Regular Flows for Diffusions with Rough Drifts

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Abstract

According to DiPerna-Lions theory, velocity fields with weak derivatives in L^p spaces possess weakly regular flows. When a velocity field is perturbed by a white noise, the corresponding (stochastic) flow is far more regular in spatial variables; a d-dimensional diffusion with a drift in $L^{r,q}$ space (r for the spatial variable and q for the temporal variable) possesses weak derivatives with stretched exponential bounds, provided that $r/d + 2/q < 1$. As an application we show that a Hamiltonian system that is perturbed by a white noise produces a symplectic flow provided that the corresponding Hamiltonian function H satisfies $\nabla H \in L^{r,q}$ with $r/d + 2/q < 1$. As our second application we derive a Constantin-Iyer type circulation formula for certain weak solutions of Navier-Stokes equation.

1 Introduction

The velocity field of an incompressible inviscid fluid is modeled by Incompressible Euler Equation

$$u_t + (Du)u + \nabla P = 0, \quad \nabla \cdot u = 0, \tag{1.1}$$

where $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ represents the velocity field and $P : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is the pressure. Here and below we write Du and ∇P for the x-derivatives of the vector field u and the scalar-valued function P respectively. In the Lagrangian formulation of the fluid,

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we interpret u as the velocity of generic fluid particles and its flow $X : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$, defined by

$$\frac{d}{dt}X(a,t) = u(X(a,t),t), \quad X(a,0) = a, \tag{1.2}$$

plays a crucial role in understanding the regularity of solutions of the equation (1.1). Since a solution of (1.1) could be singular, we need to examine the regularity of the flow X of ordinary differential equations associated with rough vector fields. Classically, a Lipschitz continuous vector field u results in a Lipschitz flow. In a prominent work [DL], DiPerna and Lions constructed a unique flow for (1.2) provided that $u \in W^{1,p}$ and $\nabla \cdot u \in L^\infty$, for some $p \geq 1$. In 2004, Ambrosio [A] extended this result to the case of a vector field u of bounded variation. Recently DeLellis and Crippa [CD] obtained a logarithmic control on the L^p–modulus of continuity of the flow in spatial variable provided that $p > 1$.

In the case of an incompressible viscid fluid, the velocity field u satisfies the celebrated Navier-Stokes equation

$$u_t + (Du)u + \nabla p(x,t) = \nu \Delta u, \quad \nabla \cdot u = 0. \tag{1.3}$$

In the corresponding Lagrangian description, a fluid particle motion is now modeled by a stochastic differential equation (SDE) of the form

$$dX = u(X,t)\ dt + \sigma dB, \tag{1.4}$$

where $\sigma = \sqrt{2\nu}$ and B denotes the standard Brownian motion. Since the regularity of solutions to Navier-Stokes equation is a long-standing open problem, we would like to study the regularity of the stochastic flow of SDE (1.4) and use such regularity to study (1.3). As it turns out, the flow of SDE (1.4) is far more regular than its inviscid analog (1.2). To state the main result of this article, let us define

$$\|f\|_{r,q} := \|f\|_{L^{r,q}} := \left[\int_0^T \left(\int_{\mathbb{R}^d} |f(x,t)|^r dx\right)^{q/r} dt\right]^{1/q} = \left[\int_0^T \|f(\cdot,t)\|^q_{L^r(\mathbb{R}^d)} dt\right]^{1/q}.$$

The space of functions with $\|u\|_{r,q} < \infty$ is denoted by $L^{r,q}$. We write $\mathbb{P}$ and $\mathbb{E}$ for the probability measure and expectation associated with SDE (1.4).

Theorem 1.1 *Assume that $\sigma > 0$ and $u \in L^{r,q}$ for some $q \in (2, \infty], r \in (d, \infty]$, satisfying*

$$\delta_1 := \frac{1}{2} - \frac{d}{2r} - \frac{1}{q} > 0. \tag{1.5}$$

Then SDE (1.4) has a flow X that is weakly differentiable with respect to the spatial variable. Moreover, there exist positive constants $C_1 = C_1(r,q)$ and $C_0 = C_0(r,q)$ such that for every $p \geq 1$,

$$\sup_a \mathbb{E}\left[|D_aX(a,t)|^p + \left|(D_aX(a,t))^{-1}\right|^p\right] \leq C_0^p \exp\left(C_1 \sigma^{-\frac{1}{\delta_1}\left(1+\frac{d}{r}\right)} p^{\frac{1}{\delta_1}} \|u\|_{r,q}^{\frac{1}{\delta_1}}\ t\right). \tag{1.6}$$

The following consequence of Theorem 1.1 allows us to go beyond the p-th moment and gives an almost Lipschitz regularity of the flow in the spatial variable.

Corollary 1.1 *There exist positive constants $C_1' = C_1'(r,q)$ and $C_2 = C_2(r,q;\ell)$ such that*

$$\sup_a \mathbb{P}\left(|D_a X(a,t)| \geq \lambda\right) \leq \exp\left(-C_1' \sigma^{\frac{1}{1-\delta_1}\left(1+\frac{d}{r}\right)} \|u\|_{r,q}^{-\frac{1}{1-\delta_1}}\ t^{-\frac{\delta_1}{1-\delta_1}} \left(\log^+ \frac{\lambda}{C_0}\right)^{\frac{1}{1-\delta_1}}\right), \tag{1.7}$$

$$\mathbb{E} \sup_{\substack{|a-b|\leq\delta \\ |a|,|b|\leq\ell}} |X(a,t)-X(b,t)| \leq C_2\ \delta \exp\left(C_2 \left(\log\frac{\ell}{\delta}\right)^{1-\delta_1} \sigma^{-\left(1+\frac{d}{r}\right)} \|u\|_{r,q}\ t^{\delta_1}\right), \tag{1.8}$$

for every u and X as in Theorem 1.1.

As another application of (1.6), we can show that the flow is jointly Hölder continuous in both x and t variables. Define

$$S_{T,\ell}(X;\delta) := \sup_{\substack{|s-t|\leq\delta \\ 0\leq s,t\leq T}} \sup_{\substack{|a-b|\leq\delta \\ |a|,|b|\leq\ell}} |X(a,t)-X(b,t)|$$

Corollary 1.2 *For every $\alpha \in (0,1/2)$, there exists a constant $C_2' = C_2'(r,q;\ell,T;\alpha)$ such that*

$$\mathbb{E}\ S_{T,\ell}(X;\delta) \leq C_2'\ \delta^\alpha \exp\left(C_2' \sigma^{-\frac{1}{\delta_1}\left(1+\frac{d}{r}\right)} \|u\|_{r,q}^{\frac{1}{\delta_1}}\right), \tag{1.9}$$

for every $\delta > 0$, and u and X as in Theorem 1.1.

Remark 1.1 Clearly our bounds (1.6) and (1.7) are vacuous when $\delta_1 = 0$. Nonetheless we conjecture that some variants of these bounds would still be true when $\delta_1 = 0$, or even when $\delta_1 < 0$. Though we do not expect to have bounds that are uniform in a. □

One of our main motivation behind Theorem 1.1 is its potential applications in Symplectic Topology. It also allows us to formulate Navier-Stokes Equation geometrically. To explain this note that (1.6) allows us to sense of the pull-back $X_t^*\beta$, for any differential form β, where $X_t(\cdot) = X(\cdot,t)$. Let us explain this in the case of a 1-form. If $\beta = w\cdot dx$, or equivalently $\beta(x;v) = w(x)\cdot v$, then

$$X_t^*\beta(x;v) = \beta(X_t(a); D_a X_t(a) v) = (D_a X_t(a))^*\, w(X_t(a))\cdot v,$$

is all well defined. In fact if $w \in L^\infty_{loc}$, then $X_t^* w := (DX_t)^*\big(w\circ X_t\big) \in L^p_{loc}$ for every $p\in[1,\infty)$. In the case that $w\in C^2$, we can make sense of $\mathcal{A}_u\beta$, where $\mathcal{A}_u = \mathcal{L}_u + \nu\Delta$ with $\mathcal{L}_u$ denoting the Lie derivative and

$$\Delta\left(\sum_i w^i\ dx^i\right) = \sum_i (\Delta w^i)\ dx^i.$$

The following theorem explains the role of the operator $\mathcal{A}_u$.

