) \lrcorner \mu^T$$

for $\mu^T$ as in Theorem 4.5.

Incidently, we now see why it was not necessary for us to put any non-degeneracy hypotheses on $\theta$, namely that in adding various rational $\psi$'s to $\theta$ (to get $\sigma\gamma$) we can obtain non-degenerate forms. In fact, if integral $\psi$'s are added, $\sigma\gamma$ does not change at all, while the effect on the Chern character can be seen to simply involve replacement of $\mu$ by another integral decomposable form.

5.7 *Definition*. Let $\theta \in \bigwedge^2 L$, and let

$$\sigma(x, y) = \bar{e}(\theta(x, y)/2)$$

as above. Then by a *standard* $S_\theta$-module we mean any projective right $S_\theta$-module which is isomorphic to a direct sum of modules of the form $V \otimes \Xi$ where $\Xi$ is the Hilbert space of a finite-dimensional right unitary $\bar{\gamma}$-representation of $D$ (where $\gamma(x, y) = \bar{e}(\psi(x, y)/2)$ for some rational $\psi \in \bigwedge^2 L$), and $V$ is an elementary $S_{\theta+\psi}$-module, that is, is constructed from an embedding map $T$ as described in the previous section. (Different summands of a standard module may have different $\psi$'s.) By a *standard* $A_\theta$-module we mean the completion of any standard $S_\theta$-module, or equivalently, any module isomorphic to a finite direct sum of modules of form ${}^{\Xi}V$ where $\Xi$ is as above and $V$ is an elementary $A_{\theta+\psi}$-module.

Thus the standard $A_\theta$-modules are the ones which we know how to construct, by the methods of this section and the last. Furthermore, we have seen that we are able to calculate the Chern characters of standard modules, and thus determine the elements of $K_0$ which they represent since the Chern character is faithful on $K_0(A_\theta)$.

In calculating the Chern characters of standard modules we need to know the dimension of $\Xi$. But, given a rational $\psi$, it does not seem possible in general to see immediately from its coefficients for the standard basis what the dimension will be of a corresponding irreducible cocycle representation. One must first "diagonalize" $\psi$ and then inspect the denominators of the new coefficients. For our present purposes, however, we will only need to use $\psi$'s of a fairly special form, for which the dimension can be immediately determined. Specifically, in the next section we will need:

5.8 Proposition. *Let $\psi$ be a rational element of $\bigwedge^2 L$. Suppose that there is an integral basis for $L$ such that, when $\psi$ is expressed as a linear combination of the corresponding basis elements for $\bigwedge^2 L$, all of the coefficients are integers except one, which is of form $p/q$ for $p$ and $q$ relatively prime. Let $\gamma$ be the corresponding cocycle on $D$ defined by*

$$\gamma(x, y) = \bar{e}(\psi(x, y)/2).$$

*Then there is an irreducible right unitary $\bar{\gamma}$-representation of $D$ of dimension $|q|$. (And in fact all irreducible $\bar{\gamma}$-representations of $D$ will have dimension $|q|$.)*

*Proof.* Let $\{F_i\}$ be the given basis, arranged so that it is the coefficient of $F_1 \wedge F_2$ which is $p/q$, and let $\{\bar{F}_i\}$ be the dual basis for $L^*$. Since all other coefficients of $\psi$ are integers, $\gamma(\bar{F}_i, \bar{F}_j) = 1$ unless $i, j \leqq 2$. As the space $\Xi$ of the representation we take $l^2(\mathbf{Z}/\mathbf{Z}q)$. We let $F_1$ act by translation by 1, and we let $\bar{F}_2$ act by pointwise multiplication by the function

$$(m + Zq) \mapsto e(mp/q).$$

We let all the other $\bar{F}_i$ act as the identity operator. This is essentially the Heisenberg representation of $M \times \hat{M}$ where $M = \mathbf{Z}/\mathbf{Z}q$, and so is irreducible. Then routine calculations show that for $\xi \in \Xi$ and $x, y \in D$ we have

$$(\xi x)y = e(x_2 y_1 p/q)\xi(x + y),$$

where

$$x = \sum x_i \bar{F}_i \quad \text{and} \quad y = \sum y_j \bar{F}_j.$$

We recognize the cocycle here as being essentially the Heisenberg cocycle. The skew bicharacter which is cohomologous to it is easily seen to be $\bar{\gamma}$. Adjusting the action above by the corresponding coboundary, we obtain the desired $\bar{\gamma}$-representation.

The fact that all irreducible representations are of dimension $|q|$ follows from Proposition 34 of [**18**], though we will early in Section 7 give a short proof of it for our special setting.

**6. The positive cone of $K_0(A_\theta)$.** The objective of the present section is to show that for $\theta$ not rational, every element of $K_0(A_\theta)$ of positive trace is represented by a standard module, so that the positive cone of $K_0(A_\theta)$ consists exactly of its elements of positive trace. As soon as we have proven in Section 7 that cancellation holds (for $\theta$ not rational), it will follow that every projective $A_\theta$-module is isomorphic to a standard module.

For the proof of cancellation we need the additional fact that positive elements of $K_0(A_\theta)$ are represented by modules having as direct summands arbitrarily high multiples of standard modules. The proof of this fact must be carried through the inductive arguments used in this section. Thus the main theorem of this section is:

6.1 THEOREM. *Let $\theta \in \wedge^2 L$, and assume that $\theta$ is not rational. Then every element of $K_0(A_\theta)$ with strictly positive trace is represented by a standard module, so that the positive cone of $K_0(A_\theta)$ consists of its elements of strictly positive trace, together with zero. Furthermore, for any integer $m > 0$ every positive element of $K_0(A_\theta)$ is represented by a standard module which has as a direct summand $m$ copies of a (non-zero) standard module.*

Of crucial importance for the proof of Theorem 6.1 is the work of George Elliott [**16**], in which he describes the range of the Chern character on $K_0(A_\theta)$, and shows that the Chern character is injective on $K_0(A_\theta)$. This reduces our task to showing that for every element of $\wedge^e L^*$ which is in the range of the Chern character and which has positive 0th component (the trace), we can find a standard module with that given element as its Chern character (for $\theta$ not rational). We now recall Elliott's specific results, with the slightly more explicit notation which we will need (and with the modification of conventions which we discussed at the end of Section 4).

We let $\wedge^e L$ be the even part of the exterior algebra of $L$, so that $\wedge^e L$ is a commutative finite-dimensional graded algebra under the exterior product. Then $\theta$, as an element of the algebra $\wedge^e L$, is nilpotent, and consequently $\exp(\theta)$ is defined by a finite series (where here $\exp(\theta)$ should not be confused with the composition of $\theta$ with the function $t \mapsto \exp(2\pi i t)$ ). Viewing $D$ as the integral lattice in $L^*$, we can view $\wedge^e D$ as the integral part of $\wedge^e L^*$, and it thus makes sense to pair elements of $\wedge^e D$ with $\exp(\theta)$ to get real numbers. For $\mu \in \wedge^e D$ we denote this pairing by $\langle \exp(\theta), \mu \rangle$. Then Elliott shows that the range of the trace on $K_0(A_\theta)$ is exactly the set

$$\langle \exp(\theta), \wedge^e D \rangle$$

of real numbers. (For a proof of this fact using Connes' $n$-traces see [**36**].) In interpreting this expression, we must pair the various terms in the series for $\exp(\theta)$ with the elements of $\wedge^e D$ of the same degree.

More generally, we can contract elements of $\wedge^e L^*$ by $\theta$, giving a nilpotent endomorphism of $\wedge^e L^*$ which lowers degree by 2. Exponentiating this endomorphism, we obtain an automorphism of $\wedge^e L^*$, which Elliott denotes by $\exp(1 \wedge \theta)$. Elliott then shows that the range of the Chern character on $K_0(A_\theta)$, which will be a subset of $\wedge^e L^*$, is exactly the image of $\wedge^e D$ under $\exp(1 \wedge \theta)$. Since $\exp(1 \wedge \theta)$ is easily seen to be just contraction by $\exp(\theta)$, we will often find it convenient to denote the image under $\exp(1 \wedge \theta)$ of $\mu \in \wedge^e L^*$ by $(\exp \theta) \lrcorner \mu$. It is easily seen that the 0-degree term of $(\exp \theta) \lrcorner \mu$ is exactly the trace term $\langle \exp(\theta), \mu \rangle$ indicated above, and that the term of degree $2k$ is defined, for $X_1, \ldots, X_{2k} \in L$, by

$$\begin{aligned} &\langle X_1 \wedge \ldots \wedge X_{2k}, (\exp \theta) \lrcorner \mu \rangle \\ &= \sum_{p=k}^{t} \langle X_1 \wedge \ldots \wedge X_{2k} \wedge \theta^{p-k}, \mu_p \rangle / (p-k)! \end{aligned}$$

where by $\mu_p$ we mean the term of $\mu$ of degree $2p$, and where $t$ is $n/2$ or $(n-1)/2$ according to whether the dimension, $n$, of $L$ is even or odd.

Our objective then, is to show (for $\theta$ not rational) that, given any $\mu \in \wedge^e D$ for which $\langle \exp(\theta), \mu \rangle > 0$ (the trace condition), we can construct a standard module whose Chern character is $(\exp \theta) \lrcorner \mu$.

To do this, we must see how the elementary and standard modules constructed earlier fit in with Elliott's results. Now from Theorem 4.5 it is clear that the elementary modules correspond exactly to the $\mu$'s which are homogeneous (i.e., concentrated in one degree) and decomposable over $\mathbf{Z}$. Now let $\psi$ be a rational element of $\wedge^2 L$ and let $\gamma$ be the corresponding cocycle as in Section 5. Let $\Xi$ be the Hilbert space for an irreducible finite dimensional right unitary $\bar{\gamma}$-representation for $D$, and let $V$ be an elementary $S_{\theta+\psi}$-module, so that $V \otimes \Xi$ is a standard $S_\theta$-module. Let $d_\psi$ denote the dimension of $\Xi$, which is the same as the dimension of any other irreducible $\gamma$-representation, by Proposition 34 of [**18**] or an argument we give early in Section 7. Let $\mu$ be the homogeneous decomposable element of $\wedge^e D$ corresponding to $V$. Then by Theorem 5.6 and the discussion immediately after, the Chern character of $V \otimes \Xi$ is

$$d_\psi(\exp(\theta + \psi)) \lrcorner \mu = (\exp \theta) \lrcorner (d_\psi(\exp \psi) \lrcorner \mu).$$

(One can show that $d_\psi(\exp \psi) \lrcorner \mu$ is in $\wedge^e D$.) Consequently, standard $S_\theta$-modules have as Chern characters finite sums of such terms (satisfying the positive trace condition).

Conversely, Theorems 4.5 and 5.6 show that for any homogeneous decomposable $v \in \wedge^e D$ we can construct an $S_\theta$-module whose Chern character is

$$(\exp \theta) \lrcorner (d_\psi(\exp \psi) \lrcorner v),$$

provided that the trace condition

$$\langle \exp(\theta + \psi), v \rangle > 0$$

is satisfied. We can rewrite this trace condition as

$$\langle \exp(\theta), d_\psi(\exp \psi) \lrcorner v \rangle > 0.$$

It now becomes evident that the proof of Theorem 6.1 can be reduced to proving a statement just about elements of $\wedge^e D$, and we can forget (for this purpose) that there are any $C^*$-algebras or modules involved. Specifically, it is sufficient to prove:

6.2 THEOREM. *Let $\theta \in \wedge^2 L$ and assume that $\theta$ is not rational. Let $\mu \in \wedge^e D$. If $\langle \exp(\theta), \mu \rangle > 0$, then $\mu$ can be expressed as a finite sum of terms of the form $d_\psi(\exp \psi) \lrcorner v$ for which*
(1) *$v \in \wedge^e D$, and $v$ is homogeneous and decomposable over $\mathbf{Z}$,*
(2) *$\psi \in \wedge^2 L$ and $\psi$ is rational,*
(3) *$\langle \exp(\theta), d_\psi(\exp \psi) \lrcorner v \rangle > 0$.*
*Furthermore, for any positive integer $m$ we can arrange that one of these terms occurs $m$ times in the sum.*

For $\mu \in \wedge^e L^*$ we will denote its component of degree $2k$ by $\mu_k$. By the height of $\mu$ we will mean the largest integer $k$ for which $\mu_k \neq 0$. From Theorem 4.5 and the discussion shortly before Lemma 4.4, this corresponds to the definition of height given in Definition 4.1. We will prove Theorem 6.2 by induction on the height of $\mu$. For given height $k$, we first treat the case in which $\mu_k$ is decomposable over $\mathbf{Z}$, and then show how to deal with the general case.

