Kinetic Description of Hamilton-Jacobi PDE I

Fraydoun Rezakhanlou

Department of Mathematics
UC Berkeley

PDE/Probability Student Seminar

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Outline

Motivation

Convex Duality

Tessellation and Triangulation

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Poisson-Laguerre Point Process

Motivation

In many models of interest we encounter an interface that separates different phases and is evolving with time. The interface at a location x and time t changes with a rate that depends on (x, t), and the inclination of the interface at that location. If the interface is represented by a graph of a function $(x, t) \mapsto u(x, t), \quad u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}, \quad$ then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x), \qquad u(x, 0) = g(x).$$

(In discrete setting some of the variables x, t or u are discrete.) H is often random (hence u is random), and we are interested in various scaling limits of solutions.

Motivation

In many models of interest we encounter an interface that separates different phases and is evolving with time. The interface at a location x and time t changes with a rate that depends on (x, t), and the inclination of the interface at that location. If the interface is represented by a graph of a function $(x, t) \mapsto u(x, t), \quad u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R},$ then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x), \qquad u(x, 0) = g(x).$$

(In discrete setting some of the variables x, t or u are discrete.) H is often random (hence u is random), and we are interested in various scaling limits of solutions.

Motivation

In many models of interest we encounter an interface that separates different phases and is evolving with time. The interface at a location x and time t changes with a rate that depends on (x, t), and the inclination of the interface at that location. If the interface is represented by a graph of a function $(x, t) \mapsto u(x, t)$, $\quad u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$, then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x), \qquad u(x, 0) = g(x).$$

(In discrete setting some of the variables x, t or u are discrete.) H is often random (hence u is random), and we are interested in various scaling limits of solutions.

Motivation

In many models of interest we encounter an interface that separates different phases and is evolving with time. The interface at a location x and time t changes with a rate that depends on (x, t), and the inclination of the interface at that location. If the interface is represented by a graph of a function $(x, t) \mapsto u(x, t), \quad u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}, \quad$ then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x), \qquad u(x, 0) = g(x).$$

(In discrete setting some of the variables x, t or u are discrete.) H is often random (hence u is random), and we are interested in various scaling limits of solutions.

Motivation

In many models of interest we encounter an interface that separates different phases and is evolving with time. The interface at a location x and time t changes with a rate that depends on (x, t), and the inclination of the interface at that location. If the interface is represented by a graph of a function $(x, t) \mapsto u(x, t), \quad u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}, \quad$ then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x), \qquad u(x, 0) = g(x).$$

(In discrete setting some of the variables x, t or u are discrete.) H is often random (hence u is random), and we are interested in various scaling limits of solutions.

A Natural Question

Select g according to a (reasonable) probability measure μ^0.

Let us write μ^t for the law of $u(\cdot, t)$ at time t. Note: If Φ_t is the flow (in other words $u(\cdot, t) = (\Phi_t g)(\cdot)$), then $\mu^t = \Phi_t^* \mu^0$.

Question: Can we find a nice/tractable/explicit evolution equation for μ^t?

We may also keep track of $\rho = u_x$ (more natural). The law of $\rho(\cdot, t)$ is denoted by ν^t. Equilibrium Measure: $\nu^t = \nu^0$.

A Natural Question

Select g according to a (reasonable) probability measure μ^0. Let us write μ^t for the law of $u(\cdot, t)$ at time t. Note: If Φ_t is the flow (in other words $u(\cdot, t) = (\Phi_t g)(\cdot)$), then $\mu^t = \Phi_t^* \mu^0$.

Question: Can we find a nice/tractable/explicit evolution equation for μ^t?

We may also keep track of $\rho = u_x$ (more natural). The law of $\rho(\cdot, t)$ is denoted by ν^t. Equilibrium Measure: $\nu^t = \nu^0$.

A Natural Question

Select g according to a (reasonable) probability measure μ^0. Let us write μ^t for the law of $u(\cdot, t)$ at time t. Note: If Φ_t is the flow (in other words $u(\cdot, t) = (\Phi_t g)(\cdot)$), then $\mu^t = \Phi_t^* \mu^0$.
Question: Can we find a nice/tractable/explicit evolution equation for μ^t?
We may also keep track of $\rho = u_x$ (more natural). The law of $\rho(\cdot, t)$ is denoted by ν^t. Equilibrium Measure: $\nu^t = \nu^0$.

