

# SOLUTIONS TO HOMEWORK #1, MATH 54 SECTION 001, SPRING 2012

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1. Beware of typos. In fact, your solutions may be better than mine.

2. Some problems have “scratch work” and “solutions.” The scratch work says how I came up with my solution and the solution is what I would actually turn in.

1. Ex. 1.1.6: Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

$$\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{bmatrix}$$

*Solution.* First add  $-3$  times Row 3 to Row 4 to turn the 3 in Row 4 into a 0. That will change Row 4 to  $[0 \ 0 \ 0 \ -5 \ 15]$ , so then multiply the new Row 4 by  $-\frac{1}{5}$  to turn the  $-5$  into a 1.

Multiplying Row 2 by  $\frac{1}{2}$  to turn the 2 into a 1 is also reasonable. □

2. Ex. 1.1.10: In Exercises 7-10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

*Solution.*

$$\left[ \begin{array}{ccccc} \begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} & \begin{array}{l} \xrightarrow{R_1 \rightarrow -3R_4 + R_1} \\ \xrightarrow{R_2 \rightarrow 4R_4 + R_2} \end{array} & \begin{bmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} & \xrightarrow{R_1 \rightarrow 2R_2 + R_1} & \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \end{array} \right]$$

The solution is  $(x_1, x_2, x_3, x_4) = (-3, -5, 6, -3)$ . □

3. Ex. 1.1.13: Solve the systems in Exercises 11-14.

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

*Solution.*

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{R_2 \rightarrow -2R_1 + R_2} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \xrightarrow{R_3 \rightarrow -2R_2 + R_3} \\ & \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\begin{array}{l} R_1 \rightarrow 3R_3 + R_1 \\ R_2 \rightarrow -5R_3 + R_2 \end{array}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{aligned}$$

The solution is  $(x_1, x_2, x_3) = (5, 3, -1)$ . □

4. Ex. 1.1.16: Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

$$\begin{aligned} x_1 \qquad \qquad -2x_4 &= -3 \\ 2x_2 + 2x_3 \qquad &= 0 \\ \qquad \qquad \qquad x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5 \end{aligned}$$

Solution.

$$\begin{aligned} \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{array} \right] &\xrightarrow{R_4 \rightarrow 2R_1 + R_4} \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array} \right] &\xrightarrow{R_2 \rightarrow \frac{1}{2}R_2} \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array} \right] \\ &\xrightarrow{R_4 \rightarrow -3R_2 + R_4} \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{array} \right] &\xrightarrow{R_4 \rightarrow R_3 + R_4} \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

This matrix corresponds to the system

$$\begin{aligned} x_1 \qquad \qquad -2x_4 &= -3 \\ x_2 + x_3 \qquad &= 0 \\ \qquad \qquad \qquad x_3 + 3x_4 &= 1 \end{aligned}$$

A solution can be found by picking some numerical value for  $x_4$ , for example  $x_4 = 0$ , and then using it to find  $x_3$ ,  $x_2$ , and  $x_1$ , so the system is consistent. □

5. Ex. 1.1.20: In Exercises 19-22, determine the value(s) of  $h$  such that the matrix is the augmented matrix of a consistent linear system.

$$\left[ \begin{array}{ccc} 1 & h & -3 \\ -2 & 4 & 6 \end{array} \right]$$

Solution.

$$\left[ \begin{array}{ccc} 1 & h & -3 \\ -2 & 4 & 6 \end{array} \right] \xrightarrow{R_2 \rightarrow 2R_1 + R_2} \left[ \begin{array}{ccc} 1 & h & -3 \\ 0 & 2h + 4 & 0 \end{array} \right]$$

This matrix corresponds to the system:

$$\begin{aligned} x_1 + \qquad \qquad hx_2 &= -3 \\ (2h + 4)x_2 &= 0 \end{aligned}$$

Then  $(x_1, x_2) = (-3, 0)$  is always a solution, no matter what  $h$  is, so the corresponding system is consistent for all numbers  $h$ . □

In §1.2 you learned another way to see that  $\left[ \begin{array}{ccc} 1 & h & -3 \\ 0 & 2h + 4 & 0 \end{array} \right]$  corresponds to a consistent system—no matter what  $h$  is, the matrix is in echelon form and does not have a row of the form  $[0 \ 0 \ b]$  where  $b$  is nonzero.

