Math 54 Homework 4 Solution

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Question 3.2.8

$$\det\begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ -3 & -7 & -5 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & -10 \end{pmatrix} = 0$$

Question 3.2.12

$$\begin{aligned}\det\begin{pmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{pmatrix} &= (-6)\cdot\det\begin{pmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 4 & 2 & 4 \end{pmatrix} + 3\cdot\det\begin{pmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 5 & 4 & 6 \end{pmatrix} \\ &= (-6)\cdot\det\begin{pmatrix} -1 & 2 & 3 \\ 0 & 10 & 12 \\ 0 & 10 & 16 \end{pmatrix} + 3\cdot\det\begin{pmatrix} -1 & 2 & 3 \\ 0 & 10 & 12 \\ 0 & 14 & 21 \end{pmatrix} \\ &= (-6)(-1)(40) + 3(-1)(42) = 114\end{aligned}$$

Question 3.2.20

$$\det\begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix} = \det\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right) = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 7$$

Question 3.2.21

$$\det\begin{pmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix} = (-4)\det\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} + (1)\det\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = -4 + 3 = -1 \neq 0$$

Since the determinant is not 0, the original matrix is invertible.

Question 3.2.26

We compute the determinant of the matrix A where the columns of A are the vectors given:

$$\det\begin{pmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{pmatrix} = (-3)\det\begin{pmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{pmatrix} = (-3)[(-6)(-14) + 3(-28)] = 0$$

Since the determinant is zero, the vectors are linearly dependent.

Question 3.2.28

1. True. $\det(B) = (-1)(-1)\det(A) = \det(A)$.
2. False. $\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0 \neq 1$
3. False. $\det\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 0$, but no two rows or columns are the same, and none of them are zero.
4. False. $\det(A^T) = \det(A)$

Question 3.2.32

Suppose A is $n \times n$, then we apply the row operation that multiplies a row by the constant r to each row separately, which gives us $det(rA) = (r^n)det(A)$.

Question 3.2.33

$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

Question 3.2.34

$\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(P^{-1})\det(A) = \det(PP^{-1})\det(A) = \det(A)$

Question 3.2.35

Since $1 = \det(I) = \det(U^TU) = \det(U^T)\det(U) = \det(U)\det(U) = (\det(U))^2$, we see that $\det(U)$ satisfy the equation $x^2 = 1$, and hence $\det(U) = \pm 1$.

Question 3.3.6

We rewrite the system of equations as a matrix equation:

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{pmatrix} \mathbf{x} = \mathbf{b}$$

Since the determinant of the matrix is 4, the matrix is invertible, and so we can apply Cramer's rule. This gives

$$x_1 = \frac{1}{4}\det\begin{pmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{pmatrix} = \frac{-16}{4} = -4$$

$$x_2 = \frac{1}{4}\det\begin{pmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{pmatrix} = \frac{52}{4} = 13$$

$$x_3 = \frac{1}{4}\det\begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix} = \frac{-4}{4} = -1$$

Question 3.3.15

$$\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{pmatrix}$$

Therefore, theorem 8 tells us that

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{pmatrix}$$

Question 3.3.20

$$\left| \det \begin{pmatrix} -1 & 4 \\ 3 & -5 \end{pmatrix} \right| = |-7| = 7$$

So the area of the parallelogram is 7.

Question 3.3.30

Substracting (x_3, y_3) from all three points (this translates the point (x_3, y_3) to the origin), we have $(x_1 - x_3, y_1 - y_3), (x_2 - x_3, y_2 - y_3), (0, 0)$. Then