To start the induction we need to know that the theorem is true for height 0. Now for this case, the first part of the theorem is obvious, as $\mu$ is trivially already homogeneous and decomposable. However, the multiplicity statement at the end of the theorem is not evident, and to prove it we must clearly go beyond height 0, and use the irrationality of $\theta$. For future purposes it is convenient for us to consider a slightly more general case.

6.3 LEMMA. *For any $\theta$ in $\wedge^2 L$ the conclusions of Theorem 6.2 are true whenever $\mu$ is of the form $\mu_0 + a\bar{F}_1 \wedge \bar{F}_2$ where $\bar{F}_1$ and $\bar{F}_2$ are part of an integral oriented basis $\{\bar{F}_i\}$ for $D$ such that $\theta_{12} = \langle \theta, \bar{F}_1 \wedge \bar{F}_2 \rangle$ is irrational. (We permit $a = 0$, to take care of the case of height 0.)*

*Proof.* Let $v = \bar{F}_1 \wedge \bar{F}_2$, let $\{F_i\}$ denote the dual basis to $\{\bar{F}_i\}$ for $L$, and let $\psi = (p/q)F_1 \wedge F_2$ where $p$ and $q$ are integers yet to be chosen. Then

$$\langle \exp(\theta), q(\exp \psi) \lrcorner v \rangle = p + q\theta_{12}.$$

Since $\theta_{12}$ is irrational, we can choose $p$ and $q$ such that $q > 0$ and

$$0 < m(p + q\theta_{12}) < \langle \exp(\theta), \mu \rangle = \mu_0 + a\theta_{12},$$

where $m$ is the desired multiplicity as in the statement of Theorem 6.2. If $\mu_0 - mp > 0$, then the decomposition

$$\mu = mq(\exp \psi) \lrcorner v + (\mu_0 - mp)$$

has almost the desired form. If $\mu_0 - mp \leqq 0$, then $mq\theta_{12} < a\theta_{12}$, so that $a \neq mq$. Let $\epsilon = \operatorname{sign}(a - mq)$, let $v' = \epsilon v$, and let

$$\psi' = \epsilon(\mu_0 - mp)/(a - mq)F_1 \wedge F_2.$$

Then

$$\epsilon(a - mp)(\exp \psi') \lrcorner v' = \mu_0 - mp + (a - mp)\bar{F}_1 \wedge \bar{F}_2,$$

which when paired with $\exp \theta$ is positive, so that

$$\mu = mq(\exp \psi) \lrcorner v + \epsilon(a - mp)(\exp \psi') \lrcorner v'$$

gives a sum of almost the desired form. Now $p/q$ may not be a reduced fraction, but in any event $d_\psi$ will divide $q$, by Proposition 5.8, so the first term above is a sum of copies of $d_\psi(\exp \psi) \lrcorner v$. The factor $\epsilon(a - mp)$, which is positive because of the $\epsilon$, is handled in the same way. We thus obtain a sum of the desired form.

We find it necessary to treat separately also the case of height 1, because there is not yet enough height to maneuver very freely, and because $D$ may have a small number of generators. In fact this case is the most complicated one.

6.4 Lemma. *Let $\theta \in \bigwedge^2 L$, and assume that $\theta$ is not rational. Then the conclusions of Theorem 6.2 hold for any $\mu$ of height 1.*

*Proof.* Since $\mu_1$ is in $\bigwedge^2 D$, there is, according to Lemma 5 on page 71 of [**24**], a basis, $\{\bar{F}_i\}$, for $D$ for which $\mu_1$ has the special form

$$\mu_1 = \sum^r a_i \bar{F}_{2i-1} \wedge \bar{F}_{2i}$$

where the $a_i$ are non-zero integers. The number, $r$, of $a_i$'s is called the rank of $\mu_1$. (Equivalently, $r$ is the smallest integer such that $(\mu_1)^{r+1} = 0$.) We will argue by induction on this rank.

To begin the induction we must prove the conclusion of Theorem 6.2 when $r = 1$, so $\mu_1$ is of the form $a\bar{F}_1 \wedge \bar{F}_2$. We must treat three successive cases. The first is that in which $\langle\theta, \mu_1\rangle$ is irrational, so that $\theta_{12} = \langle\theta, \bar{F}_1 \wedge \bar{F}_2\rangle$ is also. But we treated this case in Lemma 6.3. The next case is that in which $\theta_{12}$ is rational but there is some basis vector $\bar{F}_i$ with $i \geqq 3$ such that $\theta_{1i} = \langle\theta, \bar{F}_1 \wedge \bar{F}_i\rangle$ is irrational (or, by similar arguments, $\langle\theta, \bar{F}_2 \wedge \bar{F}_i\rangle$ is irrational). Then for any integer $p$, yet to be chosen, we have

$$\mu_1 = a\bar{F}_1 \wedge \bar{F}_2 = p\bar{F}_1 \wedge (a\bar{F}_2 + (1-p)\bar{F}_i) \\ + (1-p)\bar{F}_1 \wedge (a\bar{F}_2 - p\bar{F}_i).$$

Let

$$v_1 = p\bar{F}_1 \wedge (a\bar{F}_2 + (1-p)\bar{F}_i).$$

Then for an integer $v_0$ yet to be chosen, and for $v = v_0 + v_1$, we have

$$\langle \exp(\theta), v\rangle = v_0 + pa\theta_{12} + p(1-p)\theta_{1i}.$$

By Weyl's Theorem 9 in **[59]** we can choose $p$ and $v_0$ so that $p \neq 0, 1$ and

$$0 < \langle \exp(\theta), v\rangle < \langle \exp(\theta), \mu\rangle.$$

Let $v' = \mu - v$, so that $0 < \langle \exp(\theta), v'\rangle$ and

$$v_1' = (1-p)\bar{F}_1 \wedge (a\bar{F}_2 - p\bar{F}_i).$$

Then

$$\langle \theta, v_1'\rangle = (1-p)a\theta_{12} - (1-p)p\theta_{1i},$$

which is irrational. Both $v$ and $v'$ are clearly of rank 1, and so by Lemma 5 on page 71 of **[24]** they can be put in the form to which Lemma 6.3 applies. That is, we have reduced this second case to the first.

Finally, we must consider the case in which $\langle \theta, \bar{F}_1 \wedge \bar{F}_i\rangle$ and $\langle \theta, \bar{F}_2 \wedge \bar{F}_i\rangle$ are rational for all $i$. We reduce this case also to the first. Since $\theta$ is not rational, there is some pair of basis vectors, which by rearrangement we can assume to be $\bar{F}_3$ and $\bar{F}_4$, such that $\theta_{34} = \langle \theta, \bar{F}_3 \wedge \bar{F}_4\rangle$ is irrational. For integers $q$ and $v_0$ yet to be chosen let $v_1 = q\bar{F}_3 \wedge \bar{F}_4$ and $v = v_0 + v_1$, so that

$$\langle \exp(\theta), v\rangle = v_0 + q\theta_{34}.$$

Then we can choose $v_0$ and $q \neq 0$ such that

$$0 < \langle \exp(\theta), v\rangle < \langle \exp(\theta), \mu\rangle.$$

Notice that the first case then applies to $v$. Let $v' = \mu - v$, so for any integer $p$, yet to be chosen, we have

$$\begin{aligned} v_1' &= a\bar{F}_1 \wedge \bar{F}_2 - q\bar{F}_3 \wedge \bar{F}_4 \\ &= (a\bar{F}_1 + (1-p)\bar{F}_3) \wedge (p\bar{F}_2 - q\bar{F}_4) \\ &+ (a\bar{F}_1 - p\bar{F}_3) \wedge ((1-p)\bar{F}_2 + q\bar{F}_4). \end{aligned}$$

Let

$$\lambda_1 = (a\bar{F}_1 + (1-p)\bar{F}_3) \wedge (p\bar{F}_2 - q\bar{F}_4).$$

Then

$$\langle \theta, \lambda_1 \rangle = ap\theta_{12} + (1-p)p\theta_{32} - aq\theta_{14} - (1-p)q\theta_{34}.$$

Since $\theta_{12}$, $\theta_{32}$ and $\theta_{14}$ are by assumption rational, while $\theta_{34}$ is irrational, we can choose $p \neq 0, 1$ and integer $\lambda_0$ such that, with $\lambda = \lambda_0 + \lambda_1$, we have

$$0 < \langle \exp(\theta), \lambda \rangle < \langle \exp(\theta), v' \rangle.$$

Let $\lambda' = v' - \lambda$, so that

$$0 < \langle \exp(\theta), \lambda' \rangle$$

and

$$\lambda_1' = (a\bar{F}_1 - p\bar{F}_3) \wedge ((1-p)\bar{F}_2 + q\bar{F}_4).$$

Then $\langle \theta, \lambda_1 \rangle$ and $\langle \theta, \lambda_1' \rangle$ are both irrational. Thus we have

$$\mu = v + \lambda + \lambda',$$

arranged so that after an application of Lemma 5 on page 71 of [**24**], Lemma 6.3 applies to each of the three terms on the right. This concludes the proof for $r = 1$.

We now prove the induction step. That is, we assume Lemma 6.4 to be true for all $\mu_1$ of rank $r-1$ or less ($r \geqq 2$), and we show it to be true for all $\mu_1$ of rank $r$. We do this by showing that we can find $v$ and $v'$ of height $\leqq 1$ such that $\mu = v + v'$, while $v_1$ and $v_1'$ have rank $\leqq r - 1$, and $0 < \langle \exp(\theta), v \rangle$ and $0 < \langle \exp(\theta), v' \rangle$. Then by the induction hypotheses $v$ and $v'$ have expressions as sums of the desired form, including the multiplicity statement, and so $\mu$ does also. As before, by Lemma 5 on page 71 of [**24**] there is an oriented basis $\{\bar{F}_i\}$ for $D$ such that

$$\mu_1 = \sum^r a_i \bar{F}_{2i-1} \wedge \bar{F}_{2i}.$$

Throughout let $\theta_{ij} = \langle \theta, \bar{F}_i \wedge \bar{F}_j \rangle$. We must again consider several cases.

Case 1. Assume that some cross-term of $\theta$ for $\mu_1$ is irrational, that is, for some $j, k \leqq 2r$ not of the form $2i-1$ and $2i$, we have $\theta_{jk}$ irrational. Then by rearranging the basis (and possibly changing signs), we can assume that

$$\mu_1 = a\bar{F}_1 \wedge \bar{F}_2 + b\bar{F}_3 \wedge \bar{F}_4 + \rho \tag{6.5}$$

where $a \neq 0 \neq b$, while $\rho$ is of rank $r-2$, and $\theta_{23}$ is irrational. For any integer $p$, yet to be chosen, we have, much as above,

$$\begin{aligned} \mu_1 = {} & (a\bar{F}_1 + (1-p)\bar{F}_3) \wedge (p\bar{F}_2 + b\bar{F}_4) \\ & + (a\bar{F}_1 - p\bar{F}_3) \wedge ((1-p)F_2 - b\bar{F}_4) + \rho. \end{aligned} \tag{6.6}$$

Let

$$v_1 = (a\bar{F}_1 + (1 - p)\bar{F}_3) \wedge (p\bar{F}_2 + b\bar{F}_4),$$

so that

$$(6.7) \quad \langle \theta, v_1 \rangle = p^2\theta_{23} + p(a\theta_{12} - b\theta_{34} - \theta_{23}) + ab\theta_{14} + b\theta_{34}.$$

By Weyl's Theorem 9 in [**59**] we can choose $p$ and an integer $v_0$ so that, for $v = v_0 + v_1$,

$$0 < \langle \exp(\theta), v \rangle < \langle \exp(\theta), \mu \rangle.$$

Let $v' = \mu - v$, so that $v'_1$ has rank $r - 1$. Then $\mu = v + v'$ is of the desired form.

