Solutions to Problem Set 9

6.7.1

(a)

$$\|x\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{4(1)(1) + 5(1)(1)} = 3$$

$$\|y\| = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \sqrt{4(5)(5) + 5(-1)(-1)} = \sqrt{105}$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 = |4(1)(5) + 5(1)(-1)|^2 = 15^2 = 225$$

(b) (z_1, z_2) is orthogonal to $\mathbf{y}$ iff $0 = \langle (z_1, z_2), \mathbf{y} \rangle = 4(z_1)(5) + 5(z_2)(-1)$. So the vectors orthogonal to $\mathbf{y}$ coincide with the line $20z_1 - 5z_2 = 0$.

6.7.8

$$\langle q, p \rangle = (3{+}2(-1)^2)(3(-1){-}(-1)^2){+}(3{+}2(0)^2)(3(0){-}0^2){+}(3{+}2(1)^2)(3(1){-}1^2) = -10$$

$$\langle q, p \rangle = (3 + 2(-1)^2)^2 + (3 + 2(0)^2)^2 + (3 + 2(1)^2)^2 = 59$$

Thus

$$proj_p(q) = \frac{\langle q, p \rangle}{\langle p, p \rangle} p = -\frac{10}{59}(3t - t^2).$$

6.7.11 If $W = Span\{1, t, t^2 - 2\}$ then $proj_W(t^3) = a_0 + a_1 t + a_2(t^2 - 2)$ where

$$a_0 = \frac{\langle t^3, 1 \rangle}{\langle 1, 1 \rangle},$$

$$a_1 = \frac{\langle t^3, t \rangle}{\langle t, t \rangle},$$

$$a_2 = \frac{\langle t^3, t^2 - 2 \rangle}{\langle t^2 - 2, t^2 - 2 \rangle},$$

Now

$$\langle t^3, 1 \rangle = (-2)^3 + (-1)^3 + 0 + 1^3 + 2^3 = 0$$

and

$$\langle t^3, t^2 - 2 \rangle = (-2)^3((-2)^2 - 2) + (-1)^3((-1)^2 - 2 + 0 + 1^3(1^2 - 2) + 2^3(2^2 - 2) = 0.$$

So $a_0 = a_2 = 0$. Also

$$\langle t^3, t \rangle = (-2)^3(-2) + (-1)^3(-1) + 0 + 1^3 + 2^3(2) = 34$$

$$\langle t, t \rangle = (-2)(-2) + (-1)(-1) + 0 + 1 + 2(2) = 10.$$

So $proj_W(t^3) = \frac{17}{5} t$.

6.7.16 We see that

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}+(-\mathbf{v})\|^2 = \|\mathbf{u}\|^2 + \|-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + |-1|^2\|\mathbf{v}\|^2 = 2.$$

The second equality follows from the Pythagorian Theorem and the fact that $\langle \mathbf{u}, -\mathbf{v}\rangle = -\langle \mathbf{u}, \mathbf{v}\rangle = 0$. The last equality follows from the fact that $\mathbf{u}$ and $\mathbf{v}$ have length 1. Finally, we can take positive square roots of both sides (since lengths are non-negative) to prove the claim.

6.7.17 It suffices to show that

$$4\langle \mathbf{u}, \mathbf{v}\rangle = \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2.$$

But by the axioms of an inner product we see that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle,$$

and similarly

$$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle.$$

Subtracting the second from the first we see that the claim is true.

7.1.12 As mentioned on pg. 331, any square matrix with orthonormal columns is orthogonal. It is straightforward to check that the columns of this matrix each have unit length and that they are pairwise orthogonal. Thus this is an orthogonal matrix and by definition its inverse is equal to its transpose.

7.1.24 We first check the eigenvalues for $\mathbf{v}_1$ and $\mathbf{v}_2$. We see that

$$A\mathbf{v}_1 = (-20, 20, 10) = 10\mathbf{v}_1$$

$$A\mathbf{v}_2 = (1, 1, 0) = \mathbf{v}_2.$$

We now find bases for the corresponding eigenspaces:

$$A - 10I = \begin{bmatrix} -5 & -4 & -2 \\ -4 & -5 & 2 \\ -2 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis for the $\lambda = 10$ eigenspace is $\{(-2, 2, 1)\}$.

$$A - I = \begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So a basis for the $\lambda = 1$ eigenspace is $\{(1, 1, 0), (.5, 0, 1)\}$. Also, since A is symmetric, part (b) of the spectral theorem (pg 392) tells us that there are no other eigenvalues.

We now orthogonalize the $\lambda = 1$ basis which we label by $\mathbf{x}_1 = (1,1,0), \mathbf{x}_2 = (.5,0,1)$. We let $\mathbf{v}_1 = \mathbf{x}_1$, and let

$$\mathbf{v}_2 = \mathbf{x}_2 - proj_{\mathbf{v}_1}(\mathbf{x}_2) = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (.5,0,1) - \frac{1}{4}(1,1,0) = (.25,-.25,1).$$

We let $\mathbf{w}_1 = (-2,2,1), \mathbf{w}_2 = (1,1,0)$, and $\mathbf{w}_3 = (.25,-.25,1)$, and normalize these.

$$\|\mathbf{w}_1\| = \sqrt{4+4+1} = 3;$$
$$\|\mathbf{w}_2\| = \sqrt{1+1+0} = \sqrt{2};$$
$$\|\mathbf{w}_3\| = \sqrt{\frac{1}{16} + \sqrt{116} + 1} = \frac{3\sqrt{2}}{4}.$$

So

$$\|\hat{\mathbf{w}}_1\| = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$
$$\|\hat{\mathbf{w}}_2\| = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$
$$\|\hat{\mathbf{w}}_3\| = \left(\frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right).$$

We conclude that $A = PDP^{-1}$ is an orthogonal diagonalization where

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}.$$

7.1.26

(a) True. By Theorem 2 and the fact that every symmetric matrix is square.

(b) True. Since $P^T = P^{-1}$ we have both that $B = PDP^T = PDP^{-1}$ and that P is an orthogonal matrix. So, by definition, B is orthogonally diagonalizable (and also square, being a product of square matrices). Thus Theorem 2 tells us B must be symmetric.

(c) False. Recall the matrix from Exercise 7.1.12 was orthogonal, but it is clearly not symmetric and thus by Theorem 2 not orthogonally diagonalizable.

(d) True. Theorem 3(b).

7.1.30 Since A and B are both orthogonally diagonalizable (and thus square), thus they are both symmetric and thus $A^T = A$ and $B^T = B$. Thus $(AB)^T = B^T A^T = BA = AB$ and so AB is symmetric. Thus AB is orthogonally diagonalizable by Theorem 2.