Kinetic Description of Hamilton-Jacobi PDE IV

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PDE/Probability Student Seminar

Outline

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Discrete Gauss Curvature and Alexandrov Maps

Optimal Transport Formulation and Monge-Kantorovich Duality

Hamilton-Jacobi Dynamics: Free Motion, Coagulation, and Collision

Hamilton-Jacobi Dynamics: Directed Secondary Polytope

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Dual Tessellations/Legendre Transform

Given a finite P and a map $f : P \to \mathbb{R}$, we define two piecewise linear convex functions:

$$u(x) = f^*(x) = \sup_{\rho \in P}(x \cdot \rho - f(\rho))$$

$$u^*(\rho) = f^{**}(\rho) = \sup_x(x \cdot \rho - u(x)) = f^o(\rho) = \text{ convex hull of f.}$$

Domains of the linearity of u yield a Laguerre tessellation:

$$\mathbf{X}(f) := \{X(\rho) : \ \rho \in \mathbb{R}^d\}, \quad X(\rho) = \partial u^*(\rho).$$

Domains of the linearity of u^* yield a weighted Delaunay tessellation:

$$\mathbf{P}(f) := \{P(x) : \ x \in \mathbb{R}^d\}, \quad P(x) = \partial u(x).$$

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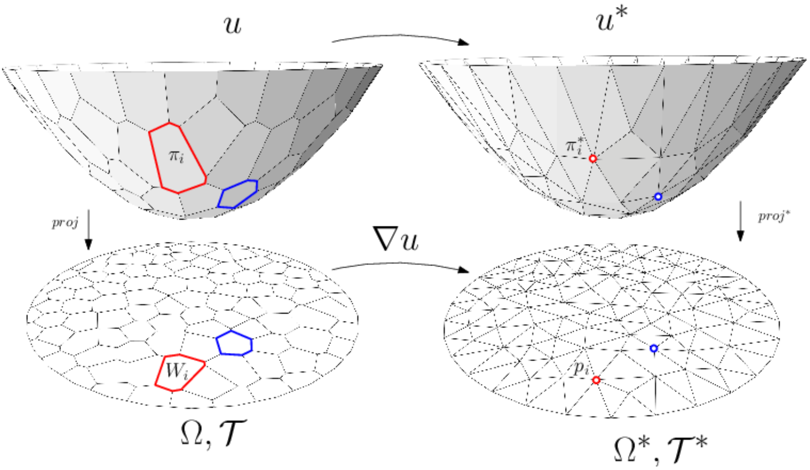
Legendre Transform

For generic f:



(Courtesy of N. Lei, W. Chen, Z. Luo, X. Gu 2019)

Alexandrov Map I

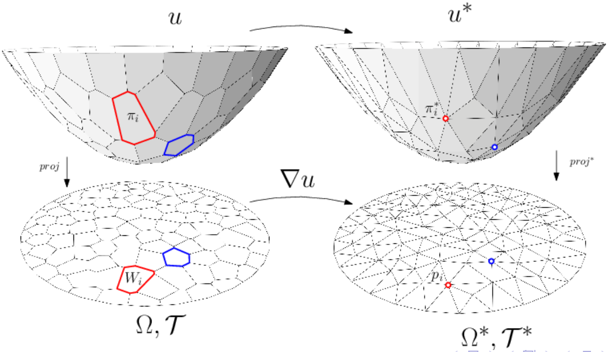
Recall that P is fixed and we only vary f. Fix a domain Ω and define $\nu : P \to [0, \infty)$, by

$$\nu(\rho) = |X(\rho) \cap \Omega|.$$

$X(\rho)$ is the set of slopes of subgradients (generized tangents) to the graph of u^* at ρ.

If ν is known, then we can recover f (and hence u) from it in Ω.

Alexandrov Map I The inverse map $\nu \mapsto u$.



