SOLUTIONS TO HOMEWORK #3, MATH 54 SECTION 001, SPRING 2012

JASON FERGUSON

Beware of typos. These may not be the only ways to solve these problems. In fact, these may not even be the best ways to solve these problems. Congratulations to you if you find better solutions!

1. Ex. 1.9.6: *In Exercises 1-10, assume that T is a linear transformation. Find the standard matrix of T. $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a horizontal shear transformation that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 3\mathbf{e}_1$.*

Solution. The problem gives that $T(\mathbf{e}_1) = \mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $T(\mathbf{e}_2) = \mathbf{e}_2 + 3\mathbf{e}_1 = \begin{bmatrix}0\\1\end{bmatrix} + 3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$, so the standard matrix of T is $\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix} = \boxed{\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}}$. □

2. Ex. 1.9.11: *A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?*

Scratch work. This exercise would be a bit easier if it said I could assume that T is linear, but it didn't. That means that I need to make sure I don't use the fact that T is linear until after I've shown it. (The answer in the back of the book makes this mistake—it skips checking that T is linear.) □

Solution. From the book, reflection through the x_1-axis is linear and is given by the formula

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\-x_2\end{bmatrix}.$$

Similarly, reflection through the x_2-axis is linear and given by the formula $\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-x_1\\x_2\end{bmatrix}$. Therefore, doing T, i.e. both reflections, has the effect $\begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}x_1\\-x_2\end{bmatrix} \mapsto \begin{bmatrix}-x_1\\-x_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so T is linear with matrix $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$.

From your book, the rotation by θ radians counterclockwise is linear with standard matrix $\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$. If $\theta = \pi$, then this matrix becomes $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, so T is a rotation by $\boxed{\pi \text{ radians counterclockwise}}$, or 180 degrees clockwise, or 3π radians counterclockwise, or... □

3. Ex. 1.9.16: *In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.*

$$\begin{bmatrix}? & ?\\? & ?\\? & ?\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1 - x_2\\-2x_1 + x_2\\x_1\end{bmatrix}.$$

Scratch Work. It's not hard to see that $\begin{bmatrix}1 & -1\\-2 & 1\\1 & 0\end{bmatrix}$ works, but the question is asking for more than this. I also need to explain why no other matrices work. □

Solution. Write the missing entries as $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. By computation, $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix}$. Plugging in $x_1 = 1$ and $x_2 = 0$ into $\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$ gives $\begin{bmatrix} a \\ c \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Plugging in $x_1 = 0$ and $x_2 = 1$ gives $\begin{bmatrix} b \\ d \\ f \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, so the matrix is $\boxed{\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}}$. To check, $\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ really does equal $\begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$ for all x_1 and x_2. ☐

4. Ex. 2.1.2: *In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let*

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

$A + 2B, \quad 3C - E, \quad CB, \quad EB.$

Solution. $A + 2B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} = \boxed{\begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}}$.

$3C$ is 2×2, but E is 2×1. You can't subtract an $m \times n$ matrix from a $p \times q$ matrix unless $m = p$ and $n = q$, so $3C - E$ $\boxed{\text{does not exist}}$.

$$CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 1 & 1(-5) + 2(-4) & 1 \cdot 1 + 2(-3) \\ (-2)7 + 1 \cdot 1 & (-2)(-5) + 1(-4) & (-2)1 + 1(-3) \end{bmatrix} = \boxed{\begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}}.$$

E is 2×1, but B is 2×3. You can't multiply an $m \times n$ matrix with a $p \times q$ matrix unless $n = p$, so EB $\boxed{\text{does not exist}}$. ☐

5. Ex. 2.1.4: *Compute $A - 5I_3$ and $(5I_3)A$, when $A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$.*

Solution. $A - 5I_3 = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}}$.