6. Ex. 1.1.28: Suppose  $a$ ,  $b$ ,  $c$ , and  $d$  are constants such that  $a$  is not zero and the system below is consistent for all possible values  $f$  and  $g$ . What can you say about the numbers  $a$ ,  $b$ ,  $c$ , and  $d$ ? Justify your answer.

$$\begin{aligned} ax_1 + bx_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

*Scratch Work.* Since  $a$  is nonzero, we can divide by it to row-reduce:

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{a}R_1} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ c & d & g \end{bmatrix} \xrightarrow{R_2 \rightarrow (-c)R_1 + R_2} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & d - \frac{bc}{a} & g - \frac{cf}{a} \end{bmatrix}$$

(If you are worried that  $c$  might be zero, you don't need to. When you multiply a row by a constant, that constant has to be nonzero. But adding a constant multiple of one row to a different row works even when that constant is zero.)

If  $d - \frac{bc}{a} \neq 0$ , the system is consistent no matter what  $f$  and  $g$  are. Even if  $d - \frac{bc}{a} = 0$ , the system is still consistent for *some* numbers  $f$  and  $g$ . For example, if  $d - \frac{bc}{a} = 0$  and  $f = g = 0$ , then the augmented matrix becomes:

$$\begin{bmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

But the problem says that the system is consistent for *all* numbers  $f$  and  $g$ , not just *some* numbers. For example, if  $d - \frac{bc}{a} = 0$  and  $f = 0$  and  $g = 1$ , the augmented matrix becomes:

$$\begin{bmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which corresponds to an inconsistent system. □

*Solution.*

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{a}R_1} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ c & d & g \end{bmatrix} \xrightarrow{R_2 \rightarrow (-c)R_1 + R_2} \begin{bmatrix} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & d - \frac{bc}{a} & g - \frac{cf}{a} \end{bmatrix}$$

If  $d - \frac{bc}{a} \neq 0$ , the system is consistent for all numbers  $f$  and  $g$ . For example, we can divide the equation  $(d - \frac{bc}{a})x_2 = g - \frac{cf}{a}$  by  $d - \frac{bc}{a}$  to find  $x_2$ , and then use  $x_1 + \frac{b}{a}x_2 = \frac{f}{a}$  to solve for  $x_1$ .

If instead  $d - \frac{bc}{a} = 0$ , then if  $f = 0$  and  $g = 1$  the augmented matrix becomes:

$$\begin{bmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which corresponds to an inconsistent system.

Therefore, we can say  $\boxed{d - \frac{bc}{a} \neq 0}$  and not much else. □

Because  $a \neq 0$ , you can rewrite the final answer as  $ad - bc \neq 0$ . The equation “ $ad - bc \neq 0$ ” is also the correct answer even when  $a = 0$ , but you will need completely different steps in your row reduction to show it. In Chapters 2 and 3 you'll learn techniques that don't depend on whether  $a$  is zero and that will give you the final answer “ $ad - bc \neq 0$ ” with almost no work.

**7. Ex. 1.2.2:** *In Exercises 1 and 2, determine which matrices are in reduced echelon form and which are only in echelon form.*

$$\begin{array}{ll} \text{a. } \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{b. } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ \text{c. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} & \text{d. } \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

*Solution.* The leftmost nonzero entries in the nonzero rows of the matrices are marked with boxes:

$$\begin{array}{ll} \begin{bmatrix} \boxed{1} & 1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} & \begin{bmatrix} 0 & \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

a. The zero row is below the nonzero rows. The boxed entries are all 1 and go from top-left to bottom-right, and all entries directly above and below the boxed entries are 0, so the matrix is in reduced echelon form.

b. The boxed entries in the nonzero rows go from top-left to bottom-right, and all entries directly below the boxed entries are 0, so the matrix is in echelon form.

However, there is a 1 above the boxed entry in Row 2, so the matrix is only in echelon form.

c. The boxed entry in Rows 2 is not to the right of the boxed entry in Row 1, so the matrix is not in echelon form, and therefore is also not in reduced echelon form.

d. The zero row is below the nonzero rows. The boxed entries go from top-left to bottom-right, and all entries directly below the boxed entries are 0, so the matrix is in echelon form.