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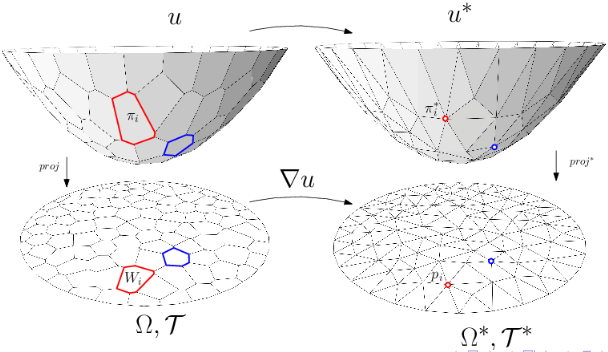

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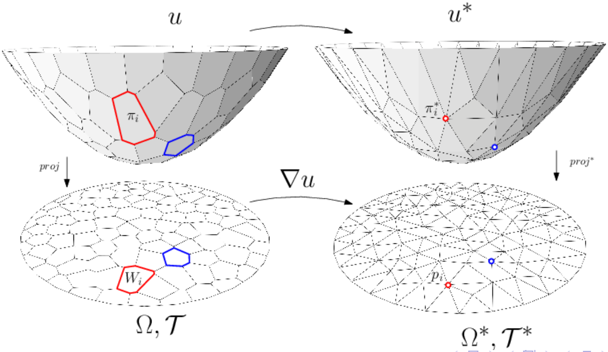

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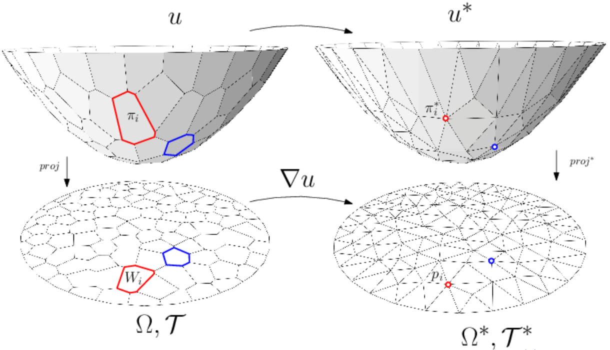

Discrete Gauss Curvature

Let $x \in X(\rho)$, then

$$N(x) = (1 + |x|^2)^{-1/2}(x, -1),$$

is normal to a face of the graph.
Define

$$\hat{X}(\rho) = \{N(x) : \; x \in X(\rho) \cap \Omega\} \subset \mathbb{S}^d_-.$$

Think of $\rho \mapsto \hat{X}(\rho)$ as a discrete Gauss map. Define

$$\alpha(\rho) = \sigma(\hat{X}(\rho)),$$

where σ is the d-dimensional (surface) area on the sphere. When $\Omega = \mathbb{R}^d$, then $\alpha(\rho)$ is our candidate for the Gauss curvature at ρ.

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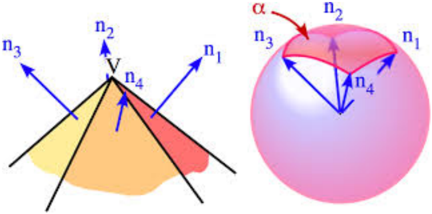
If α is known, then we can recover f (and hence u) from it in Ω.

Alexandrov Map II The inverse map $\alpha \mapsto u$.



Write λ_1 for the Lebesgue measure on Ω.

Write λ_2 for the pull back of σ with respect to $x \mapsto N(x)$.

Important Observation

1. The locally constant $\rho = \nabla u$ pushes forward λ_1 to

$$\mu_1 = \sum_{\rho \in P} \nu(\rho)\delta_\rho.$$

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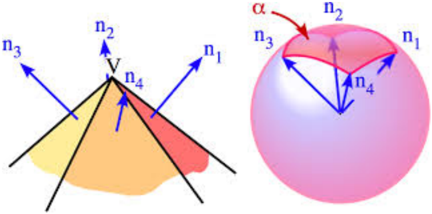

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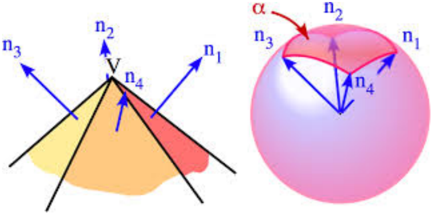

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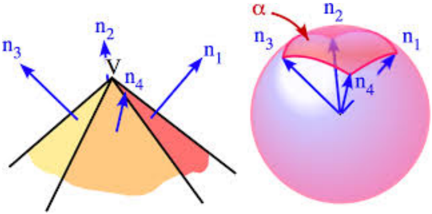