$(5I_3)A = 5(I_3A) = 5A = \boxed{\begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}}$. ☐

6. Ex. 2.1.7: *If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?*

Solution. Suppose B is $m \times n$. In order for AB to exist, m has to be 3. In that case, AB is $5 \times n$, so $n = 7$ and B is $\boxed{3 \times 7}$. ☐

7. Ex. 2.1.11: *Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.*

Scratch work. The only tricky part is finding a matrix B other than 0 or I_3 for which $AB = BA$. There are two choices of B that some people will see right away. Here's one way to get to those two answers even if you don't see them right away.

One way to find B is to substitute $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ into $AB = BA$, multiply everything out, and set the coefficients equal to each other. This gives a system of 9 linear equations in 9 unknowns, that can be solved with a lot of work. But this will give all matrices B that commute with A, and we don't need that. Maybe we can try a simpler example to see if there is some kind of pattern.

What if we let $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and try to find a 2×2 matrix $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $MN = NM$? Multiplying out gives $MN = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $NM = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$. Setting MN and NM equal to each other gives $c = b$, $d = a$, $a = d$, and $b = c$. So N must be equal to:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = aI_2 + bM.$$

In other words, the only matrices N for which $MN = NM$ are linear combinations of I_2 and M. In particular, M will commute with itself, and any constant multiple of I_2 will commute with M.

Looking back at the original problem, we see that of course A will commute with A, and also any constant multiple of I_3, say $2I_3$ or $-I_3$, will also commute with A. These are the two guesses that some people will see right away. There are other matrices that work, too, for example A^2, but these are the easiest to see. ☐

Solution. $AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 & 1 \cdot 3 & 1 \cdot 5 \\ 1 \cdot 2 & 2 \cdot 3 & 3 \cdot 5 \\ 1 \cdot 2 & 4 \cdot 3 & 5 \cdot 5 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}}.$

$$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 1 & 2 \cdot 1 \\ 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 4 & 5 \cdot 5 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}}.$$

Multiplying A on the right by D multiplies the first, second, and third columns by 2, 3, and 5, resp.

Multiplying A on the left by D multiplies the first, second, and third rows by 2, 3, and 5, resp.

Since $AA = AA$, picking $\boxed{B = A}$ gives a 3×3 matrix other than 0 and I_3 for which $AB = BA$. Also, for any constant c,

$$(cI_3)A = c(I_3A) = c(AI_3) = cA,$$

so picking $\boxed{B = cI_3 \text{ for any number } c \text{ other than 0 or 1, say } c = 2}$, will give a 3×3 matrix other than 0 and I_3 that commutes with A. ☐

If you use brute-force computation to solve the system of 9 equations in 9 unknowns (later theorems will remove some of the work), you can show that the only matrices B that commute with A are those that can be written as $aA^2 + bA + cI_3$ for some numbers a, b, c. Since A^3 and A^{-1} also commute with A, this means that each of A^3 and A^{-1} can be written as $aA^2 + bA + cI_3$ for some numbers a, b, c. Using either brute-force or advanced linear algebra theorems that are in your book but will not be taught in this class, you can show that $A^3 = 8A^2 - 3A - 2I_3$ and $A^{-1} = -\frac{1}{2}A^2 + 4A - \frac{3}{2}I_3$.

8. Ex. 2.1.16: *Exercises 15 and 16 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.*

a. *If A and B are 3×3 and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, then $AB = \begin{bmatrix} A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \end{bmatrix}$.*
b. *The second row of AB is the second row of A multiplied on the right by B.*
c. $(AB)C = (AC)B$
d. $(AB)^T = A^T B^T$
e. *The transpose of a sum of matrices equals the sum of their transposes.*

Scratch work. Statement (c) appears false, since the rule your book gives is that $(AB)C = A(BC)$ and not $(AB)C = (AC)B$. However, there are some matrices A, B, and C for which $A(BC)$ equals $(AC)B$, for example $A = B = C = I$. That means to prove that this statement is false, I need to find specific matrices A, B, and C for which $(AB)C$ and $(AC)B$ both exist, but do not equal each other. If I pick $A = I$, then all I need to do is find two matrices B and C for which $BC = CB$. I will pick $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since in my scratch work to Problem 2.1.11 I already showed that not all matrices commute with this matrix, and I even found which matrices commute with this matrix.