Theorem 1.2 *Let X be the flow of SDE (1.4) with $u \in L^{r,q}$ for some r and q satisfying (1.5). Given $\beta^t = w(\cdot,t) \cdot dx$, with $w(\cdot,t) \in C^2$ and $w(x,\cdot) \in C^1$, the process*

$$M^t = X_t^* \beta^t - \beta^0 - \int_0^t X_s^* \left[\dot{\beta}^s + \mathcal{A}_u \beta^s \right] \, ds,$$

is a martingale. (Here $\dot{\beta}^t = w_t(\cdot,t) \cdot dx$.) More precisely,

$$M^t = \int_0^t \sum_{i=1}^d X_s^* \gamma_i^s \, dB^i(s), \tag{1.10}$$

where $\gamma_i^s = w_{x^i}(\cdot,s) \cdot dx$.

Let us be more precise about the meaning of martingales in our setting. Observe that M^t is a 1-form for each t and we may regard $M^t = M^t(x)$ as a vector-valued function for each t. By Theorem 1.1, this function is locally in L^p for every $p \in [1,\infty)$. Now M^t is a martingale in the following sense: If $V(x)$ is a C^1 vector field of compact support, then the process

$$M_t(V) := \int_{\mathbb{R}^d} M^t(x) \cdot V(x) \, dx, \tag{1.11}$$

is a martingale. As we will see in Section 5, the expression $\mathcal{A}_u \beta^s$ is well-defined weakly; only after an integration by parts we can make sense of $M_t(V)$. To explain this, recall that by Cartan's formula

$$\mathcal{L}_u \beta = \hat{d} i_u \beta + i_u \hat{d} \beta$$

where we are writing $\hat{d}$ for the exterior derivative and i_u denotes the contraction operator in the direction u. (To avoid confusion with stochastic differential, we use a hat for exterior derivative.) Since $w \in C^1$, we have no problem to define $i_u \hat{d}\beta$. However we need differentiability of u to make sense $\hat{d} i_u \beta$ classically. The differentiability of u can be avoided if we integrate against a C^1 function because

$$\int \hat{d} i_u \beta(V) \, dx = - \int \beta(u) \, (\nabla \cdot V) \, dx.$$

Let us write

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where I_n denotes the $n \times n$ identity matrix. As a straight forward consequence of Theorem 1.2, we have Corollary 1.1.

Corollary 1.3 *Assume that $d = 2n$, and $u = J\nabla_x H$ in* (1.4) *for a function $H : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ that is weakly differentiable in x-variable and $\nabla_x H \in L^{q,r}$, for some q and r satisfying* (1.5). *Then the flow X_t is symplectic.*

Remark 1.2 In *Hofer Geometry* [H], if $\{H_n\}$ is a $L^{\infty,1}$-Cauchy sequence of Hamiltonian functions of compact support, the the corresponding flows $\{\phi^{H_n}\}$ is a Cauchy sequence of symplectic flows with respect to the *Hoper metric*. Completion of the group of such symplectic transformations with respect to the Hofer metric is not understood. In view of Corollary 1.2, we may wonder whether or not some kind of a limit exists for the family of the flows $\{X = X^\sigma : \sigma > 0\}$ as $\sigma \to 0$. □

As our next application, let us assume that u is a solution of the backward Navier-Stokes Equation:

$$u_t + (Du)u + \nabla P(x,t) + \nu\Delta u = 0, \quad \nabla \cdot u = 0. \tag{1.12}$$

(We use backward equation (1.12) instead of the forward equation (1.3) to simplify our presentation.) A more geometric formulation of (1.12) is achieved by writing an equation for the evolution of the 1-form $\alpha^t = u(\cdot,t)\cdot dx$:

$$\dot{\alpha}^t + \mathcal{A}_u\alpha^t - \hat{d}L^t = 0, \tag{1.13}$$

where $L^t(x) = \frac{1}{2}|u(x,t)|^2 - P(x,t)$. A natural way to approximate Navier-Stokes Equation is via *Camassa-Holm*-type equations of the form

$$v_t + (Dv)w + (Dw)^*v - \nabla_x\bar{L}(x,t) + \nu\Delta v = 0, \quad \nabla\cdot v = 0, \quad w = v *_x \zeta, \tag{1.14}$$

where $\zeta(x)$ is a smooth function. In the classical Camassa-Holm Equation, $v = w - \varepsilon\Delta w$ which leads to $u = v *_x \zeta^\varepsilon$. In this case both $v = v^\varepsilon$ and $w = w^\varepsilon$ depend on ε and according to a classical result of Foias et al. [FHT], the sequences $(w^\varepsilon, v^\varepsilon)$ are precompact in low ε limit and if (u,u) is any limit point, then u is a weak-solution of (1.12). We say u is a (r,q)-regular solution of (1.12) if it can be approximated by a sequence of solutions $(v^\varepsilon, w^\varepsilon)$ of Camassa-Holm equation such that

$$\sup_{\varepsilon>0}\|w^\varepsilon\|_{r,q} < \infty. \tag{1.15}$$

Theorem 1.3 *Let u be a (r,q)-regular solution of Navier-Stokes Equation* (1.12) *for some r and q satisfying* (1.5). *Then for any smooth divergence free vector field Z of compact support, the process*

$$\int_{\mathbb{R}^d} X_t^*\alpha^t(Z)\ dx, \tag{1.16}$$

is a martingale. Moreover, if $Du \in L^2$, then the process $X_t^*\hat{d}\alpha^t$ is also a martingale.

Remark 1.3 According to a classical result of Serrin [S], a weak solution of (1.3) is smooth if $u(\cdot, 0) \in L^2$ and $u \in L^{r,q}$ for some r and q satisfying (1.5). We may also use Theorem 1.3 to show that any (r, q)-regular solution is smooth. If we have equality in (1.5) and $r < \infty$, the regularity of solutions can be found in the work of Fabes, Jones and Riviere [FJR]. Based on this, it is natural to ask what type of regularity for the flow X is available in the extreme case $\delta_1 = 0$ (see Remark 1.1 above). We leave this for future investigation. $\square$

Here is a short review of various classical and recent results on SDE (1.4):

1. Classical Ito's theorem guarantees that (1.4) has a unique (strong) solution if u is Lipschitz continuous in spatial variable, uniformly in time.

2. By a yet another classical work of Bismut, Elworthy and Kunita (see for example [RW] or [K]), (1.4) has a smooth flow with smooth inverse if u is smooth.

3. Zvonkin [Z] in 1974 showed that (1.4) has a unique solution if $d = 1$ and $u \in L^{\infty,\infty}$. This result was extended to higher dimension by Veretynikov [V] in 1979.

4. Flandoli et al. [FGP] (2010) have shown that if u is Hölder-continuous of Hölder exponent α in spatial variable, then the flow X is also Hölder-continuous of Hölder exponent α' in spatial variable, for any $\alpha' < \alpha$.

5. Fedrizzi and Flandoli [FF] (2010) establish $X \in W^{1,p}$ for every $p \geq 2$, provided that $u \in L^{r,q}$ for some r and q satisfying (1.5). Though no bound on $D_a X$ is given in [FF].