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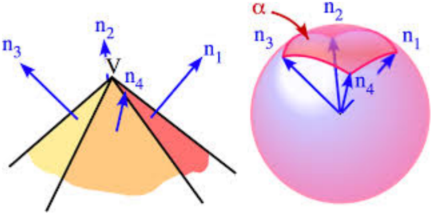

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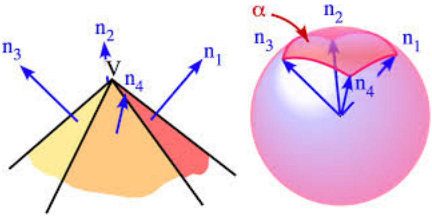

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Monge-Kantorovich Problem and Duality

Brenier: Given two measures λ and μ, there exists a unique (modulo a constant) convex function $u : \Omega \to \mathbb{R}$ such that $\rho = \nabla u$ pushes forward λ to μ.

Moreover ρ is a minimizer in

$$I(\mu)(= I(\lambda, \mu)) := \inf \frac{1}{2} \int_\Omega |x - \rho(x)|^2 \, \lambda(dx).$$

Infimum over maps ρ that pushes forward λ to μ.

Dual Formulation There is a dual presentation that is achieved by introducing a Lagrange multiplier and applying the minimax principle:

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Dual Problem

$$I(\mu) = \sup \left\{ \int \phi(x)\lambda(dx) + \int \psi(\rho)\, \mu(d\rho) \right\},$$

where the supremum is over pairs (ϕ, ψ) such that

$$\varphi(x) + \psi(\rho) \leq \frac{1}{2}|x - \rho|^2 \text{ for all } (x, \rho).$$

For each pair (φ, ψ), we define (u, v) as

$$u(x) = \frac{1}{2}|x|^2 - \varphi(x), \quad v(\rho) = \frac{1}{2}|\rho|^2 - \psi(\rho).$$

We then define

$$\hat{I}(\mu) = \sup \left\{ -\int u(x)\lambda(dx) - \int v(\rho)\, \mu(d\rho) \right\}$$

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Dual Problem

These optimization problems are equivalent:

$$I(\mu) = \hat{I}(\mu) + \int \frac{1}{2}|x|^2 \, \lambda(dx) + \int \frac{1}{2}|\rho|^2 \, \mu(d\rho).$$

The maximizing pair (u, v) satisfies $u = v^*$, and u is the desired convex function.

This suggests a functional

$$E(v) = \int v^*(x) \, \lambda(dx),$$

which is convex. In terms of this functional,

$$\hat{I}(\mu) = \sup_v \left(-\mu \cdot v - E(v)\right) = E^*(-\mu).$$

In summary the inverse of the map $f \mapsto \nu$ is given by $\nu = -\nabla E(f)$.

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Hamilton-Jacobi Dynamics

We are interested in the PDE $u_t = H(u_x)$ with $u(x,0)$ convex and piecewise linear.

Write $\mathcal{C}(P)$ for the set of functions of the form $u = f^*$ where $f : P \to \mathbb{R}$.

Write Φ_t for the flow associated with the PDE:

$$\Phi_t u(\cdot, 0) = u(\cdot, t).$$

The set $\mathcal{C}(P)$ is invariant under the flow by Hopf's theorem:

$$\Phi_t(\mathcal{C}(P)) \subset \mathcal{C}(P).$$

Indeed, if $f^t = f - tH$, then by Hope's formula,

$$\Phi_t(f^*) = (f^t)^*.$$

Set

$$\mathbf{X}_t = \mathbf{X}(f^t), \quad \mathbf{T}_t = \mathbf{T}(f^t).$$

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We are interested in the PDE $u_t = H(u_x)$ with $u(x, 0)$ convex and piecewise linear.
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We wish to understand the dynamics of $t \mapsto \mathbf{X}_t$ and $t \mapsto \mathbf{T}_t$.
Without loss of generality we may assume that P is finite.
(Speed of propagation is finite.)
Main Theorem: There are times