Case 2. Assume that no cross-terms of $\theta$ for $\mu_1$ are irrational, but some $\theta_{2i-1,2i}$ is irrational, where $i \leqq r$. Again by rearranging the basis we can assume that $\mu_1$ is of the form (6.5) above where now $\theta_{12}$ is irrational while $\theta_{23}$ and $\theta_{14}$ are rational. We then express $\mu_1$ as in (6.6) and define $v_1$ as done there, so that we obtain (6.7). We then see that as long as $a\theta_{12} - b\theta_{34}$ is irrational, we can again apply Weyl's theorem and proceed as in Case 1. Thus we only need to deal with the situation in which $a\theta_{12} - b\theta_{34}$ is rational. But note that if we then add any non-zero integral multiple of $\theta_{12}$ to $a\theta_{12} - b\theta_{34}$ we will obtain an irrational number. This suggests that for a yet to be chosen integer $q$ we write

$$\mu = q\bar{F}_1 \wedge \bar{F}_2 + (a - q)\bar{F}_1 \wedge \bar{F}_2 + b\bar{F}_3 \wedge \bar{F}_4 + \rho$$

and let $\lambda_1 = q\bar{F}_1 \wedge \bar{F}_2$. Since $\theta_{12}$ is irrational we can choose $q \neq 0$ and an integer $\lambda_0$ such that, for $\lambda = \lambda_0 + \lambda_1$,

$$0 < \langle \exp(\theta), \lambda \rangle < \langle \exp(\theta), \mu \rangle.$$

Now $\lambda_1$ is of rank 1 and so $\lambda$ has an expression as a sum of the desired form. Let $\mu' = \mu - \lambda$. Then $\mu'$ still has rank $\leqq r$, but $(a - q)\theta_{12} - b\theta_{34}$ is now irrational, so that we can apply to $\mu'$ the argument given at the beginning of Case 2.

Case 3. Assume that all $\theta_{ij}$ for $i, j \leqq 2r$ are rational, but that $\theta_{ik}$ is irrational for some $i \leqq 2r$ and some $k \geqq 2r + 1$. By rearranging the basis we can assume that $i = 1$, and that $\mu_1$ has form

$$\mu_1 = a\bar{F}_1 \wedge \bar{F}_2 + \rho,$$

where the rank of $\rho$ is $\leqq r - 1$. For an integer $q$ yet to be chosen write

$$\mu_1 = q\bar{F}_1 \wedge \bar{F}_k + \bar{F}_1 \wedge (a\bar{F}_2 - q\bar{F}_k) + \rho$$

and let $\lambda_1 = q\bar{F}_1 \wedge \bar{F}_k$. Since $\theta_{1k}$ is irrational we can choose $q \neq 0$ and an integer $\lambda_0$ so that, for $\lambda = \lambda_0 + \lambda_1$,

$$0 < \langle \exp(\theta), \lambda \rangle < \langle \exp(\theta), \mu \rangle.$$

Now $\lambda_1$ is of rank 1 and so $\lambda$ has an expression as a sum of the desired form. Let $\mu' = \mu - \lambda$. Then $\mu'$ still has rank $r$, but it is now of the form to which Case 2 applies.

Case 4. Assume that all $\theta_{ij}$ for which either $i \leqq 2r$ or $j \leqq 2r$ are rational. Since $\theta$ is not rational, we can find $i, j \geqq 2r + 1$ such that $\theta_{ij}$ is irrational. For an integer $q$ yet to be chosen write

$$\mu_1 = q\bar{F}_i \wedge \bar{F}_j + a\bar{F}_1 \wedge \bar{F}_2 + (-q)\bar{F}_i \wedge \bar{F}_j + \rho$$

where $\rho$ is of rank $r - 1$, and let

$$\lambda_1 = q\bar{F}_i \wedge \bar{F}_j + a\bar{F}_1 \wedge \bar{F}_2.$$

Since $\theta_{ij}$ is irrational, we can choose $q \neq 0$ and an integer $\lambda_0$ such that, for $\lambda = \lambda_0 + \lambda_1$,

$$0 < \langle \exp(\theta), \lambda \rangle < \langle \exp(\theta), \mu \rangle.$$

Let $\lambda' = \mu - \lambda$. Then $\lambda_1$ and $\lambda_1'$ are both of rank $\leqq r$, and are in the form to which Case 2 applies, so that we obtain the desired expression for $\mu$.

This concludes the proof of the induction step, and so of Lemma 6.4.

The induction step in the proof of Theorem 6.2 has two stages. The first deals with the case in which $\mu_k$ is decomposable.

6.8 LEMMA. *Let $\theta \in \wedge^2 L$, and assume that $\theta$ is not rational. Suppose that for some fixed $k \geqq 2$ it is known that the conclusions of Theorem 6.2 are true for all $\mu$ of height $\leqq k - 1$. Then the conclusions of Theorem 6.2 are true for all $\mu$ of height $k$ for which $\mu_k$ is decomposable.*

*Proof.* Notice that $\dim(L) \geqq 4$, since otherwise there are no $\mu$ of height $\geqq 2$. The condition on $\mu_k$ means that there is an oriented basis $\{\bar{F}_i\}$ for $D$ such that

$$\mu_k = a\epsilon\bar{F}_1 \wedge \ldots \wedge \bar{F}_{2k},$$

where $a$ is a positive integer and $\epsilon = \pm 1$. We must consider three cases (which are somewhat parallel to the three cases treated in the first half of the proof of Lemma 6.4).

Case I. We suppose that $\theta$ is not rational on the linear span of $\bar{F}_1, \ldots, \bar{F}_{2k}$. For convenience we rearrange these basis elements so that $\theta_{2k-1,2k}$ is irrational. Let

$$v_k = \epsilon\bar{F}_1 \wedge \ldots \wedge \bar{F}_{2k},$$

so that $\mu_k = av_k$, and let $v = v_k$, so that all the lower order terms of $v$ are 0. We wish to find a rational $\psi \in \wedge^2 L$ such that $d_\psi = a$ and

$$0 < a\langle \exp(\theta), (\exp \psi) \lrcorner v \rangle < \langle \exp(\theta), \mu \rangle.$$

For if we then set $\lambda = \mu - d_\psi(\exp \psi) \lrcorner v$, we see that $\lambda$ is of height $\leqq k - 1$ and $0 < \langle \exp(\theta), \lambda \rangle$, so that by the induction hypothesis $\lambda$ has an expression as a sum of the desired form, including the multiplicity requirement for $m$. Since $d_\psi(\exp \psi) \lrcorner v$ already is of the desired form, it will then follow that $\mu$ has the desired expression as a sum.

It is notationally convenient to treat first the case $k = 2$. This will also give a good indication of how the general argument works. Thus we assume for the moment that $\theta_{34}$ is irrational and that

$$\epsilon v = \bar{F}_1 \wedge \bar{F}_2 \wedge \bar{F}_3 \wedge \bar{F}_4.$$

We look for $\psi$ of the form

$$\psi = rF_1 \wedge F_2 + pF_1 \wedge F_3 + qF_2 \wedge F_4$$

where $p$ and $q$ are integers but $r = c + 1/a$ for an integer $c$, so that $d_\psi = a$ by Proposition 5.6. Notice that

$$\psi \wedge \psi = -2pqF_1 \wedge F_2 \wedge F_3 \wedge F_4,$$

so that $r$ does not occur in this expression. Then

$$(\exp \psi) \lrcorner \epsilon v = -pq + r\bar{F}_3 \wedge \bar{F}_4 - p\bar{F}_2 \wedge \bar{F}_4 - q\bar{F}_1 \wedge \bar{F}_3 + \epsilon v,$$

so that

$$\begin{aligned} &\langle \exp(\theta), (\exp \psi) \lrcorner \epsilon v \rangle \\ &= -pq + r\theta_{34} - p\theta_{24} - q\theta_{13} + (\theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23}) \\ &= r\theta_{34} - p(q + \theta_{24}) - q\theta_{13} + \text{rest}. \end{aligned}$$

Since $\theta_{34}$ is irrational, we can choose $q$ so that $\theta_{34}$ and $q + \theta_{24}$ are linearly independent over the rationals. Since $r = c + 1/a$, we can then choose $c$ and $p$ such that

$$0 < a\langle \exp(\theta), (\exp \psi) \lrcorner v \rangle < \langle \exp(\theta), \mu \rangle,$$

as desired.

For the general case with $k \geqq 3$ we look for $\psi$ of the form

$$\psi = rF_1 \wedge F_2 + pF_1 \wedge F_{2k-1} + qF_2 \wedge F_{2k} + n\phi,$$

where

$$\phi = F_3 \wedge F_4 + F_5 \wedge F_6 + \ldots + F_{2k-3} \wedge F_{2k-2}$$

and $r = c + 1/a$, for $c, n, p, q$ integers. Then

$$(1/k!)\psi^k = -pqn^{k-2}F_1 \wedge \ldots \wedge F_{2k},$$

so that

$$\langle \exp(\theta), (1/k!)\psi^k \lrcorner \epsilon v \rangle = -pqn^{k-2}.$$

Notice that $r$ does not occur in this expression. Next, we see that

$$\begin{aligned} &(1/(k-1)!)\psi^{k-1} \\ &= rn^{k-2}F_1 \wedge \ldots \wedge F_{2k-2} + pn^{k-2}F_1 \wedge F_3 \wedge \ldots \wedge F_{2k-1} \\ &+ qn^{k-2}F_2 \wedge \ldots \wedge F_{2k-2} \wedge F_{2k} \end{aligned}$$

$$+ pqn^{k-3}F_1 \wedge F_{2k-1} \wedge F_2 \wedge F_{2k} \wedge \phi^{k-3}/((k-3)!),$$

so that

$$\begin{aligned}&(1/(k-1)!)\psi^{k-1} \lrcorner \epsilon v \\ &= rn^{k-2}\bar{F}_{2k-1} \wedge \bar{F}_{2k} + pn^{k-2}\bar{F}_2 \wedge \bar{F}_{2k} + qn^{k-2}\bar{F}_1 \wedge \bar{F}_{2k-1} \\ &\quad - pqn^{k-3}(\bar{F}_3 \wedge \bar{F}_4 + \ldots + \bar{F}_{2n-3} \wedge \bar{F}_{2n-2}),\end{aligned}$$

and

$$\begin{aligned}&\langle \exp(\theta), (1/(k-1)!)\psi^{k-1} \lrcorner \epsilon v\rangle \\ &= rn^{k-2}\theta_{2k-1,2k} + pn^{k-2}\theta_{2,2k} + qn^{k-2}\theta_{1,2k-1} \\ &\quad - pqn^{k-3}(\theta_{3,4} + \ldots + \theta_{2n-3,2n-2}).\end{aligned}$$

Notice the occurrence of the irrational $\theta_{2k-1,2k}$. For $j \leqq k-2$ we see in much the same way that $\psi^j$ will be of the form

$$rn^{j-1}\xi_1 + pn^{j-1}\xi_2 + qn^{j-1}\xi_3 + pqn^{j-2}\xi_4 + n^j\xi_5$$

where the $\xi_i$ are elements of $\wedge^j L$. Thus

$$\langle \exp(\theta), (1/j!)\psi^j \lrcorner v\rangle$$

will be a homogeneous polynomial in $r$, $p$, $q$ and $n$ of degree $j$ of form

$$rn^{j-1}s_1 + pn^{j-1}s_2 + qn^{j-1}s_3 + pqn^{j-2}s_4 + n^j s_5$$

where the $s_i$ are real numbers. Adding up these various expressions, we find that

$$\begin{aligned}&\langle \exp(\theta), (\exp \psi) \lrcorner v\rangle \\ &= rP_1(n) + pP_2(n) + qP_3(n) + pqP_4(n) + P_5(n)\end{aligned}$$

where the $P_i$ are polynomials of degree $k-2$ or less, with the coefficient of $n^{k-2}$ being irrational for $P_1$, while it is rational for $P_4$. We can rewrite this more specifically as

$$r(\alpha n^{k-2} + Q_1(n)) + p((\beta - q)n^{k-2} + Q_2(n)) + qP_3(n) + P_5(n)$$

where $Q_1$ and $Q_2$ are polynomials of degree $\leqq k-3$, $\alpha$ is irrational $(= \theta_{2k-1,2k})$, and $\beta$ is real. We can thus choose $q$ so that $\alpha$ and $\beta - q$ are independent over the rationals. For this choice set

$$Q_3(n) = qP_3(n) + P_5(n) \quad \text{and} \quad \gamma = \beta - q,$$

so that we obtain

$$r(\alpha n^{k-2} + Q_1(n)) + p(\gamma n^{k-2} + Q_2(n)) + Q_3(n).$$

We claim now that we can choose $n$ so that $\alpha n^{k-2} + Q_1(n)$ and $\gamma n^{k-2} + Q_2(n)$ are independent over the rationals. Dividing through by $\gamma$

(noting that $\alpha/\gamma$ is irrational), we see that to show this it suffices to prove the following result, which may well be known, but for which I have not found a reference.

