Math 250B

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1 (4 pts.) State Hilbert's Theorem 90 for finite cyclic Galois extensions.

2 (6 pts.) Let K be a finite extension of $\mathbb{Q}$ for which $[K\colon \mathbb{Q}]$ is odd. Show that among the field embeddings $\sigma\colon K \to \mathbb{C}$, there is at least one which maps K into $\mathbb{R}$. If $K/\mathbb{Q}$ is an odd degree Galois extension, show that *all* σ map K into $\mathbb{R}$.

3 (6 pts.) In the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, prove that $1 \otimes i + i \otimes 1$ is not of the form $x \otimes y$.

4 (6 pts.) Let S be an entire ring (integral domain), and let R be a subring of S. Suppose that all elements of S are integral over R. Show by direct argument that S is a field if and only if R is a field.

5 (10 pts.) Let

$$0 \to A \to B \to C \to 0 \qquad (*)$$

be an exact sequence of R-modules, and let M be an R-module.

(3 pts.) a. Show by example that the induced sequence

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0 \qquad (**)$$

need not be exact.

(2 pts.) b. Show that $(**)$ is exact if $(*)$ is split.

(3 pts.) c. Show that $(**)$ is exact if M is projective.

(2 pts.) d. Show that $(**)$ is exact if $R = \mathbb{Z}$ and $M = \mathbb{Q}$.

6 (5 pts.) Let K be a finite Galois extension of $\mathbb{Q}$, and let B be the integral closure of $\mathbb{Z}$ in K. Let p be a prime number. Show that the number of prime ideals $\mathfrak{p} \subseteq B$ which lie over $p\mathbb{Z}$ is a divisor of $[K\colon \mathbb{Q}]$.