Similarly for statement (d). The rule your book gives is $(AB)^T = B^T A^T$, but to prove the statement is false, I need to find specific matrices A and B for which $B^T A^T$ and $A^T B^T$ are both defined, but not equal to each other. □

Solution. a. $\boxed{\text{False}}$. AB is a 3×3 matrix, but $\begin{bmatrix} A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \end{bmatrix}$ is a 3×1 matrix, so they cannot be equal. For example, pick $A = B = I_3$. Then $\begin{bmatrix} A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, which is not equal to $AB = I_3$. (The correct formula is $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$.)

b. $\boxed{\text{True}}$. The last centered equation on p.114 says that in general, the ith row of AB is the ith row of A times B.

c. $\boxed{\text{False}}$. Let $A = I_2$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then:

$$(AB)C = BC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(AC)B = CB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 1 & 0 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

so $(AB)C \neq A(CB)$ in this case. (The correct formula is $A(BC) = (AB)C$.)

d. $\boxed{\text{False}}$. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and in my answer to (c) I already showed that $A^T B^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^T A^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Because $(AB)^T$ always equals $B^T A^T$, we have shown that $(AB)^T \neq A^T B^T$ in this case.

e. $\boxed{\text{True}}$. This is Theorem 3b, but written in words instead of a formula. □

9. Ex. 2.1.24: *Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint: *Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.*

Solution. Let $\mathbf{b}$ be any vector in $\mathbb{R}^m$. Then $A(D\mathbf{b}) = (AD)\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$, so $\mathbf{x} = D\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$, as needed.

Since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$, the row-reduced form of A has a pivot in every row. Since each column has at most one pivot, this means that the row-reduced form of A cannot have more rows than columns. Therefore, the same is true of A. □

10. Ex. 2.1.34: *Give a formula for $(AB\mathbf{x})^T$, where $\mathbf{x}$ is a vector and A and B are matrices of appropriate sizes.*

Solution. Denote the columns of B by $\mathbf{b}_1, \ldots, \mathbf{b}_p$ and the entries in $\mathbf{x}$ by $x_1, \ldots, x_p$. Then $AB\mathbf{x} = x_1 A\mathbf{b}_1 + \cdots + x_p A\mathbf{b}_p$, so:

$$(AB\mathbf{x})^T = \boxed{x_1(A\mathbf{b}_1)^T + \cdots + x_p(A\mathbf{b}_p)^T}.$$

Another acceptable answer is as follows: Suppose A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in $\mathbb{R}^p$. (The dimensions need to be this form for $AB\mathbf{x}$ to be defined.) Then the i, j entry of AB is $\sum_{k=1}^n a_{ik} b_{kj}$, so the ith entry of $(AB)\mathbf{x}$ is $\sum_{l=1}^p x_l \left(\sum_{k=1}^n a_{ik} b_{kl} \right) = \sum_{l=1}^p \sum_{k=1}^n a_{ik} b_{kl} x_l$. But $AB\mathbf{x}$ is a column vector, so $(AB\mathbf{x})^T$ is a row vector. So, the same description works:

$$\boxed{\text{for each } 1 \leq i \leq m, \text{ the } i\text{th entry in the row vector } (AB\mathbf{x})^T \text{ is } \sum_{l=1}^p \sum_{k=1}^n a_{ik} b_{kl} x_l}.$$ □

11. Ex. 2.2.6: *Use the inverse found in Exercise 3 to solve the system*

$$\begin{aligned} 8x_1 + 5x_2 &= -9 \\ -7x_1 - 5x_2 &= 11 \end{aligned}$$

(For reference, Ex. 2.2.3 asked to find the inverse of $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$.)