6.9 SUBLEMMA. *Let $\alpha$ be a real (or complex) number, and let $P$ and $Q$ be monic polynomials. If $\alpha P(n)/Q(n)$ is rational for at least* (degree($P$) + degree($Q$) + 1) *distinct integers, then $\alpha$ is rational.*

*Proof.* The proof is by induction on (degree($P$) + degree($Q$)). The case for which this sum is 0 is clear. Let $S$ be a set of distinct integers for which $\alpha P(n)/Q(n)$ is rational.

Case 1. Suppose that degree($Q$) < degree($P$). Fix $m \in S$. Then for all $n \in S$

$$\alpha(P(n)/Q(n) - P(m)/Q(m))$$

is rational. We rewrite this as

$$\alpha(P(n) - (P(m)/Q(m))Q(n))/Q(n),$$

and note that the numerator is still monic in $n$ because degree($Q$) < degree($P$). Furthermore the numerator is 0 when $n = m$, and so we can factor out a term $n - m$ and rewrite our expression as

$$\alpha(n - m)R(n)/Q(n)$$

where $R$ is a polynomial, still monic, with degree($R$) = degree($P$) − 1. For $n \neq m$ we can divide by $n - m$ to find that $\alpha R(n)/Q(n)$ is rational for all $n \in S \backslash \{m\}$. By the induction hypothesis it follows that $\alpha$ is rational.

Case 2. Suppose that degree($P$) < degree($Q$) and that $P(m) = 0$ for some $m \in S$. Then we can factor as $\alpha(n - m)R(n)/Q(n)$ and conclude that $\alpha$ is rational as above.

Case 3. Suppose that degree($P$) < degree($Q$) and that $P(m) \neq 0$ for all $m \in S$. Then, assuming that $\alpha \neq 0$, we see that $\alpha^{-1}Q(n)/P(n)$ satisfies the hypotheses of Case 1, so that $\alpha^{-1}$ and hence $\alpha$ is rational.

Case 4. Suppose that degree($P$) = degree($Q$). If $P$ and $Q$ are equal then we are clearly done. If they are not equal, then in view of the size of $S$, there must exist an $m \in S$ such that $P(m) \neq Q(m)$. For this $m$ consider as above

$$\alpha(P(n) - (P(m)/Q(m))Q(n))/Q(n).$$

Now the coefficient of the term of highest degree in the numerator is $1 - P(m)/Q(m)$, since $P(m) \neq Q(m)$. We can factor out this term, as well as $n - m$, to obtain

$$\alpha(1 - P(m)/Q(m))(n - m)R(n)/Q(n),$$

where $R$ is monic and degree($R$) = degree($Q$) − 1. Then for any $n \in S \backslash \{m\}$,

$$\alpha(1 - P(m)/Q(m))R(n)/Q(n)$$

is rational. By the induction hypothesis we conclude that

$$\alpha(1 - P(m)/Q(m))$$

is rational. But by assumption $\alpha P(m)/Q(m)$ is rational, so that $\alpha$ is rational.

Returning to the proof of Case I of Lemma 6.8, we fix $n$ as claimed. It is then clear that we can choose $c$ and $p$ (recalling that $r = c + 1/a$) so that

$$0 < \langle \exp(\theta), (\exp \psi) \lrcorner\, v \rangle < \langle \exp(\theta), \mu \rangle,$$

as desired.

Case II. We suppose now that $\theta$ is rational on the linear span of $\bar{F}_1, \ldots, \bar{F}_{2k}$, but that $\theta_{ij}$ is irrational for some $i \leqq 2k$ and some $j \geqq 2k + 1$. For convenience we reorder $\bar{F}_1, \ldots, \bar{F}_{2k}$ so that $\theta_{1j}$ is irrational for a fixed $j \geqq 2k + 1$. Then we can rewrite $\mu_k$ as

$$\mu_k = a\epsilon[2\bar{F}_1 \wedge (\bar{F}_2 - \bar{F}_j) - \bar{F}_1 \wedge (\bar{F}_2 - 2\bar{F}_j)] \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k}.$$

For integers $v_0$ and $p$ yet to be chosen let

$$v = v_0 + p\bar{F}_1 \wedge \bar{F}_j + 2a\epsilon\bar{F}_1 \wedge (\bar{F}_2 - \bar{F}_j) \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k}.$$

Then

$$\langle \exp(\theta), v \rangle = v_0 + p\theta_{1j} + \text{constant},$$

so that we can choose $v_0$ and $p$ such that

$$0 < \langle \exp(\theta), v \rangle < \langle \exp(\theta), \mu \rangle.$$

Notice that $\bar{F}_1, (\bar{F}_2 - \bar{F}_j), \bar{F}_3, \ldots, \bar{F}_{2k}$ forms part of an integral basis for $L^*$, and that

$$\langle \theta, \bar{F}_1 \wedge (\bar{F}_2 - \bar{F}_j) \rangle = \theta_{12} - \theta_{1j}$$

is irrational, since $\theta_{12}$ is assumed rational while $\theta_{1j}$ is assumed irrational.

Let $v' = \mu - v$, so that $0 < \langle \exp(\theta), v' \rangle$ and

$$v'_k = -a\epsilon\bar{F}_1 \wedge (\bar{F}_2 - 2\bar{F}_j) \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k}.$$

Notice that $\bar{F}_1, (\bar{F}_2 - 2\bar{F}_j), \bar{F}_3, \ldots, \bar{F}_{2k}$ forms part of an integral basis for $L^*$, and that

$$\langle \theta, \bar{F}_1 \wedge (\bar{F}_2 - 2\bar{F}_j) \rangle = \theta_{12} - 2\theta_{1j}$$

is irrational.

Thus $v$ and $v'$ are both in exactly the form to which Case I applies, and so they can be expressed as a sum of terms of the desired form, including the multiplicity statement.

Case III. We suppose now that $\theta_{ij}$ is rational if either $i \leqq 2k$ or $j \leqq 2k$. Since $\theta$ is assumed not to be rational, we can find $i, j \geqq 2k + 1$ such that $\theta_{ij}$ is irrational. For integers $v_0$ and $p$ yet to be chosen let

$$v = v_0 + p\bar{F}_i \wedge \bar{F}_j + \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k} \wedge \bar{F}_i \wedge \bar{F}_j.$$

Then

$$\langle \exp(\theta), v \rangle = v_0 + p\theta_{ij} + \text{constant},$$

so that we can choose $v_0$ and $p$ such that

$$0 < \langle \exp(\theta), v \rangle < \langle \exp(\theta), \mu \rangle.$$

Notice that $v$ is in exactly the form to which Case I applies, and so can be expressed as a sum of terms of the desired form.

Let $v' = \mu - v$, so that $0 < \langle \exp(\theta), v' \rangle$ and

$$v'_k = (a\epsilon\bar{F}_1 \wedge \bar{F}_2 - \bar{F}_i \wedge \bar{F}_j) \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k},$$

which can be rewritten as

$$[(a\epsilon\bar{F}_1 - \bar{F}_i) \wedge (2\bar{F}_2 - \bar{F}_j) + (a\epsilon\bar{F}_1 - 2\bar{F}_i) \wedge (-\bar{F}_2 + \bar{F}_j)] \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k}.$$

For integers $\lambda_0$ and $q$ yet to be chosen let

$$\lambda = \lambda_0 + q\bar{F}_i \wedge \bar{F}_j + (a\epsilon\bar{F}_1 - \bar{F}_i) \wedge (2\bar{F}_2 - \bar{F}_j) \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k}.$$

Then

$$\langle \exp(\theta), \lambda \rangle = \lambda_0 + q\theta_{ij} + \text{constant},$$

so that we can choose $\lambda_0$ and $q$ such that

$$0 < \langle \exp(\theta), \lambda \rangle < \langle \exp(\theta), v' \rangle.$$

Notice that $(a\epsilon\bar{F}_1 - \bar{F}_i)$, $(2\bar{F}_2 - \bar{F}_j)$, $\bar{F}_3, \ldots, \bar{F}_{2k}$ form part of an integral basis for $L^*$, and that

$$\langle \theta, (a\epsilon\bar{F}_1 - \bar{F}_i) \wedge (2\bar{F}_2 - \bar{F}_j) \rangle = 2a\epsilon\theta_{12} - a\epsilon\theta_{1j} - 2\theta_{i2} + \theta_{ij}$$

is irrational, since $\theta_{12}$, $\theta_{1j}$ and $\theta_{i2}$ are assumed rational while $\theta_{ij}$ is assumed irrational.

Let $\lambda' = v' - \lambda$, so that $0 < \langle \exp(\theta), \lambda' \rangle$, and

$$\lambda'_k = (a\epsilon\bar{F}_1 - 2\bar{F}_i) \wedge (-\bar{F}_2 + \bar{F}_j) \wedge \bar{F}_3 \wedge \ldots \wedge \bar{F}_{2k}.$$

If $a$ is odd, then $a\epsilon$ and 2 are relatively prime, and so $(a\epsilon\bar{F}_1 - 2\bar{F}_i)$, $(-\bar{F}_2 + \bar{F}_j)$, $\bar{F}_3, \ldots, \bar{F}_{2k}$ form part of an integral basis for $L^*$. If $a$ is even, then after factoring out a 2 from $(a\epsilon\bar{F}_1 - 2\bar{F}_2)$, the terms again form part of an integral basis. Furthermore $\theta$ is not rational on the linear span of these terms, for reasons similar to those given above for $\lambda$.

Thus $\lambda$ and $\lambda'$ are both in exactly the form to which Case I applies, and so they can be expressed as a sum of terms of the desired form, including the multiplicity statement. Since $\mu = \upsilon + \lambda + \lambda'$, we are done.

The final stage of the induction step is given by:

6.10 LEMMA. *Assume that $\theta$ is not rational. Suppose that for some fixed $k \geqq 2$ it is known that the conclusions of Theorem 6.2 are true for all $\mu$ of height $k$ for which $\mu_k$ is decomposable. Then the conclusions of Theorem 6.2 are true for all $\mu$ of height $k$.*

*Proof.* Since $\theta$ is not rational, we can choose an oriented basis $\{\bar{F}_i\}$ for $D$ such that $\theta_{12}$ is not rational. Any $\mu_k \in \wedge^{2k}D$ can be expressed as a linear combination of the (decomposable) basis elements for $\wedge^{2k}D$ which come from the basis $\{\bar{F}_i\}$. In analogy with the rank which was used in the proof of Lemma 6.4, we define the length of a $\mu_k$ to be the number of non-zero terms in its expression as a linear combination of these basis elements. Our proof is by induction on the length, the case of length 1 being just the hypothesis of the lemma.