6. Mohammad et al. [MNP] (2014) establish $\mathbb{E}|D_a X(a, t)|^p < \infty$ for every $p \geq 1$ provided that $u \in L^{\infty,\infty}$. $\square$

An important ingredient for the work of Mohammad et al. is a bound of Davie (see Theorem 2.1) that works for $u \in L^{\infty,\infty}$. In this paper we adopt [MNP] approach and achieve Theorem 1.1 by generalizing Davie's bound to the case $u \in L^{r,q}$ with r and q satisfying (1.5). In fact Davie proves such a bound by reducing it to a certain double integral. It is worth mentioning that such a reduction is applicable only if we assume a stronger condition

$$\delta_2 := \frac{1}{4} - \frac{d}{2r} - \frac{1}{q} > 0, \tag{1.17}$$

We refer to Subsection 4.2 for more details.

The organization of the paper is as follows:

- In Section 2 we establish Theorem 1.1 and its corollaries, assuming that a Davie-type bound (Theorem 2.1) is available under the assumption $\delta_1 > 0$.
- In Section 3 we reduce the proof of Theorem 2.1 to bounding certain block-type integrals (Theorem 3.1).
- Section 4 is devoted to the proof of Theorem 3.1.
- In Section 5 we discuss symplectic diffusions and prove Theorems 1.2 and 1.3.

2 Proof of Theorem 1.1 and Its Corollaries

As a preparation for the proof of Theorem 1.1, we state one theorem and two lemmas. We write $x^1, \dots, x^d$ for coordinates of x and f_{x^i} for the partial derivative of f with respect to x^i.

Theorem 2.1 *For every r and q satisfying (1.5), there exists a constant $C_3 = C_3(r, q)$ such that for any continuously differentiable functions $b^1, \dots, b^n : \mathbb{R}^d \times [0, 1] \to \mathbb{R}$ of compact support, and indices $\alpha_1, \dots, \alpha_n \in \{1, \dots, d\}$, we have*

$$\left| \mathbb{E} \int_{\Delta^n} \prod_{i=1}^n b^i_{x^{\alpha_i}}(a + \sigma B(t_i), t_i) dt_i \right| \le C_3^n \sigma^{-n\left(\frac{d}{r}+1\right)} n^{-n\delta_1} t^{n\delta_1} \prod_{i=1}^n \|b^i\|_{q,r}, \tag{2.1}$$

where

$$\Delta^n = \Delta^n(t) = \{(t_1, \dots, t_n) \ : \ 0 \le t_1 \le \dots \le t_n \le t\}.$$

Lemma 2.1 *For every r and q satisfying (1.5), there exists a constant $C_4 = C_4(r, q)$ such that*

$$\sup_a \mathbb{E} \ \exp \left[\lambda \int_0^t |u|^2 (a + \sigma B(s), s) \ ds \right] \le C_4 \exp \left[C_4 \sigma^{-\frac{d}{r\delta_1}} \lambda^{\frac{1}{2\delta_1}} \|u\|_{r,q}^{\frac{1}{\delta_1}} \ t \right]. \tag{2.2}$$

Lemma 2.2 *For every $\beta \in (0, 1)$ and $p > (d + 1)\beta^{-1}$, we can find a constant $C_5 = C_5(p, \beta)$ such that for every continuous function $X \in \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$,*

$$S_{T,\ell}(X; \delta) \le C_5 \delta^{\beta - \frac{d+1}{p}} \ \left\{ \int_{s,t \in [0,T]} \int_{|x|,|y| \le \ell} \frac{|X(x, t) - X(y, t)|^p}{|(x, t) - (y, s)|^{\beta p + d + 1}} \ dxdydtds \right\}^{\frac{1}{p}}. \tag{2.3}$$

Lemma 2.2 is the known as Garsia-Rodemich-Rumsey Inequality and its proof can be found in [SV]. Inequality (2.2) is a Khasminskii type bound and its proof will be given at the end of this section. Theorem 2.1 is the main ingredient for the proof of Theorem 1.1 and was established by Davie when $q = r = \infty$. The proof of Theorem 2.1 will be given in the next section.

Before embarking on the proof of Theorem 1.1, let us outline our strategy.

- **(i)** We first assume that u is a smooth function of compact support. This guarantees that the flow X is a diffeomorphism in spatial variables. For such a drift u, we establish (1.6). Note that the right-hand side of (1.6) depends only $\|u\|_{r,q}$ norm and is independent of the smoothness of u.

- **(ii)** Given $u \in L^{r,q}$ with (r, q) satisfying (1.5), we choose a sequence of smooth functions $\{u_N\}_{N=1}^\infty$ of compact supports such that $\|u_N - u\|_{r,q} \to 0$ in large N limit. Writing $\mathcal{P}_N$ for the law of the corresponding flow X^N, we use Corollary 1.2 to show that the sequence $\{\mathcal{P}_N\}_{N=1}^\infty$ is tight. Then by standard arguments we can show that any limit point of $\{\mathcal{P}_N\}_{N=1}^\infty$ is a law of a flow X that satisfies (1.4) and that the bounds (1.6)-(1.9) are valid.

For the proof of (1.6) we follow [MNP] closely; Step 1 and part of Step 2 are almost identical to the proof of Lemma 7 in [MNP].

Proof of Theorem 1.1. Step 1. We prove (1.6) assuming that u is a smooth vector field of compact support as in Part (i) of the above outline. We leave Part (ii) for Section 5 where Theorem 1.2 is established. From

$$dX(a,t) = u(X(a,t),t)\ dt + \sigma dB, \qquad X(a,t) = a,$$

we can readily deduce

$$\frac{d}{dt}D_aX(a,t) = D_aX_t(a,t) = D_xu(X(a,t),t)D_aX(a,t), \qquad D_aX(a,0) = I, \tag{2.4}$$

where I denotes the $d \times d$ identity matrix. Regarding (2.4) as an ODE for $D_aX(a,t)$, this equation has a unique solution and this solution is given by

$$D_aX(a,t) = I + \sum_{n=1}^{\infty} \int_{\Delta^n(t)} D_xu(X(a,t_n),t_n)\dots D_xu(X(a,t_1),t_1)\ dt_1\dots dt_n, \tag{2.5}$$

provided that this series is convergent. (Δ^n was defined right after (2.1).)

As for the inverse $(D_aX)^{-1}$, observe

$$\frac{d}{dt}\left(D_aX(a,t)\right)^{-1} = -\left(D_aX(a,t)\right)^{-1}\frac{d}{dt}\left(D_aX(a,t)\right)\left(D_aX(a,t)\right)^{-1}.$$

This and (2.4) yields

$$\frac{d}{dt}\left(D_aX(a,t)\right)^{-1} = -\left(D_aX(a,t)\right)^{-1} D_xu(X(a,t),t), \qquad \left(D_aX(a,0)\right)^{-1} = I.$$

Regarding this as an ODE for $(D_aX(a,t))^{-1}$, this equation has a unique solution and this solution is given by

$$D_aX(a,t) = I + \sum_{n=1}^{\infty}(-1)^{-n} \int_{\Delta^n(t)} D_xu(X(a,t_1),t_1)\dots D_xu(X(a,t_n),t_n)\ dt_1\dots dt_n, \tag{2.6}$$

provided that this series is convergent.

We use (2.5) to bound $\left|D_aX(a,t)\right|$. (In the same fashion, we may use (2.6) to bound $\left|(D_aX(a,t))^{-1}\right|$.) This is achieved by bounding the summand in (2.5), which in the end verifies the convergence of the series and the validity of (2.5).