$$t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = \infty,$$

such that
1. In $(t_i, t_{k=1})$, we have a free motion.
2. At transition

$$t_i- \to t_i+,$$

we either have a coagulation or collision.
3. For $t > t_k$, the triangulation associated with f^t is very special (stable). We call it anti-H triangulation.
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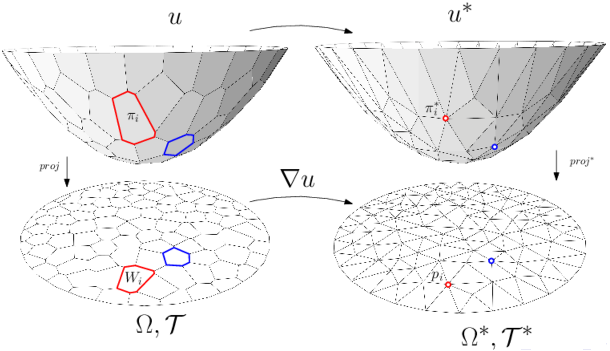
Hamilton-Jacobi Dynamics: Free Motion

During a free motion interval:

u^*: The triangulation (domains of linearity of u^*) $\mathbf{T}_t$ stays put, but the slopes of the graph of u^* change linearly with a velocity that will be described shortly.

u: The slopes of the graph stay put. The vertices of $\mathbf{X}_t$ travel according to their velocities. If t, t' are two times in the interval, then the corresponding faces in $\mathbf{X}_t$ and $\mathbf{X}_{t'}$ are parallel. Angles do not change.



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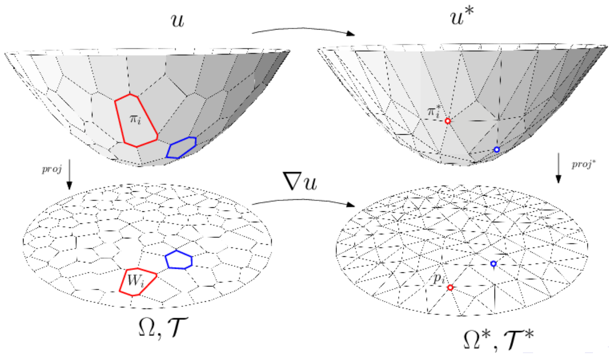

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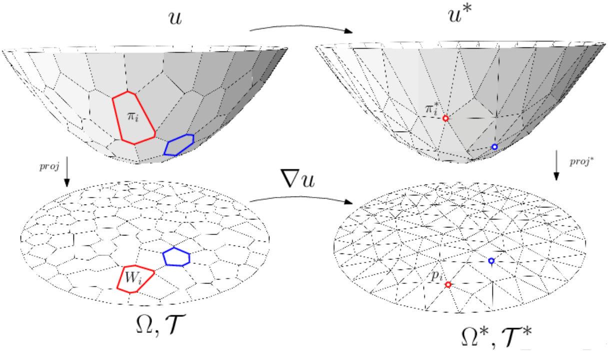

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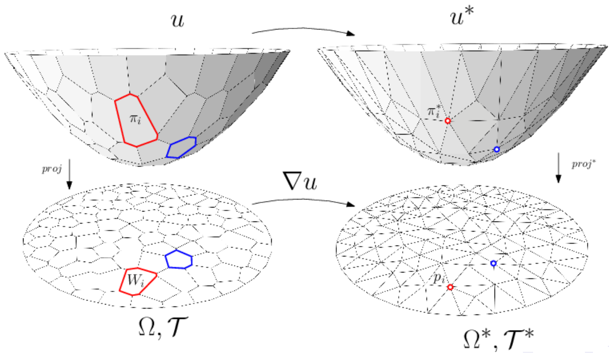
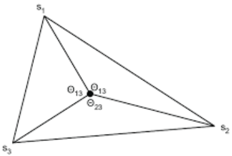


Hamilton-Jacobi Dynamics: Coagulation

u^*: Before t_i, there is a subtraingulation with $d+1$ triangles/simplexes as in the figure:



After t_i the $d+1$ simplexes are replaced with one simplex (their union).

u: Before t_i one cell in the tessellation $\mathbf{X}_t$ is a simplex/triangle. This cell shrinks before t_i. At t_i the cell collapses to a vertex.