Solution.

$$\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}^{-1} = \frac{1}{8(-5) - 5(-7)} \begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix},$$

so the solution to $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$ is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} -9 + 11 \\ \frac{63}{5} - \frac{88}{5} \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -5 \end{bmatrix}}.$$

□

12. Ex. 2.2.10: *In Exercises 9 and 10, mark each statement True or False. Justify each answer.*

a. *A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of the inverses in the same order.*

b. *If A is invertible, then the inverse of A^{-1} is A itself.*

c. *If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$, then A is not invertible.*

d. *If A can be row reduced to the identity matrix, then A must be invertible.*

e. *If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.*

Scratch work. Statement (a) is saying $(AB)^{-1} = A^{-1}B^{-1}$. But the usual formula is $(AB)^{-1} = B^{-1}A^{-1}$, so to show statement (a) is false, we need to find specific invertible matrices A and B for which $B^{-1}A^{-1} \neq A^{-1}B^{-1}$.

In general, the elementary row operations that that reduce the invertible $n \times n$ matrix A to the identity I_n will also reduce the arbitrary $n \times p$ matrix B to $A^{-1}B$. Therefore, the row operations that reduce A to I_n should reduce A^{-1} to $A^{-1}A^{-1}$. To show that part (e) is false, we need to look for an invertible matrix A for which $A^{-1}A^{-1}$ is not equal to I_n. □

Soluton. a. $\boxed{\text{False}}$. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then:

$$A^{-1} = \frac{1}{0 \cdot 0 - 1 \cdot 1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B^{-1} = \frac{1}{1 \cdot 1 - 1 \cdot 0} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

so:

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + (-1)1 & 1 \cdot 1 + (-1)0 \\ 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^{-1}B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0(-1) + 1 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 0 & 1(-1) + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Therefore, $(AB)^{-1} \neq A^{-1}B^{-1}$ in this case. (The correct rule is $(AB)^{-1} = B^{-1}A^{-1}$.)

b. $\boxed{\text{True}}$. This is exactly Theorem 6a.

c. $\boxed{\text{True}}$. Theorem 4 says that if $ad - bc = 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible.

d. $\boxed{\text{True}}$. This is part of Theorem 7.

e. $\boxed{\text{False}}$. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The elementary row operation that reduces A to I_2 is "add -1 times Row 2 to Row 1." Therefore, A is invertible, and A^{-1} is what you get by applying this operation to I_2, i.e. $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Applying this one elementary row operation to A^{-1} gives $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, which is not I_2. (The correct rule is, the row operations that turn A into I_n also turn I_n into A^{-1}, or more generally B into $A^{-1}B$.) □

13. Ex. 2.2.15: *Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)D = I$ and $D(ABC) = I$.*

Solution. Let $D = C^{-1}B^{-1}A^{-1}$. All of the dimensions match, so D exists. Also,

$$(ABC)D = AB(CC^{-1})B^{-1}A^{-1} = ABI_nB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n,$$
$$D(ABC) = C^{-1}B^{-1}(A^{-1}A)BC = C^{-1}B^{-1}I_nBC = C^{-1}B^{-1}BC = C^{-1}I_nC = C^{-1}C = I_n,$$

so ABC is invertible with inverse D, as needed. □

The way the problem asked you to solve it isn't the simplest way to do it. Theorem 6b says that since A and B are invertible, AB is invertible with inverse $B^{-1}A^{-1}$. If you use Theorem 6b again, you get that $(AB)C = ABC$ is invertible, with inverse $C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$.