So suppose that for some integer $m \geqq 2$ it is known that the conclusion holds for all $\mu$ of height $k$ for which $\mu_k$ has length $\leqq m - 1$. Let $\mu$ be of height $k$ with $\mu_k$ of length $m$ such that $0 < \langle \exp(\theta), \mu\rangle$. Then we can express $\mu_k$ as $\mu_k = \upsilon_k + \upsilon'_k$ where $\upsilon_k$ has length 1 and $\upsilon'_k$ has length $m - 1$. For integers $\upsilon_0$ and $p$ yet to be chosen, set

$$\upsilon = \upsilon_0 + p\bar{F}_1 \wedge \bar{F}_2 + \upsilon_k.$$

Then

$$\langle \exp(\theta), \upsilon\rangle = \upsilon_0 + p\theta_{12} + \langle \theta^k/k!, \upsilon_k\rangle.$$

Since $\theta_{12}$ is irrational, we can choose $\upsilon_0$ and $p$ so that

$$0 < \langle \exp(\theta), \upsilon\rangle < \langle \exp(\theta), \mu\rangle.$$

Let $\upsilon' = \mu - \upsilon$, so that $0 < \langle \exp\theta, \upsilon'\rangle$ and $\upsilon'_k$ has length $m - 1$. Then by the induction hypotheses both $\upsilon$ and $\upsilon'$ have expressions as sums of the desired type, including the multiplicity statement, and so $\mu$ does also.

This concludes the induction step, and so concludes the proofs of Theorems 6.1 and 6.2.

We remark that Theorem 6.1 can fail for $\theta$ rational. In fact already for $T^4$ there are elements of $K_0(C(T^4))$ with positive trace which do not come from any complex vector bundle over $T^4$, as can be seen by examining the Chern characters of line bundles [**20**, **23**].

**7. The cancellation theorem.** We use the notation of Section 6. The goal of this section is to prove:

7.1 THEOREM. *Let $\theta \in \bigwedge^2 L$, and assume that $\theta$ is not rational. Then cancellation holds for projective modules over $A_\theta$, that is, if $U$, $V$ and $W$ are projective $A_\theta$-modules such that*

$$U \oplus W \cong V \oplus W,$$

*then $U \cong V$. Equivalently, any two projective $A_\theta$-modules which are stably isomorphic (i.e., represent the same element of $K_0(A_\theta)$) are in fact isomorphic.*

Since from Theorem 6.1 we know that every element in the positive cone of $K_0(A_\theta)$ is represented by a standard module, we immediately obtain:

7.2 COROLLARY. *If $\theta$ is not rational, then every projective $A_\theta$-module is isomorphic to a standard module.*

Thus we know how to construct all projective $A_\theta$-modules, up to isomorphism.

At the end of this section we will give some further interesting consequences of Theorem 7.1.

Our proof of Theorem 7.1 parallels the proof in [**49**] for the two-generator case. In view of the information which we have amassed in the previous sections, the main fact which we still need is a bound on the topological stable rank of the endomorphism algebras of standard modules, so that we can apply the results of [**57**]. To obtain this bound, we need a convenient description of the endomorphism algebras of standard modules of the form ${}^{\Xi}V$ where $V$ is elementary.

We use notation as in Definition 5.7. Thus let $\theta$ and $\sigma$ be as earlier, let $\psi$ be a rational element of $\bigwedge^2 L$, and let $\gamma$ be the corresponding cocycle for $\psi$ on $D = \mathbf{Z}^n$. Let $\Xi$ be the Hilbert space for an irreducible finite-dimensional right unitary $\overline{\gamma}$-representation of $D$, and let $V$ be an elementary $A_{\theta+\psi}$-module.

Since $\psi$ is rational, we can, according to Lemma 5 on page 71 of [**24**], "diagonalize" $\psi$ into $2 \times 2$ blocks, that is, we can find a basis

$$f_1, \dots, f_k, g_1, \dots, g_k, h_1, \dots, h_j$$

for $D$, where $2k + j = n$ (though there may be no $h_i$'s), such that

$$\psi(f_i, g_i) = p_i/q_i \quad \text{for each } i,$$

where $p_i$ and $q_i$ are relatively prime integers, while $\psi$ on all other pairs of these basis vectors is zero. For each $i$ let $F_i$ be the cyclic group of order $q_i$, let $\beta_i$ be the Heisenberg cocycle on $F_i \times \hat{F}_i$ and map $f_i$ and $g_i$ to elements in $F_i$ and $\hat{F}_i$ respectively such that, identifying $f_i$ and $g_i$ with their images, we have

$$\beta_i(f_i, g_i) = \overline{e}(p_i/q_i)$$

(while $\beta(g_i, f_i) = 1$). Let $F$ be the product of the $F_i$'s, so that $\hat{F}$ is the product of the $\hat{F}_i$'s. Map $D$ into $F \times \hat{F}$ by means of the maps on the $f_i$ and $g_i$ used above, and by sending the $h_i$ to the identity element. Let $\beta$ denote the Heisenberg cocycle on $F \times \hat{F}$, and let $\beta$ also denote its pull-back to $D$. Then the anti-symmetrization of $\beta$ on $D$ will coincide with that of $\bar{\gamma}$ (which is $\bar{\gamma}^2$ since $\gamma$ is already anti-symmetric), where to see this we must recall the factor of 2 used in defining $\gamma$ in terms of $\psi$. Thus $\beta$ and $\bar{\gamma}$ are cohomologous, so that $\bar{\gamma}$-representations of $D$ correspond to $\beta$-representations. Let $C$ be the subgroup of $D$ spanned by the $q_i f_i$, the $q_i g_i$ and the $h_i$, so that $C$ is exactly the kernel of the map of $D$ onto $F \times \hat{F}$, and $\beta$ is trivial on $C$. Under any irreducible $\beta$-representation of $D$ the elements of $C$ will be carried to scalar multiples of the identity operator, thus defining a character, $\chi$, on $C$, which can be extended (not uniquely) to a character, also $\chi$, on $D$. Then the inner tensor product of the given representation with $\bar{\chi}$ will be a $\beta$-representation of $D$ which is trivial on $C$, and so is the pull-back of a $\beta$-representation of $F \times \hat{F}$. But up to isomorphism $F \times \hat{F}$ has only one irreducible $\beta$-representation, namely the Heisenberg representation on $L^2(F)$. Thus we have shown that any irreducible $\beta$-representation of $D$ is just the inner tensor product of the Heisenberg representation on $L^2(F)$ with some character $\chi$ of $D$.

Let $V$ be an elementary $A_{\theta+\psi}$-module. For the present purposes it is most convenient to go back to the original construction summarized in Proposition 3.2, rather than the definition given in Notation 4.2. Thus $V$ comes from an embedding of $D$ as a lattice in $N \times \hat{N}$, where $N = \mathbf{R}^p \times \mathbf{Z}^q$ with $2p + q = n$, and where the pull-back to $D$ of the Heisenberg cocycle on $N \times \hat{N}$, say $\delta$, is cohomologous to $\sigma\gamma$. Thus, taking into account the isomorphisms of $A_\delta$ with $A_{\theta+\psi}$ and of $A_{\delta\beta}$ with $A_\theta$, we see that we can assume that $V$ is as just described above, and that the finite-dimensional $\beta$-representation is of form $L^2(F) \otimes \chi$, where we abuse notation by letting $\chi$ denote also the (one-dimensional) space of the representation $\chi$. That is, standard modules for which $\Xi$ is irreducible are obtained by "inducing" $V$ of the above form by means of the bimodule determined by $L^2(F) \otimes \chi$ in the way described in Proposition 5.1. Now at the level of dense subspaces this "induced" module is just $S(N) \otimes L^2(F) \otimes \chi$, as discussed shortly before Proposition 5.4, and from this point of view it is evident that the effect of $\chi$ is the same as that of composing the action of $S_{\delta\beta}$ (that is, $S(D)$) on $S(N) \otimes L^2(F)$, with the automorphism of $S_{\delta\beta}$ corresponding to $\chi$ under the dual action. This then remains true for the completions. Now $\hat{D}$ $(= T^n)$ is path-connected, so that the automorphism of $A_{\delta\beta}$ corresponding to a $\chi \in \hat{D}$ is connected by a path to the identity automorphism. It follows that if $V$ is any projective $A_{\delta\beta}$-module, and if $V^\chi$ denotes the module obtained by composing the action of $A_{\delta\beta}$ with the automorphism from $\chi$, then $V^\chi$ is isomorphic to $V$. (This is most easily seen at the level of projections.) It follows that for our present purposes we can ignore the dual action of $\hat{D}$ on modules, and assume that $\chi \equiv 1$.

Since $F$ is finite, it is clear that

$$S(N) \otimes L^2(F) \cong S(N \times F),$$

and that the action of $S_{\delta\beta}$ corresponds to the Heisenberg action for $M = N \times F$, since $\delta$ and $\beta$ come from the Heisenberg cocycles for $N$ and $F$ respectively. If we keep in mind the passage from bicharacters to anti-symmetric bicharacters which we have used since Section 4, the above discussion can be summarized by:

7.3 THEOREM. *Let $\theta$ be any element of $\wedge^2 L$. Then every standard $A_\theta$-module is, up to isomorphism, the direct sum of modules obtained* (*as in* Section 3) *from embeddings of $D$ as a lattice in groups $M \times \hat{M}$, where $M$ is of the form $M = \mathbf{R}^p \times \mathbf{Z}^q \times F$, for $2p + q = n$ and for $F$ some finite Abelian group.*

We are now exactly in a position to apply Proposition 3.2, with the roles of $D$ and $D^\perp$ interchanged. We find that the endomorphism algebra of a module coming from an embedding of $D$ as a lattice in a group $M \times \hat{M}$ is $C^*(D^\perp, \beta)$. Now $D^\perp$ is itself a lattice in $M \times \hat{M}$ according to Lemma 3.1, and so must be of form $\mathbf{Z}^n \times F_0$ for some finite Abelian group $F_0$, for the reasons discussed after Proposition 3.6. But then $C^*(D^\perp, \bar{\beta})$ will have topological stable rank no larger than $n + 1$, according to Proposition 3.9. Thus the endomorphism algebra of each of the summands described in Theorem 7.3 will have topological stable rank no larger than $n + 1$.

What remains then is to see how topological stable rank of endomorphism algebras behaves under taking direct sums of the corresponding modules. Now according to [**21**], for $C^*$-algebras the topological stable rank is the same as the Bass stable rank. But it follows from Theorem 1.9 of [**57**] that the Bass stable rank of the endomorphism ring of a direct sum of modules is no larger than the maximum of the Bass stable ranks of the endomorphism rings of the summands. Thus the same must be true for the topological stable rank. It seems desirable to have a direct proof of this fact, not passing through the Bass stable rank, and which works for general Banach algebras:

7.4 PROPOSITION. *Let $A$ be a Banach algebra with identity element, and let $V_1$ and $V_2$ be projective $A$-modules. If*

$$\operatorname{tsr}(\operatorname{End}_A(V_j)) \leqq n \quad \textit{for } j = 1, 2,$$

*then*

$$\operatorname{tsr}(\operatorname{End}_A(V_1 \oplus V_2)) \leqq n.$$

*Proof.* Let $T = (T_1, \dots, T_n)$ be an element of

$$(\mathrm{End}_A(V_1 \oplus V_2))^n,$$

and let $N$ be a neighborhood of $T$. We must show that $N$ contains an element of

$$Lg_n(\mathrm{End}_A(V_1 \oplus V_2)),$$

in the notation of [**48**]. Now each $T_i$ can be written as a $2 \times 2$ matrix $\{t^i_{jk}\}$ where

$$t^i_{jk} \in \mathrm{Hom}_A(V_k, V_j).$$

By the hypotheses on $V_1$ we can perturb $\{t^i_{11}\}$ slightly so that the new $T$ is still in $N$ while the new $\{t^i_{11}\}$ is in $Lg_n(\mathrm{End}_A(V_1))$. Thus there is $\{s^i_{11}\}$ in $(\mathrm{End}_A(V_1))^n$ such that

$$\sum s^i_{11} t^i_{11} = I_{V_1}.$$

Then for each $k$ we have

$$t^k_{21} = \sum (t^k_{21} s^i_{11}) t^i_{11},$$

and so we can perform "elementary row operations" to make all the $t^k_{21}$ equal to 0. That is, we can find an invertible $n \times n$ matrix, $E$, with entries in $\mathrm{End}_A(V_1 \oplus V_2)$, such that, if $R = ET$ where $T$ is viewed as a column vector so that $R$ is also, and if $R = \{R_i\}$ and $R_i = \{r^i_{jk}\}$ with

$$r^i_{jk} \in \mathrm{Hom}_A(V_k, V_j),$$

then we have $r^i_{21} = 0$ for all $i$, and $r^i_{11} = t^i_{11}$ (the new ones). Now $EN$ is a neighborhood of $ET = R$. So by the hypothesis on $V_2$ we can perturb $\{r^i_{22}\}$ slightly so that the new $R$ is still in $EN$ while the new $\{r^i_{22}\}$ is in $Lg_n(\mathrm{End}_A(V_2))$. We can then again perform "elementary row operations" to make the $r^i_{12}$ equal to 0. That is, we can find an invertible $n \times n$ matrix, $F$, with entries in $\mathrm{End}_A(V_1 \oplus V_2)$, such that each $2 \times 2$ block of $P = FR$ is diagonal with entries $t^i_{11}$ and $r^i_{22}$ (the new ones). But $P$ is then clearly in

$$Lg_n(\mathrm{End}_A(V_1 \oplus V_2)),$$

so that $E^{-1}F^{-1}P$ is also. But $E^{-1}F^{-1}P$ is clearly in $N$, so we are done.