Using the matrix norm $|[a_{ij}]| = \sum_{i,j}|a_{ij}|$, we have

$$[\mathbb{E}|D_aX_t(a,t)|^p]^{\frac{1}{p}} \le d + \sum_{n=1}^{\infty} A_n^{\frac{1}{p}}, \tag{2.7}$$

where

$$A_n = \mathbb{E}\left|\int_{\Delta^n} D_x u(X(a,t_n),t_n)\dots D_x u(X(a,t_1),t_1)\ dt_1\dots dt_n\right|^p.$$

Writing $x=(x^1,\dots,x^d)$ and $u=(u^1,\dots,u^d)$, we may assert

$$A_n \le d^{(n+1)(p-1)} \sum_{i_0,\dots,i_n=1}^{d} A_n(i_0,\dots,i_n), \tag{2.8}$$

where $A_n(i_0,\dots,i_n)$ is given by

$$\mathbb{E}\left|\int_{\Delta^n} u^{i_0}_{x^{i_1}}(X(a,t_n),t_n)u^{i_1}_{x^{i_2}}(X(a,t_{n-1}),t_{n-1})\dots u^{i_{n-1}}_{x^{i_n}}(X(a,t_1),t_1)\ dt_1\dots dt_n\right|^p$$

On the other hand, for p an even integer, we can drop absolute values and express $A_n(i_0,\dots,i_n)$ as a sum of at most p^{np} terms of the form $B_{np}(j_1,k_1,\dots,j_{np},k_{np})$, that is given by

$$\mathbb{E}\int_{\Delta^{np}} u^{k_1}_{x^{j_1}}(X(a,s_{np}),s_{np})\dots u^{k_{np}}_{x^{j_{np}}}(X(a,s_1),s_1)\ ds_1\dots ds_{np},$$

for $j_1,k_1,\dots,j_{np},k_{np}\in\{1,\dots.d\}$. This is because there are at most p^{np} many ways to form $s_1\le\dots\le s_{np}$ out of p many groups of the form

$$\{t^i_1\le\dots\le t^i_n\},\qquad i=1,\dots,p.$$

(Once $s_1\le\dots\le s_\ell$ are selected, there are at most p many possibilities for our next selection $s_{\ell+1}$.)

Step 2. Writing $\mathbb{Q}^a$ for the law of $(a+\sigma B(s): s\in[0,t])$ with $B(\cdot)$ representing a standard Brownian motion that starts from 0, and applying Girsanov's formula, we may write B_{np} as

$$\int\left[\int_{\Delta^{np}} u^{k_1}_{x^{j_1}}(x(s_{np}),s_{np})\dots u^{k_{np}}_{x^{j_{np}}}(x(s_1),s_1)\ ds_1\dots ds_{np}\right]\ M(x(\cdot))\ \mathbb{Q}^a(dx(\cdot)),$$

where

$$M(x(\cdot)) = M_u(x(\cdot)) = \exp\left(\frac{1}{2\nu}\int_0^t u(x(s),s)\cdot dx(s) - \frac{1}{4\nu}\int_0^t |u(x(s),s)|^2\ ds\right).$$

This, by Schwartz' inequality, is bounded above by

$$D_{np}(j_1,k_1,\dots,j_{np},k_{np})^{\frac12}\ \left(\int M^2\ d\mathbb{Q}^a\right)^{\frac12},$$

where $D_{np}(j_1, k_1, \ldots, j_{np}, k_{np})$ is given by

$$\int \left[\int_{\Delta^{np}} u^{k_1}_{x^{j_1}}(x(s_{np}), s_{np}) \ldots u^{k_{np}}_{x^{j_{np}}}(x(s_1), s_1)\ ds_1 \ldots ds_{np} \right]^2\ \mathbb{Q}^a(dx(\cdot)).$$

As in Step 1, we may express $D_{np}(j_1, k_1, \ldots, j_{np}, k_{np})$ as a sum of at most 2^{np} many terms of the form $E_{2np}(j_1, k_1, \ldots, j_{2np}, k_{2np})$, that are defined as

$$\int \left[\int_{\Delta^{2np}} u^{k_1}_{x^{j_1}}(x(s_{2np}), s_{2np}) \ldots u^{k_{2np}}_{x^{j_{2np}}}(x(s_1), s_1)\ ds_1 \ldots ds_{2np} \right]\ \mathbb{Q}^a(dx(\cdot)).$$

By Theorem 2.1,

$$\left| E_{2np}(j_1, k_1, \ldots, j_{2np}, k_{2np}) \right| \leq C_3^{2np} (2\nu)^{-np\left(\frac{d}{r}+1\right)} (2np)^{-2np\delta_1} t^{2np\delta_1} \|u\|^{2np}_{r,q}.$$

This in turn implies

$$\left| D_{np}(j_1, k_1, \ldots, j_{np}, k_{np}) \right| \leq 2^{np} C_3^{2np} (2\nu)^{-np\left(\frac{d}{r}+1\right)} (2np)^{-2np\delta_1} t^{2np\delta_1} \|u\|^{2np}_{r,q}. \tag{2.9}$$

Furthermore, by Lemma 2.1,

$$\begin{aligned}
\int M_u^2\ d\mathbb{Q}^a &= \int \exp\left(\frac{1}{\nu} \int_0^t u(x(s), s) \cdot dx(s) - \frac{1}{2\nu} \int_0^t |u(x(s), s)|^2\ ds \right)\ d\mathbb{Q}^a \\
&= \int (M_{4u})^{\frac{1}{2}} \exp\left(\frac{3}{2\nu} \int_0^t |u(x(s), s)|^2\ ds \right)\ d\mathbb{Q}^a \\
&\leq \left(\int M_{4u}\ d\mathbb{Q}^a \right)^{\frac{1}{2}} \left(\int \exp\left(\frac{3}{\nu} \int_0^t |u(x(s), s)|^2\ ds \right)\ d\mathbb{Q}^a \right)^{\frac{1}{2}} \\
&= \left(\int \exp\left(\frac{3}{\nu} \int_0^t |u(x(s), s)|^2\ ds \right)\ d\mathbb{Q}^a \right)^{\frac{1}{2}} \\
&\leq C_4 \exp\left(c_0\ \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)} \|u\|^{\frac{1}{\delta_1}}_{r,q}\ t \right),
\end{aligned} \tag{2.10}$$

where for the third equality we use the fact that M_{4u} is a $\mathbb{Q}^a$-martingale. From (2.7)– (2.10) we deduce that for positive even integer p,

$$[\mathbb{E}|D_a X(a,t)|^p]^{\frac{1}{p}} \leq d + C_4 Z(p) \exp\left(c_0 2^{-1} p^{-1} \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)} \|u\|^{\frac{1}{\delta_1}}_{r,q}\ t \right) \tag{2.11}$$

where

$$
\begin{aligned}
Z(p) &= \sum_{n=1}^{\infty} d^{n+1} 2^{\frac{n}{2}} p^n C_3^n \nu^{-\frac{n}{2}\left(\frac{d}{r}+1\right)} (2np)^{-n\delta_1} t^{n\delta_1} \|u\|_{r,q}^n \\
&\le \sum_{n=1}^{\infty} \left(c_1 \nu^{-\frac{1}{2\delta_1}\left(\frac{d}{r}+1\right)} p^{\frac{1}{\delta_1}-1} \; t \|u\|_{r,q}^{\frac{1}{\delta_1}} \right)^{n\delta_1} n^{-n\delta_1} \\
&=: \sum_{n=1}^{\infty} W^{n\delta_1} n^{-n\delta_1}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{n=1}^{\infty} W^{n\delta_1} n^{-n\delta_1} &= \sum_{n=1}^{\infty} \left((2W)^n n^{-n}\right)^{\delta_1} 2^{-n\delta_1} \le c_2 \left(\sum_{n=1}^{\infty} (2W)^n n^{-n} \; 2^{-n\delta_1} \right)^{\delta_1} \\
&\le c_3 \left(\sum_{n=1}^{\infty} (2W)^n (n!)^{-1} \; 2^{-n\delta_1} \right)^{\delta_1} \le c_3 e^W,
\end{aligned}
$$

where we used Stirling's formula for the second inequality. From this and (2.11) we deduce,

$$
\mathbb{E}|D_a X(a,t)|^p \le \exp\left(c_4 p + c_4 \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)} p^{\frac{1}{\delta_1}} \|u\|_{r,q}^{\frac{1}{\delta_1}} \; t \right). \tag{2.12}
$$

