Math 54 Homework 8 Solution

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Question 6.1.6

$x \cdot w = 5$ and $x \cdot x = 49$, so we see that

$$\left(\frac{x \cdot w}{x \cdot x} x\right) = \frac{5}{49} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$$

Question 6.1.20

1. True. Dot product is symmetric.
2. False. It should be $|c| \|v\|$.
3. True. This is the definition of $W^{\perp}$.
4. True. This is Pythagorean Theorem (Theorem 2 on p.320).
5. True. This is Theorem 3.

Question 6.1.24

$$\|u + v\|^2 = (u + v) \cdot (u + v) = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

while

$$\|u - v\|^2 = (u - v) \cdot (u - v) = \|u\|^2 + \|v\|^2 - 2u \cdot v$$

Thus

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

Question 6.1.29

Given any vector $w \in W$, we write it as $w = a_1 v_1 + ... + a_p v_p$, then

$$x \cdot w = x \cdot (a_1 v_1 + ... + a_p v_p) = a_1(x \cdot v_1) + ... + a_p(x \cdot v_p) = 0$$

Thus x is orthogonal to w. Since w is arbitrary, we conclude that x is orthogonal to every vector in W.

Question 6.2.10

$$u_1 \cdot u_2 = 6 - 6 + 0 = 0, u_1 \cdot u_3 = 3 - 3 + 0 = 0, u_2 \cdot u_3 = 2 + 2 - 4 = 0$$

so we see that u_1, u_2, and u_3 form an orthogonal basis for $\mathbb{R}^3$. Now

$$\frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{4}{3}, \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{1}{3}, \frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{1}{3}$$

so we see that $x = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$.

Question 6.2.14

$$\hat{y} = \frac{y \cdot u}{u \cdot u}u = \frac{2}{5}u = \frac{2}{5}\begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

and

$$y - \hat{y} = \frac{1}{5}\begin{pmatrix} -4 \\ 28 \end{pmatrix}$$

so

$$y = \frac{2}{5}\begin{pmatrix} 7 \\ 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -4 \\ 28 \end{pmatrix}$$

Question 6.2.21

Since $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$ and $\|u_1\| = \|u_2\| = \|u_3\| = 1$, we see that these three vectors form an orthonormal set.

Question 6.2.24

1. True. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are orthogonal but not linearly independent.
2. False. It is not orthonormal because the length of each vector has to also be one.
3. True. Theorem 7.
4. True. $\frac{y \cdot cv}{cv \cdot cv}cv = \frac{c(y \cdot v)}{c^2(v \cdot v)}cv = \frac{y \cdot v}{v \cdot v}v$.
5. True. $U^{-1} = U^T$.

Question 6.3.2

$$\frac{v \cdot u_1}{u_1 \cdot u_1}u_1 = \frac{14}{7}u_1 = 2\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

and

$$v - \frac{v \cdot u_1}{u_1 \cdot u_1}u_1 = \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}$$

so we write

$$v = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}$$

Question 6.3.10

First we check that $u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = 0$ so they form an orthogonal set. Then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3 = \frac{1}{3} u_1 + \frac{14}{3} u_2 + \frac{-5}{3} u_3 = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

and substracting we have

$$y - \hat{y} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

Thus

$$y = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

Question 6.3.12

Since v_1, v_2 are orthogonal, we only need to compute $\hat{y}$, which is

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = 3v_1 + v_2 = \begin{pmatrix} -1 \\ -5 \\ -3 \\ 9 \end{pmatrix}$$

Question 6.3.22

1. True. If $v \in W \cap W^{\perp}$, then $v \cdot v = \|v\|^2 = 0$, so $v = 0$.
2. True. This is the Orthogonal Decomposition Theorem.
3. True. In Theorem 8, it stated that the decomposition is unique.
4. False. It should be $proj_W y$.
5. False. $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has orthogonal columns, but $UU^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not identity map on $\mathbb{R}^3$.

Question 6.4.10

Let x_1, x_2, x_3 be the columns.

$$\begin{aligned} v_1 &= x_1 \\ v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = x_2 - (-3) v_1 \\ v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = x_3 - \frac{1}{2} v_1 - \frac{5}{2} v_2 \end{aligned}$$

so the orthogonal basis is $\left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix} \right\}$.

Question 6.4.18

1. False. The orthogonal set might have the zero vector in it.
2. True. If $x - proj_W x = 0$, then $x = proj_W x \in W$.
3. True. This is Theorem 12.