$$\frac{1}{2}\det \begin{pmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{pmatrix} = \frac{1}{2}\det \begin{pmatrix} x_1 & x_2 \\ y_1 - y_3 & y_2 - y_3 \end{pmatrix} + \frac{1}{2}\det \begin{pmatrix} -x_3 & -x_3 \\ y_1 - y_3 & y_2 - y_3 \end{pmatrix}$$
$$= \frac{1}{2}\det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} + \frac{1}{2}\det \begin{pmatrix} x_1 & x_2 \\ -y_3 & -y_3 \end{pmatrix} + \frac{1}{2}\det \begin{pmatrix} -x_3 & -x_3 \\ y_1 & y_2 \end{pmatrix} + \frac{1}{2}\det \begin{pmatrix} -x_3 & -x_3 \\ -y_3 & -y_3 \end{pmatrix}$$
$$= \frac{1}{2}\left(\det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} + (-y_3)\det \begin{pmatrix} x_1 & x_2 \\ 1 & 1 \end{pmatrix} + (-x_3)\det \begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix}\right)$$
$$= \frac{1}{2}\left(\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} + (-y_3)\det \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} + (x_3)\det \begin{pmatrix} y_1 & 1 \\ y_2 & 1 \end{pmatrix}\right) = \frac{1}{2}\det \begin{pmatrix} x_1 & x_2 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

and the result follows.

Question 4.1.2

1. Yes. If $xy \geq 0$, then $(cx)(cy) = c^2 xy \geq 0$.
2. The vectors $(0, 1)$ and $(-1, 0)$ are in W, but $(0, 1) + (-1, 0) = (-1, 1)$ is not in W.

Question 4.1.5

Yes. The set is nonempty because $0 \cdot t^2 = 0$ is in this set. For any $a_1, a_2, r \in \mathbb{R}$, we have $a_1 t^2 - r \cdot a_2 t^2 = (a_1 - r \cdot a_2)t^2$, so this set is closed under addition and scaling, and thus this is a subspace for P_n where $n \geq 2$.

Question 4.1.6

No. The zero-polynomial cannot be written in the form $a + t^2$ for any $a \in \mathbb{R}$.

Question 4.1.7

No. This subset is not closed under scaling by real numbers. For example, let $c = \frac{1}{3}$ and $p(x) = x^3$, then $\frac{1}{3} \cdot x^3 = \frac{1}{3}x^3$ does not have integer coeefficient.

Question 4.1.8

Yes. The zero polynomial is zero when evaluated at zero, so the set is non-empty. Let $f(x), g(x) \in P_n$ and $r \in \mathbb{R}$, then $f(x) - rg(x)$, when evaluated at 0, gives $f(0) - rg(0) = 0 - 0 = 0$. Thus we see that this subset is closed under addition and scaling, and hence a subspace.

Question 4.1.12

$$W = \left\{ \begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix} : s, t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix} \right\}$$

Since Theorem 1 tells us that the Span of vectors in some vector space V is always a subspace of V, we see that $W \subset \mathbb{R}^4$ is a subspace of $\mathbb{R}^4$.

Question 4.1.15

W is not a vector space because $0 \notin W$.

Question 4.1.16

W is not a vector space because $0 \notin W$.

Question 4.1.17

$$W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Question 4.1.18

$$W = \text{Span} \left\{ \begin{pmatrix} 4 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Question 4.1.33

1. $H + K$ is nonempty as H and K are subsets of $H + K$. Let $w_1 = u_1 + v_1, w_2 = u_2 + v_2 \in H + K$, then for any $r \in \mathbb{R}$, $w_1 - r \cdot w_2 = (u_1 - r \cdot u_2) + (v_1 - r \cdot v_2) \in H + K$, so $H + K$ is a subspace of V.

2. H is a subset of $H+K$ because for any $h \in H$, it can be written as $h+0$ where $h \in H$ and $0 \in K$. Thus H is a subset of $H+K$. However, H itself is a vector space by assumption, so H is a subspace of $H+K$. A completly analogous argument shows that K is a subspace of $H+K$.

Question 4.2.6

$$\begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we see that there are three free variables, and the null space is

$$W = \left\{ \begin{pmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : x_3, x_4, x_5 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Question 4.2.10

$$W = \left\{ \begin{pmatrix} a \\ b \\ a+3b \\ 2a+4b \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} \right\}$$

Question 4.2.12

No. Again $0 \notin W$.

Question 4.2.16

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 4.2.24

$$\begin{pmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0$$

so we see that $w \in \text{Col}(A)$ and $w \in \text{null}(A)$.

Question 4.2.26

1. True. Theorem 2.
2. True.
3. False. It is all possible b such that $Ax = b$ has a solution.
4. True.
5. True. Essentially Theorem 3.
6. True. Example 9.