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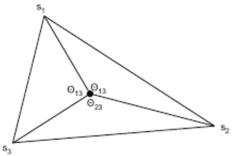

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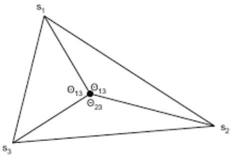

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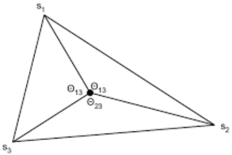

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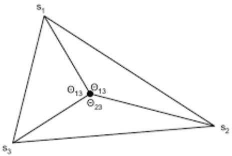

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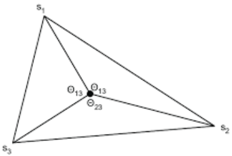

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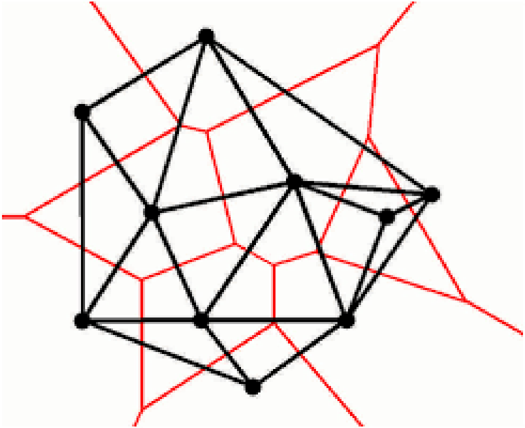
The red triangle shrinks: Triangles in $\mathbf{X}_t$ can only shrink (not true for other type of cells).



Hamilton-Jacobi Dynamics: Collision

u^*: Before t_i, there is a circuit D with $d+2$ extreme points. There are exactly two possible triangulations for this circuit, say $\mathbf{T}^{\pm}$. At t_i we switch from $\mathbf{T}^{-}$ to $\mathbf{T}^{+}$.

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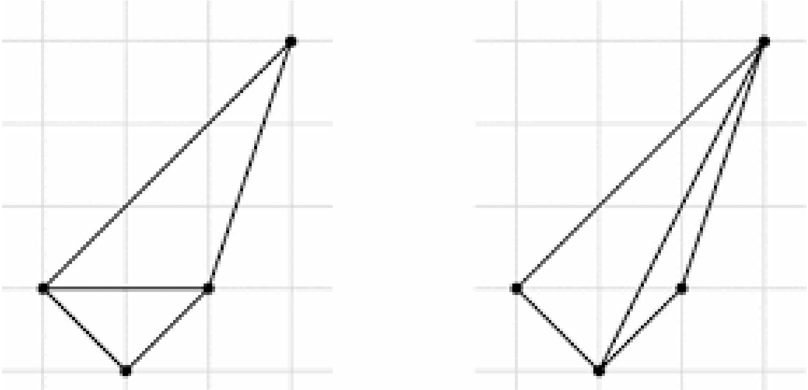
At t_i, these vertices collide and gain new velocities.

After t_i these vertices travel according to their new velocities.



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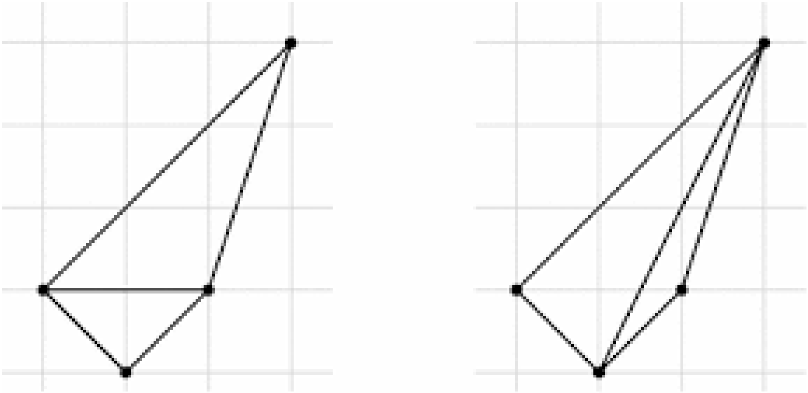