A Natural Question

Select g according to a (reasonable) probability measure μ^0. Let us write μ^t for the law of $u(\cdot, t)$ at time t. Note: If Φ_t is the flow (in other words $u(\cdot, t) = (\Phi_t g)(\cdot)$), then $\mu^t = \Phi_t^* \mu^0$.

Question: Can we find a nice/tractable/explicit evolution equation for μ^t?

We may also keep track of $\rho = u_x$ (more natural). The law of $\rho(\cdot, t)$ is denoted by ν^t. Equilibrium Measure: $\nu^t = \nu^0$.

A Natural Question

Select g according to a (reasonable) probability measure μ^0. Let us write μ^t for the law of $u(\cdot, t)$ at time t. Note: If Φ_t is the flow (in other words $u(\cdot, t) = (\Phi_t g)(\cdot)$), then $\mu^t = \Phi_t^* \mu^0$.
Question: Can we find a nice/tractable/explicit evolution equation for μ^t?
We may also keep track of $\rho = u_x$ (more natural). The law of $\rho(\cdot, t)$ is denoted by ν^t. Equilibrium Measure: $\nu^t = \nu^0$.

First Step, Main Setting

Assume $H(x,t,p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x,0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.
Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g, \; f^{**} = f.$

First Step, Main Setting

Assume $H(x, t, p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x, 0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.
Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g, \; f^{**} = f$.

First Step, Main Setting

Assume $H(x, t, p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x, 0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.
Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g, \ f^{**} = f.$

First Step, Main Setting

Assume $H(x, t, p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x, 0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.

Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g, \; f^{**} = f.$

First Step, Main Setting

Assume $H(x,t,p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x,0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.
Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g, \ f^{**} = f.$

First Step, Main Setting

Assume $H(x, t, p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x, 0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.
Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g, \; f^{**} = f.$

First Step, Main Setting

Assume $H(x, t, p) = H(p)$ depends on p only:

$$u_t = H(u_x), \quad u(x, 0) = g(x).$$

This equation does not possess classical solutions in general. The theory of viscosity solutions offers a unique generalized solution for a given Lipschitz initial g. This solution has a variational description when either g or H is convex.
Recall

$$g^*(\rho) = \sup_x (x \cdot \rho - g(x))$$

$$f^*(x) = \sup_\rho (x \cdot \rho - f(\rho)),$$

$g^{**} = g,\ f^{**} = f$.

Hopf Formula (Convex Initial Data)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)

If g is convex, then

$$u(x,t) = (g^* - tH)^*(x).$$

More explicitly

$$u^*(\rho, t) = \sup_x \left(x \cdot \rho - g(x) - tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

Hopf Formula (Convex Initial Data)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If g is convex, then

$$u(x,t) = (g^* - tH)^*(x).$$

More explicitly

$$u^*(\rho, t) = \sup_x \left(x \cdot \rho - g(x) - tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

Hopf Formula (Convex Initial Data)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If g is convex, then

$$u(x,t) = (g^* - tH)^*(x).$$

More explicitly

$$u^*(\rho, t) = \sup_x \left(x \cdot \rho - g(x) - tH(\rho)\right),$$

$$u(x, t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

Hopf Formula (Convex Initial Data)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If g is convex, then

$$u(x,t) = (g^* - tH)^*(x).$$

More explicitly

$$u^*(\rho, t) = \sup_x \left(x \cdot \rho - g(x) - tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

Hopf Formula (Convex Initial Data)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If g is convex, then

$$u(x,t) = (g^* - tH)^*(x).$$

More explicitly

$$u^*(\rho, t) = \sup_x \left(x \cdot \rho - g(x) - tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)

If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.

Remark Define the semigroup $\Phi = \left(\Phi_t : t \geq 0\right)$, by $\Phi_t g(x) = u(x,t)$.

When H is convex, then Φ_t is strongly monotone:

If $(g_a : \ a \in A)$ is a family of initial data, then

$$\Phi_t \left(\sup_{a \in A} g_\alpha \right) = \sup_{a \in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.
Remark Define the semigroup $\Phi = (\Phi_t : t \geq 0)$, by
$\Phi_t g(x) = u(x,t)$.
When H is convex, then Φ_t is strongly monotone:
If $(g_a :\ a \in A)$ is a family of initial data, then

$$\Phi_t \left(\sup_{a \in A} g_\alpha \right) = \sup_{a \in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.
Remark Define the semigroup $\Phi = \left(\Phi_t : t \geq 0\right)$, by $\Phi_t g(x) = u(x,t)$.
When H is convex, then Φ_t is strongly monotone:
If $(g_a : \ a \in A)$ is a family of initial data, then

$$\Phi_t \left(\sup_{a \in A} g_\alpha\right) = \sup_{a \in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.
Remark Define the semigroup $\Phi = \left(\Phi_t : t \geq 0\right)$, by $\Phi_t g(x) = u(x,t)$.
When H is convex, then Φ_t is strongly monotone:
If $(g_a :\ a \in A)$ is a family of initial data, then

$$\Phi_t\left(\sup_{a\in A} g_\alpha\right) = \sup_{a\in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.
Remark Define the semigroup $\Phi = \big(\Phi_t : t \geq 0\big)$, by $\Phi_t g(x) = u(x,t)$.
When H is convex, then Φ_t is strongly monotone:
If $(g_a : \ a \in A)$ is a family of initial data, then

$$\Phi_t\left(\sup_{a\in A} g_\alpha\right) = \sup_{a\in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.
Remark Define the semigroup $\Phi = \big(\Phi_t : t \geq 0\big)$, by
$\Phi_t g(x) = u(x,t)$.
When H is convex, then Φ_t is strongly monotone:
If $(g_a :\ a \in A)$ is a family of initial data, then

$$\Phi_t \left(\sup_{a \in A} g_\alpha\right) = \sup_{a \in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

Hopf-Lax-Oleinik Formula (Convex H)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
If H is convex, then

$$u(x,t) = \sup_y \left(g(y) - tL\left(\frac{y-x}{t}\right)\right),$$

where $L = H^*$ is the Legendre transform of H.
Remark Define the semigroup $\Phi = \big(\Phi_t : t \geq 0\big)$, by $\Phi_t g(x) = u(x,t)$.
When H is convex, then Φ_t is strongly monotone:
If $(g_a :\ a \in A)$ is a family of initial data, then

$$\Phi_t \left(\sup_{a\in A} g_\alpha\right) = \sup_{a \in A} \Phi_t g_a.$$

This is an immediate consequence of HLO Formula.

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
Observe $u(x,t) = x \cdot \rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x \cdot \rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho \in \mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x \cdot \rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \left(\Phi_t \ell_\rho\right)(x) = \sup_\rho \left(\ell_\rho(x) + tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
Observe $u(x,t) = x \cdot \rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x \cdot \rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho \in \mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x \cdot \rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \left(\Phi_t \ell_\rho\right)(x) = \sup_\rho \left(\ell_\rho(x) + tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot,0) = g$)
Observe $u(x,t) = x\cdot\rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x\cdot\rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho\in\mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x\cdot\rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \left(\Phi_t \ell_\rho\right)(x) = \sup_\rho \left(\ell_\rho(x) + tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x\cdot\rho - g^*(\rho) + tH(\rho)\right),$$

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
Observe $u(x,t) = x \cdot \rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x \cdot \rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho \in \mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x \cdot \rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \left(\Phi_t \ell_\rho\right)(x) = \sup_\rho \left(\ell_\rho(x) + tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
Observe $u(x,t) = x \cdot \rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x \cdot \rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho \in \mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x \cdot \rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \left(\Phi_t \ell_\rho\right)(x) = \sup_\rho \left(\ell_\rho(x) + tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot,0) = g$)
Observe $u(x,t) = x \cdot \rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x \cdot \rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho \in \mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x \cdot \rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \left(\Phi_t \ell_\rho\right)(x) = \sup_\rho \left(\ell_\rho(x) + tH(\rho)\right),$$

$$u(x,t) = \sup_\rho \left(x \cdot \rho - g^*(\rho) + tH(\rho)\right),$$

HLO implies Hopf (Both g and H Convex)

(We are solving $u_t = H(u_x),\ u(\cdot, 0) = g$)
Observe $u(x,t) = x \cdot \rho + a + tH(\rho)$ is a solution for initial $u(x,0) = x \cdot \rho + a$.
g convex means $g = g^{**}$:

$$g = \sup_{\rho \in \mathbb{R}^d} \ell_\rho, \quad \text{with} \quad \ell_\rho(x) = x \cdot \rho - g^*(\rho),$$

Assume Strong Monotonicity:

$$u(x,t) = \sup_\rho \big(\Phi_t \ell_\rho\big)(x) = \sup_\rho \big(\ell_\rho(x) + tH(\rho)\big),$$

$$u(x,t) = \sup_\rho \big(x \cdot \rho - g^*(\rho) + tH(\rho)\big),$$

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:

If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,

then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)

In words, the flow Ψ is associated with a linear motion with velocity $-H$.

Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.

Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:

If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,

then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)

In words, the flow Ψ is associated with a linear motion with velocity $-H$.

Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.

Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Assumption: H and g Convex

Write $\mathcal{C}$ for the set of convex functions. It is an invariant set for the dynamics. When $g \in \mathcal{C}$ is convex, then Hopf Formula offers a rather simple dynamics for the evolution of $\Phi_t g$:
If we define $\Psi_t h := \left(\Phi_t h^*\right)^*$,
then

$$\Psi_t h = (h - tH)^{**} =: (h - tH)^o.$$

(f^o means Convex Hull of f)
In words, the flow Ψ is associated with a linear motion with velocity $-H$.
Since $h - tH$ may not be convex, we need to take its convex hull to stay in $\mathcal{C}$.
Observe that when both H and g are convex, then it is possible that $(h - tH)^o \neq h - tH$ for every $t > 0$. Indeed this would always be the case if g is piecewise linear and H is strictly convex. Nonetheless (as will see later on), there is a kinetic description for Ψ that would give a local description of the dynamics as opposed to what is given on the right-hand side that involves a convex hull.

Convex Functions (Legendre Transform/Convex Hull)

Take a function $h : \mathbb{R}^d \to (-\infty, +\infty]$.

$$P = Dom(h) := \{\rho : \ h(\rho) \neq \infty\}.$$

Define

$$u(x) = h^*(x) = \sup_{\rho} (x \cdot \rho - h(\rho)) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)).$$

Observe u is convex and lower semicontinuous (lsc).
Also $u^* = h^{**} = h^o$.

Convex Functions (Legendre Transform/Convex Hull)

Take a function $h : \mathbb{R}^d \to (-\infty, +\infty]$.

$$P = Dom(h) := \{\rho : \ h(\rho) \neq \infty\}.$$

Define

$$u(x) = h^*(x) = \sup_{\rho} (x \cdot \rho - h(\rho)) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)).$$

Observe u is convex and lower semicontinuous (lsc).
Also $u^* = h^{**} = h^o$.

Convex Functions (Legendre Transform/Convex Hull)

Take a function $h : \mathbb{R}^d \to (-\infty, +\infty]$.

$$P = Dom(h) := \{\rho : \; h(\rho) \neq \infty\}.$$

Define

$$u(x) = h^*(x) = \sup_{\rho} (x \cdot \rho - h(\rho)) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)).$$

Observe u is convex and lower semicontinuous (lsc).
Also $u^* = h^{**} = h^o$.

Convex Functions (Legendre Transform/Convex Hull)

Take a function $h : \mathbb{R}^d \to (-\infty, +\infty]$.

$$P = Dom(h) := \{\rho : \; h(\rho) \neq \infty\}.$$

Define

$$u(x) = h^*(x) = \sup_{\rho} (x \cdot \rho - h(\rho)) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)).$$

Observe u is convex and lower semicontinuous (lsc).
Also $u^* = h^{**} = h^o$.

Convex Functions (Legendre Transform/Convex Hull)

Take a function $h : \mathbb{R}^d \to (-\infty, +\infty]$.

$$P = Dom(h) := \{\rho : \ h(\rho) \neq \infty\}.$$

Define

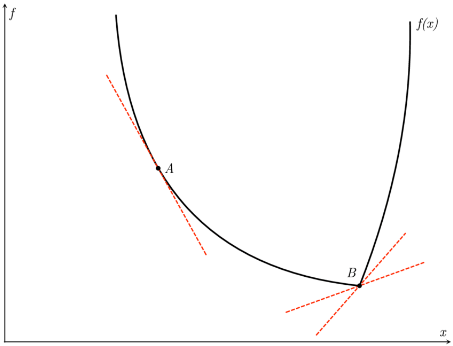
$$u(x) = h^*(x) = \sup_{\rho} (x \cdot \rho - h(\rho)) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)).$$

Observe u is convex and lower semicontinuous (lsc).
Also $u^* = h^{**} = h^o$.