The bound (2.12) is true for every even integer $p \ge 2$. By changing the constant c_4 if necessary, we can guarantee that it is also true for every real $p \in [1, \infty)$. As we mentioned earlier, with a verbatim argument we can establish the analog of (2.12) for $(D_a X)^{-1}$. □

Proof of Corollary 1.1. We start with the proof of (1.7). From (1.6) and Chebyshev's inequality we learn that for every $p, \lambda \in (1, \infty)$,

$$
\mathbb{P}\left(|D_a X(a,t)| \ge \lambda\right) \le \lambda^{-p} C_0^p e^{Ap^{\frac{1}{\delta_1}}} = \exp\left(-p \log \lambda + p \log C_0 + A p^{\frac{1}{\delta_1}} \right),
$$

where

$$
A = C_1 \sigma^{-\frac{1}{\delta_1}\left(1+\frac{d}{r}\right)} \|u\|_{r,q}^{\frac{1}{\delta_1}} \; t. \tag{2.13}
$$

We optimize this bound by choosing $\log \lambda = A \delta_1^{-1} p^{\frac{1}{\delta_1}-1} + \log C_0$;

$$
\mathbb{P}\left(|D_a X(a,t)| \ge \lambda\right) \le C_0 e^{A\left(1-\frac{1}{\delta_1}\right) p^{\frac{1}{\delta_1}}} \le \exp\left(-C_1' A^{-\frac{\delta_1}{1-\delta_1}} \left(\log^+ \frac{\lambda}{C_0} \right)^{\frac{1}{1-\delta_1}} \right),
$$

for a positive constant C_1'. This completes the proof of (1.7).

We next turn to the proof of (1.8). Set

$$\omega_\ell(\delta) = \sup_{\substack{|a-b|\le\delta \\ |a|,|b|\le\ell}} |X(a,t) - X(b,t)|.$$

By Morrey's inequality [E],

$$\omega_\ell(\delta) \le c_0 \delta^{1-\frac{d}{p}} \left(\int_{|z|\le 2\ell} |D_a X(z,t)|^p \ dz \right)^{\frac{1}{p}},$$

for every $p > d$, and for a universal constant c_0 that can be chosen to be independent of p. This, (1.6) and Hölder's inequality imply

$$\mathbb{E}\omega_\ell(\delta) \le c_0 \delta^{1-\frac{d}{p}} \left(\int_{|z|\le 2\ell} \mathbb{E}|D_a X(z,t)|^p \ dz \right)^{\frac{1}{p}} \le c_1 \delta^{1-\frac{d}{p}} \ell^{\frac{d}{p}} e^{Ap^{\frac{1}{\delta_1}-1}},$$

with A defined by (2.13). We optimize this bound by choosing

$$\log \frac{\ell^d}{\delta^d} = A \left(\frac{1}{\delta_1} - 1 \right) p^{\frac{1}{\delta_1}}.$$

For such a choice of p we deduce

$$\mathbb{E}\omega_\ell(\delta) \le c_1 \delta \exp \left(c_2 \left(\log \frac{\ell}{\delta} \right)^{1-\delta_1} A^{\delta_1} \right),$$

for a positive constant c_2. This completes the proof of (1.8). □

Proof of Corollary 1.2. We use (1.6) to assert that for $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}|X(x,t) - X(y,t)|^p &= \mathbb{E} \left| \int_0^1 D_a X(\theta x + (1-\theta)y, t) \cdot (x-y) \ d\theta \right|^p \\ &\le C_0^p \exp \left(C_1 \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)} p^{\frac{1}{\delta_1}} \|u\|_{r,q}^{\frac{1}{\delta_1}} T \right) |x-y|^p. \end{aligned} \tag{2.14}$$

On the other hand, by Girsanov's formula and Hölder's inequality,

$$\begin{aligned} \mathbb{E}|X(y,t) - X(y,s)|^p &= \int |x(t) - x(s)|^p \ M(x(\cdot)) \ \mathbb{Q}^y(dx(\cdot)) \\ &\le \left(\int |x(t) - x(s)|^{p\gamma} \ \mathbb{Q}^y(dx(\cdot)) \right)^{\frac{1}{\gamma}} \left(\int M^{\gamma'} \ d\mathbb{Q}^y \right)^{\frac{1}{\gamma'}} \\ &\le c_0 |t-s|^{\frac{p}{2}} \left(\int M^{\gamma'} \ d\mathbb{Q}^y \right)^{\frac{1}{\gamma'}} \\ &\le c_1 \exp \left(c_2 \ \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)} \|u\|_{r,q}^{\frac{1}{\delta_1}} \ T \right) \ |t-s|^{\frac{p}{2}}, \end{aligned}$$

where $1/\gamma + 1/\gamma' = 1$, $\mathbb{Q}^y$ and M were defined in the beginning of Step 2 of the proof of Theorem 1.1, and for the last inequality we follow (2.10). From this and (2.14) we deduce

$$\mathbb{E}|X(x,t) - X(y,s)|^p \le c_3^p \exp\left(c_3\ \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)}\|u\|_{r,q}^{\frac{1}{\delta_1}}\ T\right)\left(|x-y|^p + |t-s|^{\frac{p}{2}}\right).$$

This and Lemma 2.2 imply,

$$\mathbb{E}\ S_{T,\ell}(X;\delta)^p \le c_4 c_3^p \exp\left(c_3\ \nu^{-\frac{1}{2\delta_1}\left(1+\frac{d}{r}\right)}\|u\|_{r,q}^{\frac{1}{\delta_1}}\ T\right)\ \delta^{\beta p-d-1},$$

with $c_4 < \infty$ if $\beta \in (0, 1/2)$. (Here we have used $|x-y|^p \le c|x-y|^{p/2}$ for $|x|, |y| \le \ell$.) Finally we choose p so that $\beta - (d+1)/p = \alpha$ to complete the proof. $\square$

We end this section with the proof of Lemma 2.1. Let us make some preparations. We write $p(x,t) = (t\nu)^{-d/2}p(x/\sqrt{t\nu})$ with

$$p(z) = (4\pi)^{-d/2}\exp\left(-|z|^2/4\right). \tag{2.15}$$

Throughout the paper we need to bound L^r norms of $p(\cdot, s)$ and its spatial derivatives. These bounds are stated in Lemma 2.3 below. The elementary proof of this lemma is omitted.

Lemma 2.3 *For every $r \in [1,\infty]$ and nonnegative integer k, there exists a constant $C_5(k,r)$ such that if $\tilde{p}(\cdot, s)$ denotes a k-th spatial derivative of $p(\cdot, s)$, then*

$$\|\tilde{p}(\cdot, s)\|_{L^{r'}} \le C_5(s\nu)^{-\frac{d}{2r}-\frac{k}{2}}, \tag{2.16}$$

for every $s > 0$,where $r' = r/(r-1)$.

We are now ready to establish (2.2).