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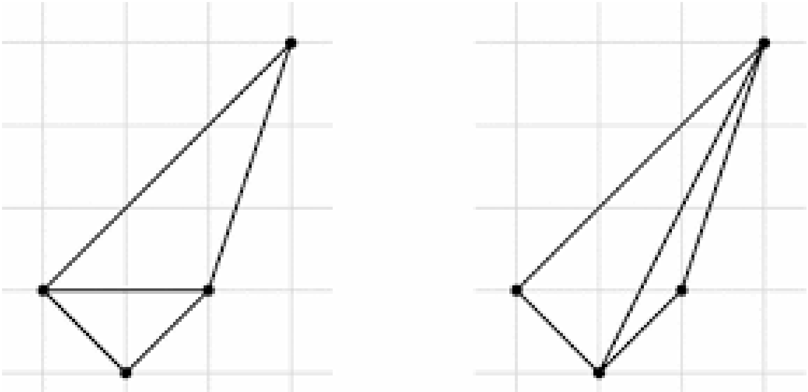

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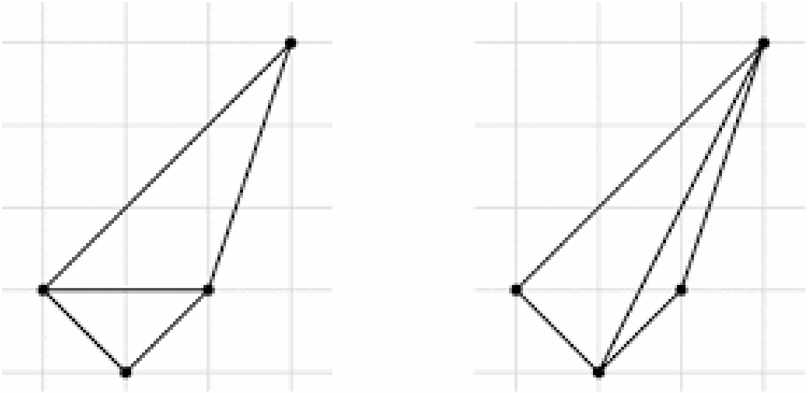
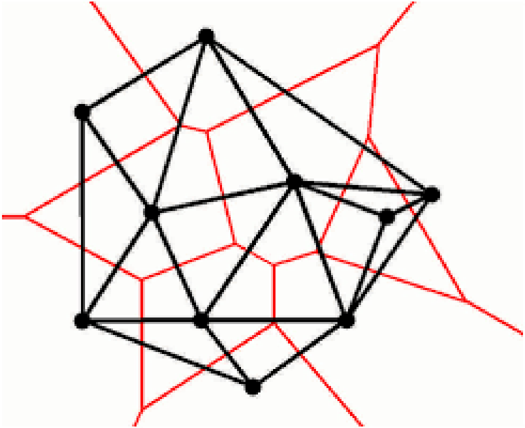


Hamilton-Jacobi Dynamics: Collision

Two red vertices may get closer or move away from each other.



Hamilton-Jacobi Dynamics: Velocities

Remarks

1. $X(\rho) \cap X(\rho')$ is a common face of $X(\rho)$ and $X(\rho')$.
The vector $\rho - \rho' \perp X(\rho) \cap X(\rho')$ (In dimension one this is known as Rankine-Hugoniot Formula).
It points from $X(\rho')$ side to $X(\rho)$ side (this is entropy condition/viscosity criteria).

2. If T is a triangle/simplex in the triangulation, then it is associated with a vertex $x(T) = x^t(T)$ that is uniquely determined from solving

$$x^t(T) \cdot (\rho - \rho') = f^t(\rho) - f^t(\rho'), \qquad \rho, \rho' \in T.$$

3. The velocity of $x^t(T)$ is $-v(T)$, where $v(T)$ is the unique solution of the linear system

$$v(T) \cdot (\rho - \rho') = H(\rho) - H(\rho'), \qquad \rho, \rho' \in T.$$

Moral: v is a vertex in the tessellation $\mathbf{X}(H)$.

