Solution to Homework 2 by Yuhao Huang

1. (1.4.6)
$$-2\begin{pmatrix}7\\2\\9\\-3\end{pmatrix}-5\begin{pmatrix}-3\\1\\-6\\2\end{pmatrix}=\begin{pmatrix}1\\-9\\12\\-4\end{pmatrix}$$

2. (1.4.10)
$$x_1\begin{pmatrix}8\\5\\1\end{pmatrix}+x_2\begin{pmatrix}-1\\4\\-3\end{pmatrix}=\begin{pmatrix}4\\1\\2\end{pmatrix}$$
$$\begin{pmatrix}8&-1\\5&4\\1&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}4\\1\\2\end{pmatrix}$$

3. (1.4.14) The system $A\mathbf{x}=\mathbf{u}$ is inconsistent, so $\mathbf{u}$ is not in the span of the columns of A.

4. (1.4.16) For example, when
$$\mathbf{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$
the system $A\mathbf{x}=\mathbf{b}$ has no solution.

   The set of all $\mathbf{b}$ such that $A\mathbf{x}=\mathbf{b}$ has a solution can be described as the span of the first two columns (or any two columns of A).

   Here is another way to describe it:
$$\left\{\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\middle|\, b_1+2b_2+b_3=0\right\}$$

5. (1.4.17) A can be row reduced to
$$\begin{pmatrix}1&3&0&3\\0&2&-1&4\\0&-4&2&-8\\2&0&3&-1\end{pmatrix}\to\begin{pmatrix}1&3&0&3\\0&2&-1&4\\0&0&0&0\\0&-6&3&-7\end{pmatrix}\to\begin{pmatrix}1&3&0&3\\0&2&-1&4\\0&0&0&1\\0&0&0&0\end{pmatrix}$$
   So it has 3 pivot rows, thus there exist some $\mathbf{b}\in\mathbb{R}^4$ such that $A\mathbf{x}=\mathbf{b}$ does not have a solution.

6. (1.5.5) The system can be reduced to
$$\begin{cases}x_1+3x_2+x_3=0\\ \quad\quad x_2+2x_3=0\end{cases}$$
   and then
$$\begin{cases}x_1\quad\quad -5x_3=0\\ \quad\quad x_2+2x_3=0\end{cases}$$

So the solutions are

$$\mathbf{x} = x_3 \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

7. (1.5.6) The system can be reduced to

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 0 \\ \qquad\; x_2 - 3x_3 = 0 \end{cases}$$

and then

$$\begin{cases} x_1 \qquad + 4x_3 = 0 \\ \qquad x_2 - 3x_3 = 0 \end{cases}$$

So the solutions are

$$\mathbf{x} = x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$$

8. (1.5.12)

$$\mathbf{x} = x_2 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

9. (1.5.29)
   (a) No, because A has 3 pivot **columns** (= number of columns).
   (b) Yes, because A has 3 pivot **rows** (= number of rows).

10. (1.5.30)
   (a) Yes, because A has 2 pivot **columns** so there is one free variable.
   (b) No, because A has 2 pivot **rows** so there is one row for possible inconsistency.

11. (1.7.6) They are linearly independent because the row reduced echelon form has 3 pivots columns (= number of columns).

12. (1.7.10)
   (a) For no values of h will $\mathbf{v}_3$ lie in the span of the first two, because $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{v}_3$ is never consistent.
   (b) They are always linearly dependent because $2\mathbf{v}_1 + \mathbf{v}_2 = 0$.

13. (1.7.18) The 4 vectors are linearly dependent, because the 3rd and the 4th lie in the span of the first two.

14. (1.7.22)
   (a) T. In this case $a\mathbf{v}_1 = b\mathbf{v}_2$ thus they are linearly dependent.
   (b) F. E.g. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ are linearly dependent.

(c) T. In this case $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{v}_3$ for some a, b and thus $a\mathbf{v}_1 + b\mathbf{v}_2 - \mathbf{v}_3 = 0$.

(d) F. E.g. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ are linearly dependent but there are only 2 vectors, while there are 3 entries in each vectors.

15. (1.7.24) $\begin{pmatrix} \blacksquare & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \blacksquare \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

16. (1.7.37) True. In this case there will be a linear relation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, which implies $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$, thus $\mathbf{v_1}, \ldots, \mathbf{v_4}$ are linearly dependent.

17. (1.8.10) Thats equivalent to solve $A\mathbf{x} = \mathbf{0}$. We get those $\mathbf{x}$ of the form

$$\mathbf{x} = x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

are mapped to $\mathbf{0}$.

18. (1.8.11) Yes because $A\mathbf{x} = \mathbf{b}$ is consistent.

19. (1.8.20)

$$A = \begin{pmatrix} -2 & 7 \\ 5 & -3 \end{pmatrix}$$

20. (1.8.33) $T(0, 0) \neq \mathbf{0}$, so it is not linear.