Proof of Lemma 2.1. The proof is based on Khasminskii's trick. We first show that there exists a constant c_1 such that

$$\sup_a \sup_{\theta\in[0,t]} \mathbb{E}\int_0^t |u|^2(a+\sigma B(s), s+\theta)\ ds \le c_1\nu^{-d/r}t^{2\delta_1}\|u\|_{r,q}^2. \tag{2.17}$$

This is a straight forward consequences of Hölder's inequality:

$$\begin{aligned}
\int_0^t\int |u|^2(a+x, s+\theta)p(x,s)\ dxds &\le \int_0^t\left(\int |u|^r(x,s)\ dx\right)^{\frac{2}{r}}\left(\int p^{r'}(x,s+\theta)\ dx\right)^{\frac{1}{r'}}\ ds\\
&\le c_0\int_0^t\left(\int |u|^r(x,s)\ dx\right)^{\frac{2}{r}}(s\nu)^{-\frac{d}{2}(1-\frac{1}{r'})}\ ds\\
&= c_0\|u\|_{r,q}^2\left(\int_0^t (s\nu)^{-\frac{d}{2}(1-\frac{1}{r'})q'}\ ds\right)^{\frac{1}{q'}}\\
&= c_1\nu^{-d/r}t^{2\delta_1}\|u\|_{r,q}^2,
\end{aligned}$$

where $\frac{2}{r} + \frac{1}{r'} = 1$ and $\frac{2}{q} + \frac{1}{q'} = 1$. Given $\lambda > 0$, choose t_0 such that

$$c_1 \nu^{-d/r} t_0^{2\delta_1} \|u\|_{r,q}^2 \lambda = \frac{1}{2} =: \alpha_0,$$

and use Khasminskii's trick (see for example [S]) to deduce

$$\sup_a \mathbb{E}\ \exp\left[\lambda \int_0^{t_0} |u|^2(a + B(s), s)\ ds\right] \le (1 - \alpha_0)^{-1} = 2,$$

from (2.17). This and Markov property yields

$$\sup_a \mathbb{E}\ \exp\left[\lambda \int_0^{\ell t_0} |u|^2(a + B(s), s)\ ds\right] \le 2^\ell.$$

This implies (2.2) after choosing $\ell = [t/t_0] + 1$. □

3 Proof of Theorem 2.1

The main ingredient for the proof of Theorem 2.1 is a bound on certain *block integrals*. In this section we state a crucial bound for block integrals and show how such bounds can be used to establish Theorem 2.1.

We say a function h is of type j if it is a spatial partial jth-derivative of p. We can readily show that if h is of type j, then

$$h(z, s) \le C_6\ (\nu s)^{-\frac{j}{2}} p(z, 2s), \tag{3.1}$$

for a constant C_6. Define

$$\Delta^k = \Delta^k(t_0, t) = \{(t_1, \dots, t_k) :\ t_0 \le t_1 \le \cdots \le t_k \le t\}.$$

For our purposes, we would like to bound *block integrals* $I^k(f_1, \dots, f_k)$, where

- $I^1(f_1) = \int_{\Delta^1} \int_{\mathbb{R}^d} f_1(z_1, t_1) p^{(1)}(z_1, t_1 - t_0)(t - t_1)^\alpha\ dz_1 dt_1$, with $p^{(1)}$ being of type 1.
- $I^2(f_1, f_2)$ is defined as

$$\int_{\Delta^2} \int_{\mathbb{R}^{2d}} f_1(z_1, t_1) f_2(z_2, t_2) p(z_1, t_1 - t_0) p^{(2)}(z_2 - z_1, t_2 - t_1)(t - t_2)^\alpha\ dz_1 dz_2 dt_1 dt_2,$$

  with $p^{(2)}$ being of type 2.

- For $k > 2$, we define $I^k(f_1, \dots, f_k)$ by

$$\int_{\Delta^k}\int_{\mathbb{R}^{kd}} f_1(z_1,t_1)p(z_1,t_1-t_0)\left\{\prod_{i=2}^{k-1} f_i(z_i,t_i)p_i^{(1)}(z_i-z_{i-1},t_i-t_{i-1})\ dz_i dt_i\right\}$$
$$\times f_k(z_k,t_k)p^{(2)}(z_k-z_{k-1},t_k-t_{k-1})(t-t_k)^\alpha\ dz_1 dz_k dt_1 dt_k,$$

  where $p^{(2)}$ is of type 2 and $p_i^{(1)}$ is of type 1 for $i = 2, \dots, k-1$.

Our main result on block integrals is Theorem 3.1.

Theorem 3.1 *There exists a constant $C_7 = C_7(r, q)$ such that*

$$|I_k(f_1,\dots,f_k)| \le C_7^k \nu^{-k\left(\frac{d}{2r}+\frac{1}{2}\right)}\gamma_k(\alpha)(t-t_0)^{\alpha+k\delta_1}\prod_{i=1}^k \|f_i\|_{r,q}, \tag{3.2}$$

where

$$\gamma_k(\alpha) = \frac{\alpha^\alpha}{(\alpha+k\delta_1)^{\alpha+k\delta_1}}.$$

(By convention, $0^0 = 1$.)

Armed with Theorem 3.1, we are now ready to give a proof for (2.1)

Proof of Theorem 2.1. Step 1. Let us write R for the left-hand side of (2.1). We certainly have

$$\begin{aligned} R &= \left|\int_{\Delta^n}\int_{\mathbb{R}^{dn}}\prod_{i=1}^n b^i_{x^{\alpha_i}}(a+y_i,t_i)p(y_i-y_{i-1},t_i-t_{i-1})dy_i dt_i\right| \\ &= \left|\int_{\Delta^n}\int_{\mathbb{R}^{dn}}\prod_{i=1}^n b^i_{y_i^{\alpha_i}}(a+y_i,t_i)p(y_i-y_{i-1},t_i-t_{i-1})dy_i dt_i\right|, \end{aligned}$$

where $y_0 = 0$, $y_i^{\alpha_i}$ denotes the α_i-th coordinate of $y_i \in \mathbb{R}^d$, and p was defined by (2.15). After some integration by parts we learn

$$R = \left|\sum_{r=1}^{2^{n-1}} \varepsilon_r I(\beta_1(r),\dots,\beta_n(r))\right|, \tag{3.3}$$

where each ε_r is either 1 or -1, the indices $\beta_1(r),\dots,\beta_n(r)$ are in $\{0,1,2\}$ and satisfy $\sum_i \beta_i(r) = n$, and the expression I has the form

$$I(\beta_1,\dots,\beta_n) = \int_{\Delta^n}\int_{\mathbb{R}^{dn}}\prod_{i=1}^n b^i(a+y_i,t_i)q^{\beta_i}(y_i-y_{i-1},t_i-t_{i-1})dy_i dt_i.$$

Here $q^0(a,t) = p(a,t)$, $q^1(a,t) = p_{a^j}(a,t)$ for some $j \in \{1, \dots d\}$ and $q^2(a,t) = p_{a^j a^k}(a,t)$ for some $j, k \in \{1, \dots d\}$. Recall that by p_{a^j} and $p_{a^j a^k}$, we mean partial derivatives with respect to coordinates a^j and a^j, a^k respectively. As a result, (2.1) would follow, if we can find a constant c_1 such that for all $\beta_1, \dots, \beta_n$,

$$|I(\beta_1, \dots, \beta_n)| \le c_1^n \nu^{-\frac{n}{2}\left(\frac{d}{r}+1\right)} (n\delta_1)^{-n\delta_1} t^{n\delta_1} \prod_{i=1}^n \|b^i\|_{q,r}. \tag{3.4}$$

By induction on n, we can readily show that the type of n-tuple $(\beta_1, \dots, \beta_n)$ that appears in (3.3) can be decomposed into blocks of sizes $n_1, \dots, n_\ell$ such that if

$$m_0 = 0, \quad m_1 = n_1, \quad m_2 = n_1 + n_2, \ \dots, \ m_\ell = n_1 + n_2 + \cdots + n_\ell = n,$$

then each block $(\beta_{m_{i-1}+1}, \dots, \beta_{m_i})$ satisfies the following conditions:

- If $n_i = 1$, then $\beta_{m_{i-1}+1} = 1$.
- If $n_i = 2$, then $\beta_{m_{i-1}+1} = 0$ and $\beta_{m_{i-1}+2} = \beta_{m_i} = 2$.
- If $n_i > 2$, then $\beta_{m_{i-1}+1} = 0$ and $\beta_{m_i} = 2$ and all β_s in between are 1.