Combining this result with the previous discussion, we obtain:

7.5 THEOREM. *Let $\theta$ be any element of $\wedge^2 L$. Then for any standard $A_\theta$-module $V$ one has*

$$\mathrm{tsr}(\mathrm{End}_{A_\theta}(V)) \leqq n + 1.$$

We now return to the proof of Theorem 7.1. We argue as in the proof of Theorem 2.2 of [**49**]. For any projective module $Y$ we denote by $[Y]$ its class in $K_0$. Let $U$ and $V$ be as in the statement of Theorem 7.1, so that

$[U] = [V]$. Because of the hypothesis on $\theta$, Theorem 6.1 is applicable, so we can represent $[V]$ by a standard module, which can be assumed to have as a direct summand $n + 1$ copies of a (non-zero) standard module $Y$. That is, we can find a projective module $Z$ such that

$$[Z \oplus Y^{n+1}] = [V].$$

What we will actually show is that $Z \oplus Y^{n+1} \cong V$. Since $V$ is an arbitrary representative of $[V]$, we will be done.

Since $Z \oplus Y^{n+1}$ and $V$ are stably isomorphic, there is an integer $m$ such that, as modules,

$$Z \oplus Y^{n+1} \oplus A_\theta^m \cong V \oplus A_\theta^m.$$

We need:

7.6 LEMMA. *Let $\theta$ be any element of $\wedge^2 L$. Then any non-zero projective $A_\theta$-module is a generator for the category of projective $A_\theta$-modules.*

Before proving the lemma, we show how to use it to complete the proof of Theorem 7.1. From the lemma we know that $Y$ is a generator, so that $A_\theta^m$ is a summand of $Y^k$ for some integer $k$. By adding the complementary module to the last equation above, we obtain

$$Z \oplus Y^{n+1+k} \cong V \oplus Y^k.$$

Now because $Y$ is a standard module, we know from Theorem 7.5 that

$$\mathrm{tsr}(\mathrm{End}_{A_\theta}(Y)) \leqq n + 1,$$

so that from Theorem 2.3 of [**48**] we have

$$\mathrm{Bsr}(\mathrm{End}_{A_\theta}(Y)) \leqq n + 1,$$

where Bsr denotes the Bass stable rank. We are thus exactly in a position to apply the cancellation theorems of Warfield, Theorems 1.2 and 1.6 of [**57**] (which are also restated as Theorem 2.1 of [**49**]; see also Proposition 1 of [**30**] ), to conclude that

$$Z \oplus Y^{n+1} \cong V.$$

This concludes the proof of Theorem 7.1, except for:

*Proof of Lemma* 7.6. This is most easily carried out by using the description of projective modules in terms of projections. For any $m$, view $A_\theta^m$ as a right $A_\theta$-module with $A_\theta$-valued inner-product, and view the endomorphism algebra of $A_\theta^m$ as being the algebra $M_m(A_\theta)$ of $m \times m$ matrices, acting on $A_\theta^m$ on the left. Then for some $m$ there is a projection, $e$, in $M_m(A_\theta)$ such that $V$ is isomorphic to the module $e(A_\theta^m)$. We note that this identification equips $V$ with an $A_\theta$-valued inner-product. As we will see shortly, the crux of the matter is to show that the span of the range of this inner-product, which is an ideal in $A_\theta$, is in fact all of $A_\theta$.

Let $\alpha$ denote not only the dual action of $T^n$ on $A_\theta$, but also its extension to $B = M_m(A_\theta)$. Fix a primitive ideal $P$ of $B$. Then the function on $T^n$ defined by

$$t \mapsto \alpha_{-t}(e) \mapsto \|\alpha_{-t}(e) + P\|_{B/P} = \|e + \alpha_t(P)\|_{B/\alpha_t(P)}$$

for $t \in T^n$, is clearly continuous. But the image of $e$ in each $B/\alpha_t(P)$ is a projection, and so its norm there is either 0 or 1. Since $T^n$ is connected, it follows that the image of $e$ in each $B/\alpha_t(P)$ is either always zero or never zero. But $\alpha$ gives a transitive action of $T^n$ on $\mathrm{Prim}(A_\theta)$ by Proposition 34 of [**18**], and so also on $\mathrm{Prim}(B)$, because the primitive ideals of $B$ correspond to those of $A_\theta$ in an evident way. Thus for any primitive ideal $Q$ of $B$ the image of $e$ in $B/Q$ is not zero, since $e \neq 0$. At the level of $A_\theta$, this means that for any primitive ideal $P$ of $A_\theta$ not all the entries of the matrix $e$ are in $P$. But these entries are contained in the range of the $A_\theta$-valued inner-product of $V$, so this range is not contained in $P$ for any primitive ideal $P$ of $A_\theta$. Consequently, the span of the range of the inner-product, which is an ideal of $A_\theta$, must be dense in $A_\theta$. But $A_\theta$ has an identity element, and so this span must be all of $A_\theta$.

In particular, there must be elements $x_1, \dots, x_m$ and $y_1, \dots, y_m$ of $V$ such that

$$\sum \langle x_i, y_i \rangle = 1_{A_\theta}.$$

But then the homomorphism of $A_\theta$ into $V^m$ defined by $a \mapsto (y_i a)$ has as left inverse the homomorphism

$$(v_i) \mapsto \sum \langle x_i, v_i \rangle_{A_\theta},$$

and so exhibits $A_\theta$ as a direct summand of $V^m$, as desired.

From Theorems 7.1 and 7.5 we obtain:

7.7 COROLLARY. *Let $\theta \in \wedge^2 L$ and assume that $\theta$ is not rational. Then for every projective $A_\theta$-module $V$ we have*

$$\mathrm{tsr}(\mathrm{End}_{A_\theta}(V)) \leqq n + 1.$$

Actually, continuing the comments made at the end of Section 3, I can show that the upper bound of $n + 1$ in this corollary can be replaced by 2.

From Theorem 7.1 we immediately obtain a generalization of Corollary 2.5 of [**49**] (see also Proposition 4.5.1 of [**4**]):

7.8 COROLLARY. *Let $\theta \in \wedge^2 L$, and assume that $\theta$ is not rational. If $p$ and $q$ are projections in some $M_n(A_\theta)$ which represent the same element of $K_0(A_\theta)$, then they are unitarily equivalent in $M_n(A_\theta)$.*

*Proof.* Since $p$ and $q$ represent the same element of $K_0(A_\theta)$, the modules $p(A_\theta)^n$ and $q(A_\theta)^n$ are stably isomorphic. But then by Theorem 7.1 they are isomorphic, and there is a partial isometry, $v$, in $M_n(A_\theta)$ with $vv^* = p$ and $v^*v = q$. But the same will apply to $1 - p$ and $1 - q$. Putting together the partial isometries for these two cases gives the desired unitary equivalence.

In the next section we will obtain an even stronger result (Theorem 8.13).

We remark that Corollary 2.5 of **[49]** was phrased only in terms of the trace on $A_\theta$, and not in terms of $K_0(A_\theta)$. It is clear that a similar rephrasing of Corollary 7.8 can be given in those cases for which the trace is faithful on $K_0(A_\theta)$. Now the condition that the trace be faithful is equivalent, by Theorem 3.1 of **[17]**, to the condition that the functional

$$\mu \mapsto \langle \exp(\theta), \mu \rangle$$

from $\wedge^e D$ to $R$ be injective. (See the discussion after the statement of Theorem 6.1.) This injectivity implies that $\theta$ is not rational even on any two-dimensional rational subspace, for if $F_1$ and $F_2$ are elements of some basis for $D$ such that

$$\langle \theta, \bar{F}_1 \wedge' \bar{F}_2 \rangle = k/m$$

for integers $k$ and $m$, then

$$\langle \exp(\theta), k - m\bar{F}_1 \wedge \bar{F}_2 \rangle = 0,$$

so that the functional is not injective. Thus injectivity of the trace implies that the hypotheses of Theorem 7.1 hold (as long as $D \neq \mathbf{Z}$), and also that $A_\theta$ is simple. Thus we obtain:

7.9 Corollary. *Suppose that $\mu \mapsto \langle \exp(\theta), \mu \rangle$ is injective from $\Lambda^e D$ to $\mathbf{R}$ (and $D \neq \mathbf{Z}$). If $p$ and $q$ are projections in some $M_n(A_\theta)$ which have the same trace, then they are unitarily equivalent in $M_n(A_\theta)$.*

As another interesting consequence of the cancellation theorem and of our characterization of the positive cone we have:

7.10 Corollary. *Assume that $\theta$ is not rational, and let $\tau$ denote the canonical normalized trace on $A_\theta$, viewed also as a homomorphism from $K_0(A_\theta)$. Let $\eta \in K_0(A_\theta)$ and suppose that $0 < \tau(\eta) < 1$. Then there is a projection, $p$, in $A_\theta$ (not just in some $M_k(A_\theta)$) such that $\eta = [p]$. Furthermore, the projections in $A_\theta$ generate all of $K_0(A_\theta)$.*

*Proof.* Let $\xi = [A_\theta] - \eta$, so that $0 < \tau(\xi) < 1$. By Theorem 6.1 there are projective $A_\theta$-modules $V$ and $W$ representing $\eta$ and $\xi$ respectively. Then $[V \oplus W] = [A_\theta]$, so that by cancellation, $V \oplus W \cong A_\theta$ as right $A_\theta$-modules. (Actually, we only need here that stably free modules are free.) For any such isomorphism the projection of $A_\theta$ on the image of $V$ will be given by a $p \in A_\theta$ with the desired property. This argument works with $[A_\theta]$ replaced by any other positive element, $\zeta$, of $K_0(A_\theta)$ for which $\tau(\eta) < \tau(\zeta)$. It follows that the projections in $A_\theta$ generate all of $K_0(A_\theta)$.