However, the boxed entry in Row 2 is 2 and not 1, so the matrix is only in echelon form.  $\square$

**8. Ex. 1.2.11:** Find the general solutions of the systems whose augmented matrices are given in Exercises 7-14.

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

*Solution.*

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \xrightarrow[\begin{smallmatrix} R_2 \rightarrow 3R_1 + R_2 \\ R_3 \rightarrow 2R_1 + R_3 \end{smallmatrix}]{\phantom{R_1 \rightarrow \frac{1}{3}R_1}} \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow \frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is  $(x_1, x_2, x_3) = (\frac{4}{3}u - \frac{2}{3}v, u, v)$ , where  $u$  and  $v$  are any numbers.  $\square$

**9. Ex. 1.2.16:** Exercises 15 and 16 use the notation of Example 1 for matrices in echelon form. [In other words, each “■” is a nonzero number, not necessarily all the same. Each “\*” is any number, not necessarily all the same.] Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

a.  $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$

b.  $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

*Solution.* a. There is no row of the form  $[0 \ 0 \ \blacksquare]$ , so the system is consistent. Each column except the rightmost column has a ■ in it, so the system has no free variables and therefore has a unique solution.

b. There is no row of the form  $[0 \ 0 \ 0 \ 0 \ \blacksquare]$ , so the system is consistent. The second column has no ■ in it, so the second variable is free. Thus, the system does not have a unique solution.  $\square$

**10. Ex. 1.2.23:** Suppose a  $3 \times 5$  coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

*Scratch Work.* So far, the only way we’ve learned to decide if a system is consistent is to row-reduce the augmented matrix and see if there is a pivot in the rightmost column. So that is what we need to do for this problem. That means we’ll need to take what the problem gives us and use it to figure out something about what happens when we row-reduce the  $3 \times 6$  augmented matrix.

The key thing to realize is, the leftmost 5 columns of the augmented matrix are the coefficient matrix. That means if you take row operations that will row-reduce the coefficient matrix and apply them to the augmented matrix, then what they will do to the rightmost column is unpredictable, but they will row-reduce the leftmost 5 columns of the augmented matrix.

Now, once we've row-reduced the leftmost 5 columns of the  $3 \times 6$  augmented matrix, we are almost completely done with row-reducing the  $3 \times 6$  matrix. In fact, we will be done row-reducing if we have already found a pivot in every row among the leftmost five columns, and in that case there will be no pivots in the rightmost column.

Here we can finally use what the problem gives us. We are given that the  $3 \times 5$  coefficient matrix has three pivot columns. That means that once you row-reduce the coefficient matrix, there will be three different pivots, and they have to be in three different rows, so by the previous paragraph, we are finished.  $\square$

*Solution.* Think about the  $3 \times 6$  augmented matrix. If you take row operations that reduce the coefficient matrix and apply them to the augmented matrix, you will get a  $3 \times 6$  matrix that I will call  $A$ , whose leftmost five columns are an echelon form of the coefficient matrix.

Each pivot column of the coefficient matrix has only one pivot in it, so the coefficient matrix has three pivots. Each of these pivots have to be in a different row, so each of the three rows of coefficient matrix have a pivot.

That means that all three rows of  $A$  are nonzero, and that the leftmost nonzero entry of each row of  $A$  is in one of the leftmost five columns. That means  $A$  is in echelon form and has no pivots in the rightmost column, so the system is consistent.  $\square$

**11.** Ex. 1.2.25: *Suppose the coefficient matrix of a system of linear equations has a pivot in every row. Explain why the system is consistent.*

*Scratch work.* If you look at the solution to the previous problem, the thing we really needed to know about the coefficient matrix was that there was a pivot in every row, not that the matrix was  $3 \times 5$ . That means that the same solution will work here.  $\square$

*Solution.* Suppose the coefficient matrix is  $m \times n$ , and think about the  $m \times (n+1)$  augmented matrix. If you take row operations that reduce the coefficient matrix and apply them to the augmented matrix, you will get an  $m \times (n+1)$  matrix that I will call  $A$ , whose leftmost  $n$  columns are an echelon form of the coefficient matrix.

