Math 250B

Professor K. A. Ribet

Final Exam

This exam is due on Friday, May 14, at 10:10 AM. Papers will be collected in 885 Evans. A stapler will be available for your use at that time. *Please work on these problems on your own.* Please leave ample space for the graders' comments.

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1. Let I be the set of positive integers with its usual ordering. For each $i \in I$, let $X_i = \mathbb{Z}$. For $j \geq i$, let $f_i^j : X_j \to X_i$ be multiplication by 2^{j-i}. Compute: (a) $\varprojlim X_i$, (b) $(\varprojlim X_i) \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$, (c) $\varprojlim (X_i \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z})$.

2. Let A be a principal ideal domain. For which $a \in A$ is the A-module A/aA semisimple?

3. Prove that the Jacobson radical of a semisimple ring is 0.

4. (a) Show that the center of a semisimple ring contains no non-zero nilpotent element. (b) Let $R = k[G]$ where G is a finite group and k is a field. Assume that the characteristic of k is a prime number p and that p divides the order of G. Prove that R is *not* semisimple. [It may help to consider $(\sum_{\sigma \in G} \sigma) \in R$.]

5. Let V and W be finite dimensional k-vector spaces. Let $V^* = \mathrm{Hom}(V, k)$ be the linear dual of V. Given $(\phi, w) \in V^* \times W$, we may define a linear map $V \to W$ by $v \mapsto \phi(v) \cdot w$. Show that this association defines a bilinear map $V^* \times W \to \mathrm{Hom}(V, W)$, and that the induced homomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ is an isomorphism of k-vector spaces.

6. Let L/K be a finite Galois extension, and let $G = \mathrm{Gal}(L/K)$. Let W be a finite-dimensional vector space over K, and let $V = W \otimes_K L$ be the associated L-vector space. Note that the map $w \mapsto w \otimes 1$ identifies W with a subset of V. For each $g \in G$, show that there is a bijection $\lambda_g : V \xrightarrow{\sim} V$ which satisfies $\lambda_g(w \otimes a) = w \otimes ga$ and which is such that $\lambda_g(cv) = g(c)\lambda_g(v)$ for $v \in V$ and $c \in L$. (We say that λ_g is "g-linear.") Prove that

$$W = \{\, v \in V \mid \lambda_g(v) = v \text{ for all } g \in G \,\}. \tag{1}$$

7. (continuation) Suppose now that V is a 1-dimensional L-vector space and that we are given a g-linear bijection $\lambda_g\colon V \xrightarrow{\sim} V$ for each $g \in G$. Assume moreover that the λ_g are compatible in the sense that $\lambda_{\sigma\tau} = \lambda_\sigma \circ \lambda_\tau$ for $\sigma, \tau \in G$. Show that the set W defined by (1) is a K-vector space and that the inclusion $W \subseteq V$ induces an isomorphism $W \otimes_K L \xrightarrow{\sim} V$. If you are ambitious, try to do the more general case where V has dimension $n \geq 1$.

8. Let $\mathbb{F}$ be a finite field with q elements. Let N_ν be the number of monic polynomials of degree ν in $\mathbb{F}[X]$ and let C_ν be the number of *irreducible* monic polynomials of degree ν in $\mathbb{F}[X]$. Compute N_ν, and show that

$$N_\nu = \sum_{d|\nu} dC_d.$$

Explain how this equation may be used to compute the C_ν. Illustrate by computing $C_1, \ldots, C_5$ when $q = 2$.

9. The discriminant D_f of a monic polynomial $f(X) = \prod_{i=1}^{n}(X - \alpha_i)$ is defined to be $\prod_{i<j}(\alpha_i - \alpha_j)^2$. Show that $D_f = (-1)^{n(n-1)/2} \prod_i f'(\alpha_i)$. If q is an odd prime number and $f(X) = X^q - 1$, conclude that $D_f = (-1)^{(q-1)/2} q^q$. Suppose now that k is a field of characteristic prime to q, and let K be the splitting field of $X^q - 1$ over k. Regard the Galois group of K/k as a subgroup of $\mathbf{S}_q$, the group of permutations of the set of roots of $X^q - 1$. Show that $\mathrm{Gal}(K/k) \subseteq \mathbf{A}_q$ if and only if $(-1)^{(q-1)/2}q$ is a square in k.

10. (continuation) Let k be the finite field $\mathbb{F}_p$, where p is an odd prime distinct from q. Using the Galois theory of finite fields, show that $\mathrm{Gal}(K/k) \subseteq \mathbf{A}_q$ if and only if p is a square in $\mathbb{F}_q$. [Think about the action of the Frobenius automorphism of K on the roots of $f(X)$.] Conclude that p is a square in $\mathbb{F}_q$ if and only if $(-1)^{(q-1)/2}q$ is a square in $\mathbb{F}_p$.