However, under the conditions that the trace, $\tau$, is faithful on $K_0(A_\theta)$, we have a stronger result:

7.11 THEOREM. *Suppose that $\mu \mapsto \langle \exp(\theta), \mu \rangle$ is injective from $\Lambda^e D$ to $\mathbf{R}$. Then we can find a totally ordered family, $S$, of projections in $A_\theta$ such that*

$$\tau(S) = (\tau(K_0(A_\theta))) \cap (0, 1].$$

*In particular, every projection in $A_\theta$ will be unitarily equivalent to some element of $S$.*

*Proof.* Since $A_\theta$ is separable, $(\tau(K_0(A_\theta))) \cap (0, 1]$ is countable. Let $\{t_k : 1 \leqq k < \infty\}$ be an enumeration of its elements, with $t_1 = 1$. We will construct a sequence $\{p_k\}$ of projections such that $\tau(p_k) = t_k$ and whenever $t_k < t_j$ then $p_k < p_j$. The construction is by induction on $k$, and we start, of course, by setting $p_1 = 1$. Suppose that $p_1, \ldots, p_{k-1}$ have been chosen. Let $t_a$ and $t_b$ be, respectively, the largest and the smallest of $t_1, \ldots, t_{k-1}$ such that $t_a < t_k$ and $t_k < t_b$. Then $p_a < p_b$, and every other of the $p_i$ is either smaller than $p_a$ or larger than $p_b$. By Corollary 7.10 we can find projections $q$ and $r$ in $A_\theta$ such that

$$\tau(q) = t_k - t_a \quad \text{and} \quad \tau(r) = t_b - t_k.$$

Then the right $A_\theta$-module $qA_\theta \oplus rA_\theta$ will represent an element of $K_0(A_\theta)$ whose trace is $t_b - t_a$, as will $(p_b - p_a)A_\theta$. Since the trace is assumed to be faithful on $K_0(A_\theta)$, these two modules are stably isomorphic. Since cancellation holds, these modules are actually isomorphic. This means that there must be a subprojection, $p'$, of $p_b - p_a$ such that $\tau(p') = t_k - t_a$. We let $p_k = p_a + p'$.

We remark that the $C^*$-subalgebra of $A_\theta$ generated by $\{p_k\}$ will be a commutative subalgebra (with Cantor spectrum) such that its injection into $A_\theta$ gives an isomorphism at the level of $K_0$. In Section 9/9 of **[25]** Kumjian showed that when $D = \mathbf{Z}^2$ one can actually embed a simple $AF$-algebra into $A_\theta$ giving an isomorphism at the level of $K_0$. It would be interesting to know if this result generalizes to the higher dimensional case.

A referee has pointed out that the above results quickly give the following answer to a question which Norbert Riedel asked me.

7.12 PROPOSITION. *Assume that $\theta$ is not rational. Then there exists in $A_\theta$ a totally ordered family, $S'$, of projections which cannot be enlarged to a totally ordered family, $S$, of projections in $A_\theta$ such that every projection in $A_\theta$ is unitarily equivalent to some element of $S$.*

*Proof.* Choose a proper projection, $p$, in $A_\theta$, and let $r = \tau(p)$, so that $0 < r < 1$. Since $\theta$ is irrational, we can find a subalgebra of $A_\theta$ to which Theorem 7.11 applies. From this we see that we can find in $A_\theta$ an infinite sequence $\{q_j\}$ of non-zero orthogonal projections such that

$$\sum \tau(q_j) > r.$$

Let $C$ denote the set of subsequences $\{q_{j_n}\}$ such that

$$\sum \tau(q_{j_n}) = r.$$

A Cantor diagonal argument using the fact that the sequence $(\tau(q_j))$ converges to 0 shows that $C$ is uncountable. Let $N$ denote the von Neumann algebra generated by $A_\theta$ on $L^2(A_\theta, \tau)$. Then each element of $C$ has a sum in $N$, the sum being a projection which may or may not be in $A_\theta$. The distance between sums for different elements of $C$ will be 2. Since $C$ is uncountable and $A_\theta$ is separable, some elements of $C$ must have sum not in $A_\theta$. Let $\{p_j\}$ be the sequence of partial sums for such an element of $C$. Then $\{p_j\}$ is a totally ordered sequence of projections in $A_\theta$, whose supremum is a projection, $e$, in $N$ which is not in $A_\theta$. Furthermore it is clear that $\tau(e) = r$. It follows that $\{p_j\}$ cannot be enlarged to a totally ordered family of projections of $A_\theta$ containing a projection of trace $r$.

**8. Consequences for $K_1$-groups.** For a unital $C^*$-algebra $A$ we let $U_m(A)$ denote the group of unitary elements of $M_m(A)$, and we let $U_m^0(A)$ denote the connected component of the identity element of $U_m(A)$, so that, by definition, $K_1(A)$ is the direct limit of the groups $U_m(A)/U_m^0(A)$ for the usual inclusions. The purpose of this section is to show that when $\theta$ is not rational, the natural maps of $U_m(A_\theta)/U_m^0(A_\theta)$ into $K_1(A_\theta)$ are isomorphisms. We then obtain an interesting consequence for projections.

We begin with the surjectivity:

8.1 THEOREM. *Let $\theta \in \bigwedge^2 L$, and assume that $\theta$ is not rational. Then for every integer $m \geqq 1$ the natural map from $U_m(A_\theta)$ to $K_1(A_\theta)$ is surjective.*

It is clear that to prove this it suffices to treat the case $m = 1$. One way to approach this would be to try to apply Theorem 10.10 of [**48**], which would require showing that $\operatorname{csr}(A_\theta) \leqq 2$ (as defined there), but I do not know how to do this (except in the two-generator case; see Corollary 8.6 of [**48**]). Instead, we will give a proof by induction on $n$ for $D = \mathbf{Z}^n$. The induction step will be based on:

8.2 PROPOSITION. *Let $A$ be a unital $C^*$-algebra, and let $\alpha$ be an automorphism of $A$ which is in the connected component of the identity automorphism. Let $\alpha$ also denote the corresponding action of $\mathbf{Z}$ on $A$. Assume that*

(1) *Every element of $K_1(A)$ is represented by an element of $U_1(A)$.*
(2) *The projections in $A$ generate $K_0(A)$.*

*Then every element of* $K_1(A \times_\alpha \mathbf{Z})$ *is represented by an element of* $U_1(A \times_\alpha \mathbf{Z})$.

*Proof.* For brevity, write $U(A)$ instead of $U_1(A)$, and similarly for $U(A \times_\alpha \mathbf{Z})$. Note that the hypothesis on $\alpha$ implies that $\alpha$ acts as the identity automorphism on both $K_0(A)$ and $K_1(A)$. Then the exact sequence of Pimsner and Voiculescu [37] for the Toeplitz extension gives the following commutative diagram with exact second row:

$$\begin{tikzcd}
 & U(A) \arrow[r, "i"] \arrow[d] & U(A \times_\alpha \mathbf{Z}) \arrow[d] & & \\
0 \arrow[r] & K_1(A) \arrow[r, "i"] & K_1(A \times_\alpha \mathbf{Z}) \arrow[r, "\mathrm{ind}"] & K_0(A) \arrow[r] & 0
\end{tikzcd}$$

where $i$ comes from the inclusion of $A$ in $A \times_\alpha \mathbf{Z}$, and "ind" is the index map for the Toeplitz extension. The assertion of the proposition is that the second vertical map is surjective. Since the first hypothesis of the proposition is that the first vertical map is surjective, it clearly suffices to show that every element of $K_0(A)$ is the index of some element of $U(A \times_\alpha \mathbf{Z})$. But from the second hypothesis we see that it suffices to show just that every projection, $p$, in $A$ is the index of some element of $U(A \times_\alpha \mathbf{Z})$. Now since $\alpha$ is path-connected to the identity automorphism, it follows that $\alpha(p)$ is path-connected to $p$ through projections, and so there is a partial isometry, $v$, such that $vv^* = p$ and $v^*v = \alpha(p)$. Let $t$ denote the unitary in $A \times_\alpha \mathbf{Z}$ corresponding to $1 \in \mathbf{Z}$. Motivated by Lemma 1.2 of [37], we set

$$u = pvt^{-1}p + (1 - p).$$

Then one checks immediately that $u$ is a unitary in $A \times_\alpha \mathbf{Z}$. Furthermore, it is easily seen, as indicated at the top of page 102 of [37], that the index of $u$ is $p$.

We remark that the analogous statement for $K_0(A \times_\alpha \mathbf{Z})$ is not true, that is, under the hypotheses of the above theorem we cannot conclude that the projections in $A \times_\alpha \mathbf{Z}$ generate $K_0(A \times_\alpha \mathbf{Z})$. For example, let $A = C(T)$ and let $\alpha$ be the trivial action, so that

$$A \times_\alpha \mathbf{Z} \cong C(T^2).$$

*Proof of Theorem* 8.1. We argue by induction on $n$, where $D = \mathbf{Z}^n$. For $n = 2$ the conclusion of the theorem was obtained by Pimsner and Voiculescu in Corollary 2.5 of [37] (or we could even start with $n = 1$ and $A = C(T)$ ). Thus we need to show that for any $n \geqq 3$ the conclusion of the theorem holds if it is known to hold for $n - 1$. Now by Proposition 3.10

$$A_\theta \cong B_\theta \times_\alpha \mathbf{Z}$$

where $B_\theta = C^*(D', \theta)$ and $D'$ is a summand of $D$ of rank $n - 1$. That $\alpha$ is in the connected component of the identity automorphism is easily seen by bringing appropriate coefficients of $\theta$ to integers. We can always arrange that the restriction of $\theta$ to $D'$ not be rational. Then by the induction hypothesis we know that $U(B_\theta)$ maps onto $K_1(B_\theta)$. This means that the first hypothesis of Proposition 8.2 holds. But the second hypothesis also holds by Corollary 7.10. Thus Proposition 8.2 applies to show that $U(A_\theta)$ maps onto $K_1(A_\theta)$.

We remark that Theorem 8.1 can fail if $\theta$ is permitted to be rational, since for $A = C(T^3)$ we have

$$UA/U^0A \cong H^1(T^3, \mathbf{Z}) \cong \mathbf{Z}^3$$

(see 11.7 of **[55]**), while $K_1(A) \cong \mathbf{Z}^4$.

8.3 THEOREM. *Let $\theta \in \bigwedge^2 L$, and assume that $\theta$ is not rational. Then for every integer $m \geqq 1$ the natural map from $U_m(A_\theta)/U_m^0(A_\theta)$ to $K_1(A_\theta)$ is injective, and so is an isomorphism.*

We will show that this theorem is in a sense a corollary of the cancellation theorem. To do this, we will, for any $C^*$-algebra $A$, let $TA$ denote the $C^*$-algebra of continuous functions from the circle, $T$, into $A$. Our proof of Theorem 8.3 is based on:

8.4 THEOREM. *Let $A$ be a unital $C^*$-algebra. Then $TA$ has cancellation (that is, the cancellation law holds for projective $TA$-modules) if and only if*

(1) *$A$ has cancellation and*

(2) *For every projective $A$-module $V$ the natural map from* $\mathrm{Aut}_A(V)/\mathrm{Aut}_A^0(V)$ *into $K_1(A)$ is injective.*

*Proof.* We will assume first that $TA$ has cancellation, and show that this implies property (2). It is this implication which will provide our proof of Theorem 8.3. We use the following familiar "clutching" construction. View $TA$ as consisting of the continuous functions from the interval $[0, 1]$ to $A$ which agree at the endpoints. Let $V$ be any projective $A$-module, and let $u \in \mathrm{Aut}_A(V)$. (If desired, we can always assume that $V$ has been equipped with an $A$-valued inner-product, and that $u$ is unitary for this inner-product.) We define a $TA$-module, $X(u)$, by

$$X(u) = \{\phi:[0, 1] \to V, \text{ continuous, with } \phi(1) = u\phi(0)\},$$

with the evident right action of $TA$ by pointwise multiplication. We state the elementary properties of this construction as a series of lemmas.

8.5 LEMMA. *If* $u_0$ *and* $u_1$ *are path-connected in* $\mathrm{Aut}_A(V)$, *then* $X(u_0) \cong X(u_1)$.

*Proof.* Let $\{u_t\}$ be a path in $\mathrm{Aut}_A(V)$ connecting $u_0$ to $u_1$. It is easily checked that the map which sends $\phi \in X(u_0)$ to $\tilde{\phi} \in X(u_1)$ defined by

$$\tilde{\phi}(t) = u_t u_0^{-1}\phi(t)$$

is an isomorphism.

8.6 LEMMA. *If* $V_1$ *and* $V_2$ *are projective A-modules, and if* $u_1 \in \mathrm{Aut}_A(V_1)$ *and* $u_2 \in \mathrm{Aut}_A(V_2)$, *then*

$$X(u_1 \oplus u_2) \cong X(u_1) \oplus X(u_2).$$

This is evident.

8.7 LEMMA. *If* $V = A^n$ *and if* $i_V$ *denotes the identity automorphism of* $V$, *then* $X(i_V)$ *is isomorphic to the free module* $(TA)^n$.