Because the coefficient matrix has a pivot in every row, every row of  $A$  is nonzero, and the leftmost nonzero entry of each row of  $A$  is in one of the leftmost  $n$  columns. That means  $A$  is in echelon form and has no pivots in the rightmost column, so the system is consistent.  $\square$

**12.** Ex. 1.3.6: *In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.*

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

*Solution.* Do the scalar multiplications and additions and then compare entries to get:

$$\begin{array}{l} -2x_1 + 8x_2 + x_3 = 0 \\ 3x_1 + 5x_2 - 6x_3 = 0 \end{array}$$

**13.** Ex. 1.3.10: *In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.*

$$\begin{array}{l} 4x_1 + x_2 + 3x_3 = 9 \\ x_1 - 7x_2 - 2x_3 = 2 \\ 8x_1 + 6x_2 - 5x_3 = 15 \end{array}$$

*Solution.* Do the scalar multiplications and additions and then compare entries to get:

$$x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}.$$

14. Ex. 1.3.11: In Exercises 11 and 12, determine if  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$ .

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

*Scratch Work.* Saying  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  is the same as saying there are numbers  $x_1$ ,  $x_2$ , and  $x_3$  for which  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ . If you rephrase this question as a system, the problem is asking if:

$$\begin{aligned} x_1 + 5x_3 &= 2 \\ -2x_1 + x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= 6 \end{aligned}$$

is consistent. So just try to solve this system. □

*Solution.*

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow{R_2 \rightarrow 2R_1 + R_2} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow{R_3 \rightarrow -2R_2 + R_3} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This means that system of equations corresponding to the vector equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$  is consistent, so  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$ . □

15. Ex. 1.3.14: In Exercises 13 and 14, determine if  $\mathbf{b}$  is a linear combination of the vectors formed by the columns of  $A$ .

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

*Scratch work.* Rephrase this question in terms of consistency of a system of linear equations, just like in the previous problem. □

*Solution.*

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \xrightarrow{R_3 \rightarrow -R_1 + R_3} \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix}$$

So if  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are the three columns of  $A$ , the system of equations corresponding to the vector equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$  is consistent, so  $\mathbf{b}$  is a linear combination of the columns of  $A$ . □

16. Ex. 1.3.18: Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$ , and  $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$ . For what value(s) of  $h$  is  $\mathbf{y}$  in the plane generated by  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ?

*Scratch work.* The plane generated by  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is the set of all points in  $\mathbb{R}^3$  that can be written as  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$  for some numbers  $x_1$  and  $x_2$ . So the question is asking to find all numbers  $h$  for which  $\mathbf{y}$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . □

*Solution.*

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \xrightarrow{R_3 \rightarrow 2R_1 + R_3} \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h - 3 \end{bmatrix} \xrightarrow{R_3 \rightarrow -2R_2 + R_3} \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h + 7 \end{bmatrix}$$

So if  $h = -\frac{7}{2}$ , the system corresponding to  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{y}$  is consistent, and otherwise it is not. Therefore,  $h = -\frac{7}{2}$  is the only number for which  $\mathbf{y}$  is in the plane generated by  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . □

17. Ex. 1.3.26: Let  $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ , let  $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$ , and let  $W$  be the set of all linear combinations of the columns of  $A$ .

a. Is  $\mathbf{b}$  in  $W$ ?

b. Show that the third column of  $A$  is in  $W$ .

*Scratch work.* According to the problem,  $W$  just means everything in  $\mathbb{R}^3$  that can be written as a linear combination of the columns of  $A$ . So asking if  $\mathbf{b}$  is in  $W$  is a roundabout way of asking if  $\mathbf{b}$  can be written as a linear combination of the columns of  $A$ .  $\square$

*Solution.* a.

$$\begin{aligned} \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 3 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{bmatrix} \xrightarrow{\begin{array}{l} R_2 \rightarrow R_1 + R_2 \\ R_3 \rightarrow -2R_1 + R_3 \end{array}} \\ \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 0 & 4 & 4 & 4 \end{bmatrix} &\xrightarrow{R_3 \rightarrow -\frac{2}{3}R_2 + R_3} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So if  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  are the three columns of  $A$ , the system of equations corresponding to the vector equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$  is consistent, so  $\mathbf{b}$  is a linear combination of the columns of  $A$ . This means  $\boxed{\mathbf{b} \text{ is in } W}$ .

b. Let the three columns of  $A$  be  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$ . Because  $\mathbf{a}_3 = 0\mathbf{a}_1 + 0\mathbf{a}_2 + 1\mathbf{a}_3$ ,  $\mathbf{a}_3$  is a linear combination of the columns of  $A$ , so  $\mathbf{a}_3$  is in  $W$ .  $\square$