14. Ex. 2.2.22: *Explain why the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible. [*Hint: *Review Theorem 4 in Section 1.4.]*

Solution. Suppose A is invertible. Let $\mathbf{b}$ be any vector in $\mathbb{R}^n$. Then the equation $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$. Therefore, $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$, so by Theorem 4 of Chapter 1, the columns of A span $\mathbb{R}^n$. □

15. Ex. 2.3.8: *Determine which of the matrices in Exercises 1-10 are invertible. Use as few calculations as possible. Justify your answers.*

$$\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Solution. This 4×4 matrix is in row-echelon form and has four pivots, so $\boxed{\text{it is invertible}}$. □

16. Ex. 2.3.12: *In Exercises 11 and 12, the matrices are all $n \times n$. Each part of the exercises is an* implication *of the form "If ⟨statement 1⟩, then ⟨statement 2⟩." Mark an implication as True if the truth of ⟨statement 2⟩* always *follows whenever ⟨statement 1⟩ happens to be true. An implication is False if there is an instance in which ⟨statement 2⟩ is false but ⟨statement 1⟩ is true. Justify each answer.*

a. *If there is an $n \times n$ matrix D such that $AD = I$, then there is also an $n \times n$ matrix C such that $CA = I$.*
b. *If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$.*
c. *If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the solution is unique for each $\mathbf{b}$.*
d. *If the linear transformation $x \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, then A has n pivot positions.*
e. *If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.*

Scratch work. Part (d) is sneaky; the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *into* $\mathbb{R}^n$ for *any* $n \times n$ matrix A. The phrase that appears in part (i) of the invertible matrix theorem is "the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^n$," i.e. the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an *onto* function from $\mathbb{R}^n$ to $\mathbb{R}^n$. □

Solution. a. $\boxed{\text{True}}$. This is the "If (k) then (j)" part of the invertible matrix theorem. (In fact, there is only one matrix C that works, and it is D, but strictly speaking that isn't part of the invertible matrix theorem.)

b. $\boxed{\text{True}}$. This is the "If (e) then (h)" part of the invertible matrix theorem.

c. $\boxed{\text{True}}$. From the "If (g) then (f)" part of the invertible matrix theorem, the linear transformatoin $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. That means that for any $\mathbf{b}$ in $\mathbb{R}^n$, there is at most one $\mathbf{x}$ in $\mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$. Since we are given that there always is a solution $\mathbf{x}$, we have shown that the solution is unique.

d. $\boxed{\text{False}}$. Let $n = 1$ and $A = \begin{bmatrix} 0 \end{bmatrix}$. Then $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^1$ *into* $\mathbb{R}^1$, but A has no pivot positions.

e. $\boxed{\text{True}}$. This is the "If not (g) then not (f)" part of the invertible matrix theorem. □

17. Ex. 2.3.14: *An $m \times n$ **lower triangular matrix** is one whose entries* above *the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.* (For reference, the matrix in Ex. 2.3.3 is $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$.)

Scratch work. This exercise isn't very hard once you know what the corresponding result for upper-triangular matrices is, since the transpose of a lower-triangular matrix is upper-triangular, and transposes do not affect whether a matrix is invertible.

However, proving the corresponding result for upper-triangular matrices is very tricky.

For a square upper-triangular matrix A, if the elements on the main diagonal of A are all nonzero, then A is in row-echelon form with n pivot positions, so A is invertible. This might make you think that if A is square and upper-triangular with a zero on the main diagonal, then A is not invertible.

This is in fact true, but the explanation in the previous paragraph doesn't work any more: if an upper-triangular matrix A has a zero on its diagonal, it may no longer be in echelon form, for example:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

One way I could work around this is to argue that once I row-reduce such a matrix, it will have a row of zeroes, but this kind of argument is quite difficult to state precisely, so I'll go about it in a different way. ☐

Solution. The final answer is: A square lower-triangular matrix is invertible if and only if all elements on its main diagonal are nonzero. To prove this, we will first show that a square upper-triangular matrix is invertible if and only if all elements on its main diagonal are nonzero.