Convex Functions (subgradient)

We write $\partial h(a)$ for the set of subgradients of h at a:

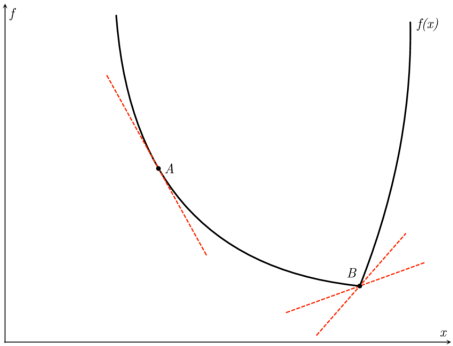
$$p \in \partial h(a) \quad \Leftrightarrow \quad h(\rho) \geq h(a) + (\rho - a) \cdot p.$$

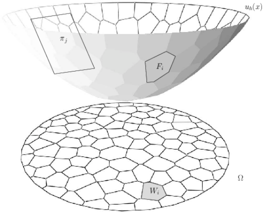


Convex Functions (subgradient)

We write $\partial h(a)$ for the set of subgradients of h at a:

$$p \in \partial h(a) \quad \Leftrightarrow \quad h(\rho) \geq h(a) + (\rho - a) \cdot p.$$



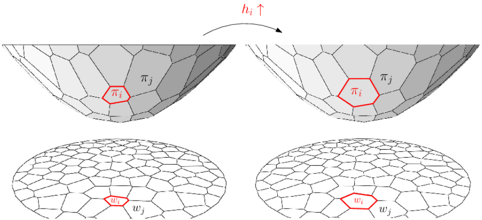


$u_h(x)$
π_j
F_i
Ω
W_i

Convex Functions (subgradient)

If h is convex and lsc, then

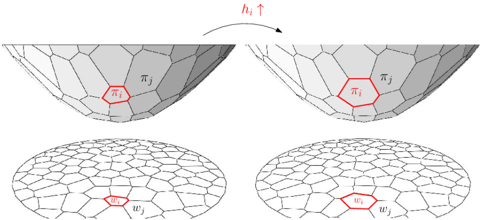
$$x \in \partial h(\rho) \quad \Leftrightarrow \quad \rho \in \partial h^*(x) \quad \Leftrightarrow \quad x \cdot \rho = h(\rho) + h^*(x).$$



Convex Functions (subgradient)

If h is convex and lsc, then

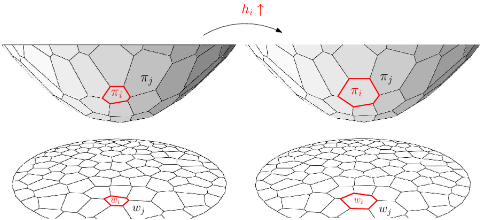
$$x \in \partial h(\rho) \quad \Leftrightarrow \quad \rho \in \partial h^*(x) \quad \Leftrightarrow \quad x \cdot \rho = h(\rho) + h^*(x).$$

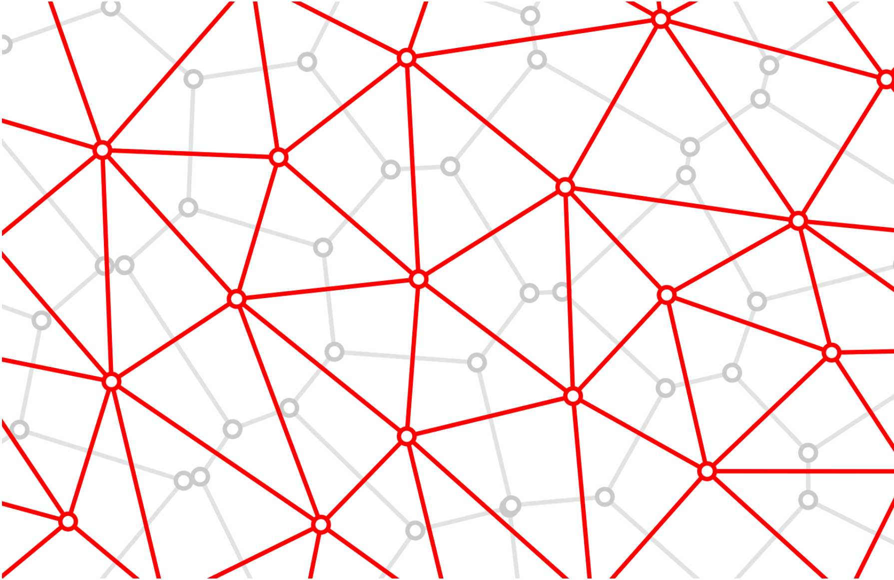


Convex Functions (subgradient)