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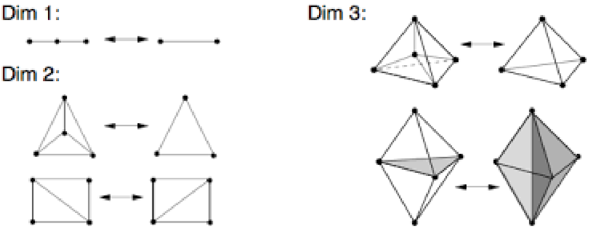
Hamilton-Jacobi Dynamics: Circuits

If R is a circuit, then there exists a function $c : R \to (0, \infty)$ and a decomposition $R = R^- \cup R^+$ such that

$$\sum_{m \in R^\pm} c(m) = 1,$$

$$a := \sum_{m \in R^-} c(m)m = \sum_{m \in R^+} c(m)m.$$



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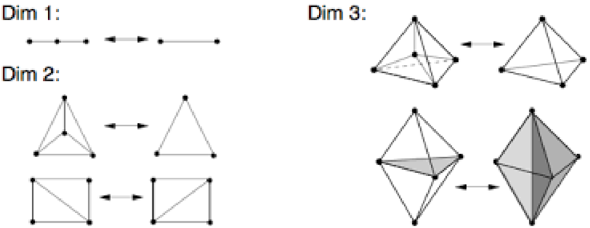

Hamilton-Jacobi Dynamics: Positive Edges

There are two triangulations:

$$\mathbf{T}^{\pm}(R) = \left\{R \setminus \{m\} : \; m \in R^{\mp}\right\}.$$

Choose $\pm$ so that

$$\hat{H}(R) = \sum_{m \in R^{+}} c(m)H(m) - \sum_{m \in R^{-}} c(m)H(m) \geq 0.$$

In this way the restriction of H to R is associated with the triangulation $\mathbf{T}^{-}(R)$.

If two triangulations $\mathbf{T}$ and $\mathbf{T}'$ are vertices of an edge of the secondary polytope, then they differ only on a circuit R.

We call the edge positive if $\mathbf{T} \to \mathbf{T}'$ means switching from $\mathbf{T}^{-}(R)$ to $\mathbf{T}^{+}(R)$.

In the HJ dynamics we can only jump across a positive edge at t_i.

Hamilton-Jacobi Dynamics: Positive Edges

There are two triangulations:

$$\mathbf{T}^{\pm}(R) = \left\{ R \setminus \{m\} : \; m \in R^{\mp} \right\}.$$

Choose $\pm$ so that

$$\hat{H}(R) = \sum_{m \in R^{+}} c(m)H(m) - \sum_{m \in R^{-}} c(m)H(m) \geq 0.$$

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In this way the restriction of H to R is associated with the triangulation $\mathbf{T}^{-}(R)$.
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Hamilton-Jacobi Dynamics: Positive Edges

There are two triangulations:

$$\mathbf{T}^{\pm}(R) = \{R \setminus \{m\} : \; m \in R^{\mp}\}.$$

Choose $\pm$ so that

$$\hat{H}(R) = \sum_{m \in R^+} c(m)H(m) - \sum_{m \in R^-} c(m)H(m) \geq 0.$$

In this way the restriction of H to R is associated with the triangulation $\mathbf{T}^-(R)$.
If two triangulations $\mathbf{T}$ and $\mathbf{T}'$ are vertices of an edge of the secondary polytope, then they differ only on a circuit R.
We call the edge positive if $\mathbf{T} \to \mathbf{T}'$ means switching from $\mathbf{T}^-(R)$ to $\mathbf{T}^+(R)$.
In the HJ dynamics we can only jump across a positive edge at t_i.

Hamilton-Jacobi Dynamics: Positive Edges

There are two triangulations:

$$\mathbf{T}^{\pm}(R) = \left\{ R \setminus \{m\} : \; m \in R^{\mp} \right\}.$$

Choose $\pm$ so that

$$\hat{H}(R) = \sum_{m \in R^{+}} c(m)H(m) - \sum_{m \in R^{-}} c(m)H(m) \geq 0.$$

In this way the restriction of H to R is associated with the triangulation $\mathbf{T}^{-}(R)$.
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Hamilton-Jacobi Dynamics: Coagulation/Collision

1. The time of a coagulation of a shrinking $f : R \to \mathbb{R}$:

$$\tau = \frac{\hat{f}(R)}{\hat{H}(R)}.$$

2. If $f : R \to \mathbb{R}$, and $\hat{f}(R) < 0$, then the triangulation induced by f is $\mathbf{T}^{+}(R)$ and there will be no collision.
3. If $f : R \to \mathbb{R}$, and $\hat{f}(R) > 0$, then the triangulation induced by f is $\mathbf{T}^{-}(R)$, and collision occurs at

$$\tau = \frac{\hat{f}(R)}{\hat{H}(R)}.$$

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