First, suppose that A is an $n \times n$ upper-triangular matrix whose diagonal entries are all nonzero. Then A is in row-echelon form with n pivots, namely the entries that lie along the main diagonal. Therefore, A is invertible. (The matrix below illustrates this with an example:)

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{3} & 4 & 5 \\ 0 & 0 & \boxed{5} & 6 \\ 0 & 0 & 0 & \boxed{7} \end{bmatrix}$$

Now suppose that A is an $n \times n$ upper-triangular matrix with at least one nonzero diagonal entry. I will explain why A is not invertible. I will show how my explanation works by applying it to the matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

but my explanation will work for any $n \times n$ matrix.

First, use elementary row operations to sort the rows of the matrix according to how far to the left the leftmost nonzero entries are. The order will not matter if there a tie:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

In the sorted matrix, there will be at least one zero entry along the main diagonal. (Otherwise, the sorted matrix must have been the original matrix A, and A will have all nonzero entries in its main diagonal.) Pick the upper-leftmost zero entry on the main diagonal of the sorted matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & \boxed{0} & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

and look at all entries above or above and to the left of it:

$$\begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{3} & 4 & 5 \\ \boxed{0} & \boxed{3} & \boxed{4} & 5 & 6 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

After you ignore the zeroes along the bottom, these entries will form a list of vectors with more vectors than entries in each vector:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Therefore, these vectors will be linearly dependent; there is some linear combination of them with weights not all zero that gives the zero vector:

$$1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The only difference between these columns and the original columns in the matrix are the extra zeroes, so the corresponding columns from the sorted matrix are linearly dependent, by using the same weights:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since some subset of the columns of the sorted matrix is linearly dependent, that means that all columns of the sorted matrix are linearly dependent:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 5 \\ 8 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 6 \\ 9 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This means that the sorted matrix is not invertible. Since the sorted matrix was obtained from A by elementary row operations, this means that A is not invertible, either.

We have proved what we wanted to for upper-triangular matrices. Now, a square lower-triangular matrix A is invertible if and only if A^T is invertible. A^T is an upper-triangular matrix, so by what we just showed, A^T is invertible if and only if its diagonal entries are all nonzero. But the diagonal entries of A are the diagonal entries of A^T, so A is invertible if and only if its diagonal entries are nonzero, which is the final result. □

There is a much slicker solution to this result using some facts about determinants that you will learn in Chapter 3: Theorem 2 from Section 3.1 says the determinant of a square lower- or upper-triangular matrix is the product of its diagonal entries, and Theorem 4 from Section 3.2 says that a square matrix is invertible if and only if its determinant is nozero.

18. Ex. 2.3.20: *If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.*

Solution. By the invertible matrix theorem, since $EF = I$, E is invertible. Multiplying both sides of $EF = I$ on the left by E^{-1} then gives: $F = E^{-1}EF = E^{-1}I = E^{-1}$. In particular, $FE = E^{-1}E = I = EF$. □

19. Ex. 3.1.10: *Compute the determinants in Exercises 9-14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.* $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$

Solution.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} \xrightarrow{\text{Row 2}} -3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix} \xrightarrow{\text{Row 3}} -3\left(5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} + 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix}\right)$$

$$= -3\Big(5\big((-2)5 - 2(-6)\big) + 4\big(1(-6) - (-2)2\big)\Big)$$

$$= -3\big(5(-10+12) + 4(-6+4)\big) = -3(10-8) = \boxed{-6}.$$

□

20. Ex. 3.1.12: $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$

Solution. The determinant of a triangular matrix is the product of its diagonal elements, so this determinant is $4(-1)3(-3) = \boxed{36}$. □

21. Ex. 3.1.22: *In Exercises 19-24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.*
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}.$

Solution. The row operation is $\boxed{\text{add } k \text{ times Row 2 to Row 1}}$. We have:

$$\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = (a+kc)d - (b+kd)c = ad - kcd - bc + kcd = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

so $\boxed{\text{the row operation does not change the determinant}}$. □

In section 3.2, you will see more generally that adding any multiple of one row to another row of any square matrix of any size will not change the determinant.