This is evident.

8.8 LEMMA. *For any* $V$ *and* $u$ *as earlier,* $X(u)$ *is (finitely generated) projective.*

*Proof.* Choose an $A$-module $W$ such that $V \oplus W \cong A^n$ for some $n$. It is well known **[55]** that $(u \oplus i_W) \oplus (u \oplus i_W)^{-1}$ is path-connected to the identity automorphism of $A^{2n}$. By the above three lemmas this exhibits $X(u)$ as a direct summand of the free $TA$-modules $(TA)^{2n}$.

8.9 LEMMA. *If* $u_1, u_2 \in \mathrm{Aut}_A(V)$ *and if* $u_1$ *and* $u_2$ *are in the same class in* $K_1(A)$, *then* $X(u_1)$ *and* $X(u_2)$ *are stably isomorphic.*

*Proof.* That $u_1$ and $u_2$ are in the same class in $K_1(A)$ means, by definition, that there is an $A$-module $W$ such that $V \oplus W$ is free and that $u_1 \oplus i_W$ and $u_2 \oplus i_W$ are path-connected in $\mathrm{Aut}_A(V \oplus W)$. This means, by the above lemmas, that

$$X(u_1) \oplus X(i_W) \cong X(u_2) \oplus X(i_W),$$

which says exactly that $X(u_1)$ and $X(u_2)$ are stably isomorphic.

8.10 LEMMA. *If* $u_1, u_2 \in \mathrm{Aut}_A(V)$ *and if* $X(u_1) \cong X(u_2)$, *then there is a* $w \in \mathrm{Aut}_A(V)$ *such that* $u_2$ *is path-connected to* $wu_1w^{-1}$ *in* $\mathrm{Aut}_A(V)$.

*Proof.* It is clear that $X(u_1)$ and $X(u_2)$ consist of the sections of locally trivial bundles over $T$ with fiber $V$. It follows that any isomorphism from $X(u_1)$ to $X(u_2)$ must be given by left multiplication by a continuous function, say $g$, from $[0, 1]$ into $\mathrm{Aut}_A(V)$. For every $\phi \in X(u_1)$ we must have

$$u_2 g(0)\phi(0) = g(1)\phi(1) = g(1)u_1\phi(0),$$

from which it follows that $u_2 g(0) = g(1)u_1$. Let $w = g(0)$. Then

$$u_2 = g(1)u_1 w^{-1},$$

which is path-connected, via $g$, to $wu_1w^{-1}$.

We now return to the proof of Theorem 8.4, and show that if $TA$ has cancellation then property (2) holds. Let $V$ be any projective $A$-module, let $u \in \mathrm{Aut}_A(V)$, and suppose that the class of $u$ in $K_1(A)$ is 1. We must show that $u \in \mathrm{Aut}_A^0(V)$. Now the fact that the class of $u$ in $K_1(A)$ is 1 means, by Lemma 8.9, that $X(u)$ and $X(i_V)$ are stably isomorphic. But $TA$ is assumed to have cancellation, so $X(u)$ and $X(i_V)$ are isomorphic. But then by Lemma 8.10 there is a $w \in \mathrm{Aut}_A(V)$ such that $u$ is path-connected to $wi_Vw^{-1} = i_V$, as desired.

We now consider property (1). To each $V$ we can associate $X(i_V)$, and it is easily seen from this that if $TA$ has cancellation then $A$ must also. We have thus proven one direction of Theorem 8.4.

To prove the converse direction of Theorem 8.4 we need:

8.11 LEMMA. *Every projective $TA$-module is isomorphic to some $X(u)$ for some projective $A$-module $V$ and some $u \in \mathrm{Aut}_A(V)$.*

*Proof.* Any projective $TA$-module is of the form $P(TA)^n$ for some $n$, where $P$ is a projection in $M_n(TA)$. But a projection in $M_n(TA)$ is the same as a continuous function (still denoted by $P$) from $[0, 1]$ into projections in $M_n(A)$ which agrees at the endpoints. Let $p = P(0)$, and let $V = pA^n$. Now $P$ is a continuous path of projections, and so, as is well-known, one can find a continuous path, $U$, of invertible elements of $M_n(A)$ such that

$$P(t) = U(t)^{-1}P(0)U(t)$$

for all $t$. In particular,

$$U(1)^{-1}P(0)U(1) = P(1) = P(0),$$

so that $U(1)$ commutes with $P(0) = p$. Let $u = pU(1)$, so that $u \in \mathrm{Aut}_A(V)$. We claim that $P(TA)^n$ is isomorphic to $X(u)$. For if $f \in P(TA)^n$, viewed as a function from $T$ to $A^n$, we can define $\phi$ by

$$\phi(t) = U(t)f(t) = P(0)U(t)f(t).$$

Then

$$\phi(1) = P(0)U(1)f(1) = u\phi(0),$$

so that $\phi \in X(u)$. It is easily seen that this correspondence of $\phi$ to $f$ gives an isomorphism.

Thus to show, for Theorem 8.4, that $TA$ has cancellation, it suffices to consider modules of the form $X(u)$. So let $U$, $V$ and $Y$ be projective

$A$-modules, let $u \in \mathrm{Aut}_A(U)$, $v \in \mathrm{Aut}_A(V)$ and $y \in \mathrm{Aut}_A(Y)$, and suppose that

$$X(u) \oplus X(y) \cong X(v) \oplus X(y).$$

From Lemma 8.6 it follows that

$$X(u \oplus y) \cong X(v \oplus y).$$

Considering this at any given point of $t$, we see that $U \oplus Y \cong V \oplus Y$. But by hypothesis 1 we are assuming that $A$ has cancellation, so that $U \cong V$. Let $r$ be a specific isomorphism, and let $u' = r^{-1}vr$, so $u' \in \mathrm{Aut}_A(U)$, and

$$X(u \oplus y) \cong X(u' \oplus y),$$

where now both $u \oplus y$ and $u' \oplus y$ are in $\mathrm{Aut}_A(U \oplus Y)$. From Lemma 8.10 it follows that there is a $w \in \mathrm{Aut}_A(U \oplus Y)$ such that $u' \oplus y$ is path-connected to $w(u \oplus y)w^{-1}$. It follows that $u$ and $u'$ are in the same class in $K_1(A)$. But by hypothesis 2 it then follows that $u$ and $u'$ are path-connected in $\mathrm{Aut}_A(V)$. Then $X(u) \cong X(u')$ by Lemma 8.5. Thus the proof of Theorem 8.4 will be complete once we have:

8.12 LEMMA. *Let $U$ and $V$ be projective $A$-modules, let $r$ be an isomorphism from $U$ to $V$, and let $v \in \mathrm{Aut}_A(V)$. Then*

$$X(r^{-1}vr) \cong X(v).$$

*Proof.* It is easily verified that the map $R$ from $X(v)$ to $X(r^{-1}vr)$ defined by

$$R(\phi)(t) = r\phi(t)$$

is an isomorphism.

It is not clear to me whether Theorem 8.4 remains true if hypothesis 2 is weakened to considering only free modules, nor whether there is a generalization to the case in which $TA$ is replaced by a crossed product with the integers.

*Proof of Theorem* 8.3. We can express $TA_\theta$ as the crossed product $A_\theta \times Z$ for the trivial action of $Z$, and from this it is clear that $TA_\theta$ is again a non-commutative torus, whose "$\theta$" will not be rational since the original $\theta$ is not. Thus $TA_\theta$ has cancellation by Theorem 7.1, and so we can apply Theorem 8.4 to obtain injectivity.

We remark that Theorem 8.3 can fail if $\theta$ is permitted to be rational. To be specific, it is known that if $A = C(T^4)$, then there is a unitary in $U_2(A)$ which is not in the connected component of the identity element, and yet whose image in $K_1(A)$ is 1. (I am indebted to Steven Hurder for explaining this to me.) Notice also that this latter fact, together with Theorem 8.4, immediately shows that cancellation fails for $C(T^5)$, since $C(T^5) \cong TC(T^4)$.

We will now see how Theorem 8.3 can be used to obtain further information about projections in non-commutative tori. If $p$ and $q$ are projections in a unital $C^*$-algebra $A$ which are in the same class in $K_0(A)$, then $1 - p$ and $1 - q$ also are in the same class. If $A$ has cancellation, then it easily follows from this that there is a unitary $u$ in $A$ such that $q = upu^*$. (See the proof of Theorem 7.8 above, or Corollary 2.5 of [**49**], or Proposition 4.5.1 of [**4**].) It is natural to ask whether $u$ can, in fact, be chosen in the connected component of the identity element, so that $p$ and $q$ are in the same path-component of the set of projections in $A$. (My interest in this particular matter was heightened by questions which M.-D. Choi asked me.) For non-commutative tori we have:

8.13 THEOREM. *Let $\theta \in \bigwedge^2 L$, and assume that $\theta$ is not rational. For any $m \geqq 1$ let $p$ and $q$ be projections in $M_m(A_\theta)$ which are in the same class in $K_0(A_\theta)$. Then there is a unitary $u$ in $U_m^0(A_\theta)$ such that $q = upu^*$, so that $p$ and $q$ are in the same path-component of the set of projections in $M_m(A_\theta)$. In particular, the elements of the positive cone of $K_0(A_\theta)$ which are no bigger than $[A_\theta]$ (the class of the free module of rank one), exactly label the path-components of the set of projections in $A_\theta$.*

*Proof.* Assume given $p$, $q$ and $u$ with $q = upu^*$. We first need to show that we can replace $u$ by a unitary whose class in $K_1(A_\theta)$ is 1, so that we can then invoke Theorem 8.3. Clearly we can assume that $p \neq 1$, so that $\bar{p} = 1 - p$ is not zero. By Theorems 6.1 and 7.1 the module $\bar{p}(A_\theta)^m$ is a standard module. Then by the last part of Theorem 6.1 there is a projection $e \leqq \bar{p}$ such that $e$ is equivalent to a sum of $n + 2$ mutually equivalent projections $\{f_i\}$, where $D = \mathbf{Z}^n$. By Theorems 6.1 and 7.5,

$$\operatorname{tsr}(\operatorname{End}_{A_\theta}(f_1(A_\theta^m))) \leqq n + 1.$$

Let $B = M_m(A_\theta)$. Then by Theorem 10.10 of [**48**], the image of

$$\operatorname{Aut}_{A_\theta}((f_1(A_\theta^m))^{n+2})$$

in $K_1(f_1Bf_1)$ is all of $K_1(f_1Bf_1)$. It follows that the image of

$$\operatorname{Aut}_{A_\theta}(eA_\theta^m)$$

in $K_1(eBe)$ is all of $K_1(eBe)$. Now $eA_\theta^m$ will be a generator for the category of projective $A_\theta$-modules by Lemma 7.6 and so $eBe$ will be Morita equivalent to $A_\theta$. Thus the inclusion of $eBe$ into $M_m(A_\theta)$ induces an isomorphism of $K$-groups. In particular, every element of $K_1(A_\theta)$ will be represented by a unitary coming from $eBe$. Thus, for the $u$ given at the beginning of the proof, we can find a unitary $v \in eBe$ such that $[\tilde{v}] = [u^*]$ where $\tilde{v} = v \oplus (1 - e)$. But $\tilde{v}p\tilde{v}^* = p$ since $e \leqq 1 - p$. Thus

$$q = (u\tilde{v})p(u\tilde{v})^*,$$

and $[u\tilde{v}]$ is $[1]$ in $K_1(A_\theta)$, as desired.

But now we can apply Theorem 8.3 to conclude that $\tilde{v}u$ is in $U_m^0(A_\theta)$.

We remark that Theorem 8.13 can fail if $\theta$ is permitted to be rational. This was pointed out to me by Terry Loring, who showed me how to construct two projections, $p$ and $q$, in $M_2(C(T^3))$, such that there is a $u$ in $U_2(C(T^3))$ with $upu^* = q$, but there is no such $u$ in $U_2^0(C(T^3))$.

*Added in proof.* For further consequences of the results of this paper, see the author's paper: *The homotopy groups of the unitary groups of non-commutative tori*, J. Operator Theory *17* (1987), 237-254.

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