Solutions to Problem Set 5

4.3.8 Letting A be the coefficient matrix with columns the vectors given. Row reducing, we see that

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & -8 & -4 \end{bmatrix}.$$

There is a pivot in every row in the reduced matrix, so since being *onto* is preserved by row equivalence, the columns of A span $\mathbb{R}^3$. There is also a free variable in the reduced matrix, and so, since the property of having linearly independent columns is preserved under row equivalence, we see that the columns of A must be linearly dependent. Thus the set is not a basis, but it does span $\mathbb{R}^3$.

4.3.10 Letting

$$A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix},$$

$Nul(A)$ consists of all solutions to the augmented matrix $[A \mid \mathbf{0}]$. Also

$$A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

Thus,

$$Nul(A) = \left\{ \mathbf{x} \in \mathbb{R}^5 : \mathbf{x} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}, x_3, x_5 \in \mathbb{R} \right\}.$$

It is now easy to check that $\begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ together give a basis for $Nul(A)$.

4.3.14 $Nul(A)$ is equal to the solution set to $[A \mid \mathbf{0}]$ which is equal to the solution set of $[B \mid \mathbf{0}]$ which is equal to that of

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So a vector parametrization of $Nul(A)$ is

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix},$$

and this gives a basis as well.

We verify from B that the pivot columns of A are the first, third, and fifth columns, and we conclude the following is a basis for $Col(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}.$$

4.3.16 The pivot columns of

$$A = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are the first, second, and third, thus a the following is a desired basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

4.3.22

(a) False. The linearly independent set S in question might not span H. For example let $H = \mathbb{R}^2$, and $S = \{\langle 1, 0 \rangle\}$.

(b) True. There are two cases to consider.

If S is nonempty, then it contains a nonzero vector $\mathbf{v}$. So $V \neq \{\mathbf{0}\}$, and we can apply part (b) of the Spanning Set Theorem.

If S is the empty set, then we know that $V = \{\mathbf{0}\}$, and S is a basis for V.

(c) True. A basis $\mathscr{B}$ is linearly independent by definition. Furthermore, since $\mathscr{B}$ spans H, no subset $\mathscr{C}$ of H containing all the elements of $\mathscr{B}$ (other than $\mathscr{B}$ itself) can be linearly independent and thus cannot be a basis. To summarize, if a subset $\mathscr{B}$ of H is a basis for H then for it is linearly independent and it cannot be increased in size and still be a basis. (In fact we will later show that something stronger is true: All bases have the same size.)

(d) False. The method of section 4.2, since it row reduces the matrix A all the way to *reduced* echelon form, gives a vector parametrization of $Nul(A)$ with vectors that are linearly independent.

(e) False. Consider the row reduction of $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ to $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

4.3.32 If the given set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ were linearly dependent, then there exist weights $w_1, \ldots w_p$ not all zero such that $w_1 T(\mathbf{v}_1) +, \ldots, + w_p T(\mathbf{v}_p)\} = 0$. Since T is a linear transformation, we then have $T(w_1\mathbf{v}_1 +, \ldots, + w_p\mathbf{v}_p)) = \mathbf{0}$, and since T is one-to-one it follows that $w_1\mathbf{v}_1 +, \ldots, + w_p\mathbf{v}_p = \mathbf{0}$ so that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent as well.

4.4.4 Letting $P_{\mathscr{B}}$ be the change of coordinates matrix for the basis $\mathscr{B}$, we have

$$\mathbf{x} = P_{\mathscr{B}}[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & -7 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$$

4.4.8 From $P_{\mathscr{B}}[\mathbf{x}]_{\mathscr{B}} = \mathbf{x}$ we know that

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix} [\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

So row reduce

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{bmatrix}$$

to get $[\mathbf{x}]_{\mathscr{B}} = \langle -2, 0, 5 \rangle$.

4.4.14 We wish to find $[p(t)]_{\mathscr{B}} = \langle w_1, w_2, w_3 \rangle$ such that

$$w_1(1 - t^2) + w_2(t - t^2) + w_3(2 - 2t + t^2) = 3 + t - 6t^2.$$

We can proceed either by equating terms of like degree and solving for $\langle w_1, w_2, w_3 \rangle$, or we can use the fact that the map $(x, y, z) \mapsto x + yt + zt^2$ is an isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$ for which reason it suffices to find a solution $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ to

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}.$$

Both approaches lead to the solution $\langle 7, -3, -2 \rangle$.

4.4.16

(a) True. Since in this case $P_{\mathscr{B}} = I_n$.

(b) False. The map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is the coordinate mapping.

(c) True. Take any plane P in $\mathbb{R}^3$ that passes through the origin. Let $\mathbf{v}$ and $\mathbf{w}$ be any two linearly independent vectors in P. Then these vectors form a basis for P and if we let $\mathscr{B} = \{\mathbf{v}, \mathbf{w}\}$ then the coordinate map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathscr{B}}$ is an isomorphism between P and $\mathbb{R}^2$.

4.4.28 Let $\mathscr{B} = \{1, t, t^2, t^3\}$ be the standard basis for $\mathbb{P}_3$, so that the coordinate mapping $w + xt + yt^2 + zt^3 \mapsto \langle w, x, y, z \rangle$ is an isomorphism between $\mathbb{P}_3$ and $\mathbb{R}^4$. Then it suffices to check whether the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix} \right\}$$

is linearly independent. Row reducing the corresponding coefficient matrix reveals a free variable, and we conclude that the set S, and therefore the given set of polynomials is linearly dependent.