Step 2. When $\ell > 1$, we set

$$J_{\ell-1}(t_{m_{\ell-1}}, y_{m_{\ell-1}}) = \int_{\Delta^{n_\ell}(t_{m_{\ell-1}}, t)} \int_{\mathbb{R}^{dn_\ell}} \prod_{i=m_{\ell-1}+1}^{n} b^i(a+y_i, t_i) q^{\beta_i}(y_i - y_{i-1}, t_i - t_{i-1}) \, dy_i dt_i.$$

In the case of $\ell > 2$, we inductively define

$$J_j(t_{m_j}, y_{m_j}) = \int_{\Delta^{n_{j+1}}(t_{m_j}, t)} \int_{\mathbb{R}^{dn_{j+1}}} J_{j+1}(t_{m_{j+1}}, y_{m_{j+1}}) \prod_{i=m_j+1}^{m_{j+1}} b^i(a+y_i, t_i) q^{\beta_i}(y_i - y_{i-1}, t_i - t_{i-1}) \, dy_i dt_i.$$

for $j = \ell - 2, \dots, 1$. This allows us to write

$$I(\beta_1, \dots, \beta_n) = \int_{\Delta^{n_1}(0,t)} \int_{\mathbb{R}^{dn_1}} J_1(t_{m_1}, y_{m_1}) \prod_{i=1}^{m_1} b^i(a+y_i, t_i) q^{\beta_i}(y_i - y_{i-1}, t_i - t_{i-1}) \, dy_i dt_i.$$

We then apply Theorem 3.1 to assert

$$\left| J_{\ell-1}(t_{m_{\ell-1}}, y_{m_{\ell-1}}) \right| \le C_7^{n_\ell} \nu^{-n_\ell\left(\frac{d}{2r}+\frac{1}{2}\right)} (n_\ell \delta_1)^{-n_\ell \delta_1} \left(t - t_{m_{\ell-1}}\right)^{n_\ell \delta_1} \prod_{i=m_{\ell-1}+1}^{n} \|b^i\|_{r,q}.$$

This allows us to express

$$J_{\ell-1}(t_{m_{\ell-1}}, y_{m_{\ell-1}}) = \hat{J}_{\ell-1}(t_{m_{\ell-1}}, y_{m_{\ell-1}}) \left(t - t_{m_{\ell-1}}\right)^{n_\ell \delta_1},$$

with $\hat{J}_{\ell-1}$ satisfying

$$\left|\hat{J}_{\ell-1}(t_{m_{\ell-1}}, y_{m_{\ell-1}})\right| \leq C_7^{n_\ell} \nu^{-n_\ell\left(\frac{d}{2r}+\frac{1}{2}\right)} (n_\ell \delta_1)^{-n_\ell\delta_1} \prod_{i=m_{\ell-1}+1}^{n} \|b^i\|_{r,q}.$$

After replacing $b^{m_{\ell-1}}$ with $b^{m_{\ell-1}}\hat{J}_{\ell-1}$, we apply Theorem 3.1 again to assert

$$\begin{aligned}|J_{\ell-2}(t_{m_{\ell-2}}, y_{m_{\ell-2}})| \leq\ & C_7^{n_\ell+n_{\ell-1}} \nu^{-(n_\ell+n_{\ell-1})\left(\frac{d}{2r}+\frac{1}{2}\right)} \left((n_\ell+n_{\ell-1})\,\delta_1\right)^{-(n_\ell+n_{\ell-1})\delta_1} \\ & \times \left(t - t_{m_{\ell-2}}\right)^{(n_\ell+n_{\ell-1})\delta_1} \prod_{i=m_{\ell-2}+1}^{n} \|b^i\|_{r,q},\end{aligned}$$

provided that $\ell > 2$. Continuing this inductively we arrive at (3.4) for $c_1 = C_7$. The bound (3.4) in turn implies (2.1) for $C_3 = 2C_7\delta_1^{-\delta_1}$. □

4 Bounding Block Integrals

4.1 Proof of Theorem 3.1

As preparation for the proof of Theorem 3.1, we establish two lemmas. The first lemma is a slight generalization of (3.2) when $k = 2$. Given $\beta \geq 0$, define $I'(f_1, f_2)$ by

$$\int_{\Delta^2(t)} \int_{\mathbb{R}^{2d}} f_1(z_1,t_1) f_2(z_2,t_2) p(z_1, 2t_1) p^{(2)}(z_2 - z_1, t_2 - t_1) t_1^{\beta} (t - t_2)^{\alpha}\ dz_1 dz_2 dt_1 dt_2,$$

where $\Delta^2 = \Delta^2(0,t)$. Also define $J(f_1, \ldots, f_\ell; z_{\ell+1}, t_{\ell+1})$ by

$$\int_{\Delta^\ell(t_{\ell+1})} \int_{\mathbb{R}^{d\ell}} p(z_1,t_1) f_1(z_1,t_1) p_{\ell+1}^{(1)}(z_{\ell+1}-z_\ell, t_{\ell+1}-t_\ell) \prod_{i=2}^{\ell} f_i(z_i,t_i) p_i^{(1)}(z_i - z_{i-1}, t_i - t_{i-1}) \prod_{i=1}^{\ell} dz_i dt_i,$$

where

$$\Delta^\ell(t_{\ell+1}) = \{(t_1, \ldots, t_\ell) : 0 \leq t_1 \leq \cdots \leq t_\ell \leq t_{\ell+1}\}.$$

Lemma 4.1 *There exists a constant $C_8 = C_8(r,q)$ such that for $\alpha, \beta \geq 0$,*

$$|I'(f_1,f_2)| \leq C_8 \nu^{-\left(\frac{d}{r}+1\right)} \zeta(\alpha,\beta) t^{2\delta_1+\alpha+\beta} \|f_1\|_{r,q} \|f_2\|_{r,q}. \tag{4.1}$$

where

$$\zeta(\alpha,\beta) = \frac{\alpha^\alpha \beta^\beta (\beta+1)^{\delta_1 - \frac{d}{2r}}}{(\alpha+\beta+2\delta_1)^{\alpha+\beta+2\delta_1}}.$$

(By convention $0^0 = 1$.)

Lemma 4.2 *There exists a constant $C_9 = C_9(r,q)$ such that*

$$|J(f_1,\ldots,f_\ell;z_{\ell+1},t_{\ell+1})| \le C_9^\ell\ \nu^{-\left(1+\frac{d}{r}\right)\frac{\ell}{2}}\ (\ell\delta_1)^{-\ell\delta_1}\ p\left(z_{\ell+1},2t_{\ell+1}\right) t_{\ell+1}^{\ell\delta_1}\prod_{i=1}^{\ell}\|f_i\|_{r,q}. \tag{4.2}$$

Before embarking on the proofs of Lemmas 4.1 and 4.2, let us recall the relationship between generalized Beta function and Gamma function Γ.

Lemma 4.3 *For every $\alpha_0\ldots\alpha_n > 0$,*

$$\int_{t_0\le t_1\le\cdots\le t_{n+1}}\prod_{i=0}^{n}(t_{i+1}-t_i)^{\alpha_i-1}\ \prod_{i=1}^{n}dt_i = (t_{n+1}-t_0)^{\sum_{i=0}^n\alpha_i-1}\ \frac{\prod_{i=0}^n\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=0}^n\alpha_i\right)}. \tag{4.3}$$

The elementary proof of Lemma 4.3 is omitted.