If h is convex and lsc, then

$$x \in \partial h(\rho) \quad \Leftrightarrow \quad \rho \in \partial h^*(x) \quad \Leftrightarrow \quad x \cdot \rho = h(\rho) + h^*(x).$$





As Hopf's formula, it is more convenient to assume our convex function can be expressed as $u = h^*$, where $h : \mathbb{R}^d \to (-\infty, \infty]$ with $Dom(h) =: P$ a closed subset of $\mathbb{R}^d$.
We always assume that the restriction of h to the set P is continuous, and

$$\lim_{|\rho| \to \infty} \frac{h(\rho)}{|\rho|} = \infty,$$

so that u is a finite-valued convex function.
We say the set P is *minimal*, if the set P cannot be replaced with any proper subset of P in

$$u(x) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)).$$

As Hopf's formula, it is more convenient to assume our convex function can be expressed as $u = h^*$, where $h : \mathbb{R}^d \to (-\infty, \infty]$ with $Dom(h) =: P$ a closed subset of $\mathbb{R}^d$.

We always assume that the restriction of h to the set P is continuous, and

$$\lim_{|\rho|\to\infty} \frac{h(\rho)}{|\rho|} = \infty,$$

so that u is a finite-valued convex function.

We say the set P is *minimal*, if the set P cannot be replaced with any proper subset of P in

$$u(x) = \sup_{\rho\in P} (x \cdot \rho - h(\rho)).$$

As Hopf's formula, it is more convenient to assume our convex function can be expressed as $u = h^*$, where $h : \mathbb{R}^d \to (-\infty, \infty]$ with $Dom(h) =: P$ a closed subset of $\mathbb{R}^d$.

We always assume that the restriction of h to the set P is continuous, and

$$\lim_{|\rho|\to\infty} \frac{h(\rho)}{|\rho|} = \infty,$$

so that u is a finite-valued convex function.

We say the set P is *minimal*, if the set P cannot be replaced with any proper subset of P in

$$u(x) = \sup_{\rho\in P} (x \cdot \rho - h(\rho)).$$

As Hopf's formula, it is more convenient to assume our convex function can be expressed as $u = h^*$, where $h : \mathbb{R}^d \to (-\infty, \infty]$ with $Dom(h) =: P$ a closed subset of $\mathbb{R}^d$.
We always assume that the restriction of h to the set P is continuous, and

$$\lim_{|\rho|\to\infty} \frac{h(\rho)}{|\rho|} = \infty,$$

so that u is a finite-valued convex function.
We say the set P is *minimal*, if the set P cannot be replaced with any proper subset of P in

$$u(x) = \sup_{\rho\in P} (x \cdot \rho - h(\rho)).$$

Laguerre Tesselation

$$\mathbb{X}(h) = \mathbb{X} = \left\{\partial u^*(\rho) : \ \rho \in \mathbb{R}^d\right\}$$

$$\mathbb{P}(h) = \mathbb{P} = \left\{\partial u(x) : \ x \in \mathbb{R}^d\right\}.$$

For each m and a, the sets $\partial u^*(m)$ and $\partial u(a)$ are convex.

Laguerre Tesselation

$$\mathbb{X}(h) = \mathbb{X} = \left\{\partial u^*(\rho) : \ \rho \in \mathbb{R}^d\right\}$$

$$\mathbb{P}(h) = \mathbb{P} = \left\{\partial u(x) : \ x \in \mathbb{R}^d\right\}.$$

For each m and a, the sets $\partial u^*(m)$ and $\partial u(a)$ are convex.

Laguerre Tesselation

$$\mathbb{X}(h) = \mathbb{X} = \left\{\partial u^*(\rho) : \ \rho \in \mathbb{R}^d\right\}$$

$$\mathbb{P}(h) = \mathbb{P} = \left\{\partial u(x) : \ x \in \mathbb{R}^d\right\}.$$

For each m and a, the sets $\partial u^*(m)$ and $\partial u(a)$ are convex.

Laguerre Tesselation

$$\mathbb{X}(h) = \mathbb{X} = \left\{\partial u^*(\rho) : \ \rho \in \mathbb{R}^d\right\}$$

$$\mathbb{P}(h) = \mathbb{P} = \left\{\partial u(x) : \ x \in \mathbb{R}^d\right\}.$$

For each m and a, the sets $\partial u^*(m)$ and $\partial u(a)$ are convex.