Proof of Lemma 4.1. We write Δ for the set $\Delta^2=\Delta^2(t)$ and set

$$f_i'(s) = \|f_i(\cdot,s)\|_{L^r(\mathbb{R}^d)}.$$

Step 1. We decompose $I' = I_1 + I_2$ where I_i is obtained from I by replacing the domain of integration $\Delta=\Delta^2$ with Δ_i. The sets Δ_1 and Δ_2 are defined by

$$\begin{aligned}\Delta_1 &= \{(t_1,t_2)\in\Delta:\ t_1\le t_2-t_1\},\\ \Delta_2 &= \{(t_1,t_2)\in\Delta:\ t_2-t_1\le t_1\}.\end{aligned}$$

The term I_1 is easily bounded with the aid of our L^r bounds on p and $p^{(2)}$: If we set

$$\eta(\alpha,\beta;q) := \left(\frac{\Gamma(\beta q'+\delta_1 q')\Gamma(\delta_1 q')\Gamma(\alpha q'+1)}{\Gamma(2\delta_1 q'+(\alpha+\beta)q'+1)}\right)^{\frac{1}{q'}},$$

with $q'=q/(q-1)$, then by Lemma 2.3 and Hölder's inequality, the expression $|I_1|$ is bounded above by

$$\begin{aligned}
&c_0\nu^{-\left(\frac{d}{r}+1\right)}\int_{\Delta_1} f_1'(t_1)f_2'(t_2)t_1^{\beta-\frac{d}{2r}}(t_2-t_1)^{-\frac{d}{2r}-1}(t-t_2)^{\alpha}\ dt_1dt_2\\
&\le c_0\nu^{-\left(\frac{d}{r}+1\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ \left(\int_{\Delta_1} t_1^{\left(\beta-\frac{d}{2r}\right)q'}(t_2-t_1)^{-\left(\frac{d}{2r}+1\right)q'}(t-t_2)^{\alpha q'}\ dt_1dt_2\right)^{\frac{1}{q'}}\\
&\le c_0\nu^{-\left(\frac{d}{r}+1\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ \left(\int_{\Delta_1} t_1^{\left(\beta-\frac{d}{2r}-\frac{1}{2}\right)q'}(t_2-t_1)^{-\left(\frac{d}{2r}+\frac{1}{2}\right)q'}(t-t_2)^{\alpha q'}\ dt_1dt_2\right)^{\frac{1}{q'}}\\
&= c_0\nu^{-\left(\frac{d}{r}+1\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ t^{2\delta_1+\alpha+\beta}\eta(\alpha,\beta;q),
\end{aligned}$$

where for the second inequality we used the fact that $t_2 - t_1 \geq t_1$ in the set Δ_1, and for the equality we used the fact

$$\left(\frac{d}{2r} + \frac{1}{2}\right) q' < 1,$$

which is the same as (1.5). In summary,

$$|I_1| \leq c_0 \nu^{-\left(\frac{d}{r}+1\right)} \eta(\alpha, \beta; q) \|f_1\|_{r,q} \|f_2\|_{r,q} \, t^{2\delta_1+\alpha+\beta}, \tag{4.4}$$

It remains to bound I_2.

Step 2. We next decompose I_2 as $I_{21} + I_{22}$, where I_{21} is obtained from I_2 by restricting the domain of $dz_1 dz_2$-integration to a set of points (z_1, z_2) such that $|z_2 - z_1|/\sqrt{\nu t_1}$ stays away from zero. Though this restriction is done so that the product structure of $f_1(z_1, t_1) f_2(z_2, t_2)$ is not destroyed. For this purpose, we decompose $\mathbb{R}^{2d}$ into cells

$$B_{k\ell}(t_1, t_2) = B_k(t_1) \times B_\ell(t_2),$$

where for $k = (k^1, \ldots, k^d)$, the set $B_k(s)$ denotes the set of $z = (z^1, \ldots, z^d)$ such that

$$z^i / \sqrt{\nu s} \in [k^i, k^i + 1),$$

for $i = 1, \ldots, d$. We now write $|k - \ell|_1 = \sum_{i=1}^d |k^i - \ell^i|$ for the L^1 distance between $k, \ell \in \mathbb{R}^d$, and set

$$I_{21} = \sum_{(k,\ell) \in \Lambda_1} I_2(k, \ell), \quad I_{22} = \sum_{(k,\ell) \in \Lambda_2} I_2(k, \ell),$$

where

$$\Lambda_1 = \left\{ (k, \ell) : \ k, \ell \in \mathbb{Z}^d, \ |k|_1 \notin \left[|\ell|_1 - 4d, \sqrt{2}|\ell|_1 + 4d \right] \right\}$$
$$\Lambda_2 = \left\{ (k, \ell) : \ k, \ell \in \mathbb{Z}^d, \ |k|_1 \in \left[|\ell|_1 - 4d, \sqrt{2}|\ell|_1 + 4d \right] \right\},$$

and $I_2(k, \ell)$ is defined by

$$\int_{\Delta_2} \int_{B_{k\ell}(t_1, t_2)} f_1(z_1, t_1) f_2(z_2, t_2) p(z_1, 2t_1) p^{(2)}(z_2 - z_1, t_2 - t_1) t_1^\beta (t - t_2)^\alpha \, dz_1 dz_2 dt_1 dt_2.$$

To bound I_{21}, assume that $(z_1, z_2) \in B_{k\ell}$ for some k, ℓ satisfying either $|k|_1 > \sqrt{2}|\ell|_1 + 4d$, or $|\ell|_1 > |k|_1 + 4d$. If the former occurs and $(t_1, t_2) \in \Delta_2$, then

$$\begin{aligned}
|z_2 - z_1|_1 &= |(z_2 - \ell\sqrt{\nu t_2}) - (z_1 - k\sqrt{\nu t_1}) + \ell\sqrt{\nu t_2} - k\sqrt{\nu t_1}|_1 \\
&\geq |\ell\sqrt{\nu t_2} - k\sqrt{\nu t_1}|_1 - d(\sqrt{\nu t_2} + \sqrt{\nu t_1}) \\
&\geq |k|_1\sqrt{\nu t_1} - |\ell|_1\sqrt{\nu t_2} - d(\sqrt{\nu t_2} + \sqrt{\nu t_1}) \\
&\geq \left(\sqrt{2}|\ell|_1 + 4d\right)\sqrt{\nu t_1} - |\ell|_1\sqrt{\nu t_2} - d(\sqrt{\nu t_2} + \sqrt{\nu t_1}) \\
&\geq \left(\sqrt{2}|\ell|_1 + 4d\right)\sqrt{\nu t_1} - |\ell|_1\sqrt{2\nu t_1} - d(\sqrt{2\nu t_1} + \sqrt{\nu t_1}) \\
&\geq d\sqrt{\nu t_1}.
\end{aligned}$$

If the latter occurs and $(t_1, t_2) \in \Delta_2$, then

$$\begin{aligned}
|z_2 - z_1|_1 &= |(z_2 - \ell\sqrt{\nu t_2}) - (z_1 - k\sqrt{\nu t_1}) + \ell\sqrt{\nu t_2} - k\sqrt{\nu t_1}|_1 \\
&\geq |\ell|_1\sqrt{\nu t_2} - |k|_1\sqrt{\nu t_1} - d(\sqrt{\nu t_2} + \sqrt{\nu t_1}) \\
&\geq (|k|_1 + 4d)\sqrt{\nu t_1} - |k|_1\sqrt{\nu t_1} - d(\sqrt{\nu t_2} + \sqrt{\nu t_1}) \\
&\geq 3d\sqrt{\nu t_1} - d\sqrt{\nu t_2} \geq d\sqrt{\nu t_1}.
\end{aligned}$$

In any case, we always have

$$|z_2 - z_1|^2 \geq d^{-1}|z_2 - z_1|_1^2 \geq d\nu t_1.$$

From this and (3.1) we learn

$$\begin{aligned}
|p^{(2)}(z_2 - z_1, t_2 - t_1)| &\leq \frac{c_1}{\nu(t_2 - t_1)} p(z_2 - z_1, 2(t_2 - t_1)) \\
&= \frac{c_1}{\sqrt{\nu(t_2 - t_1)}} \frac{|z_2 - z_1|}{\sqrt{\nu(t_2 - t_1)}} \frac{1}{|z_2 - z_1|} p(z_2 - z_1, 2(t_2 - t_1)) \\
&\leq \frac{c_2}{\sqrt{\nu(t_2 - t_1)}} \frac{1}{|z_2 - z_1|} p(z_2 - z_1, 4(t_2 - t_1)) \\
&\leq \frac{c_2}{\sqrt{\nu(t_2 - t_1)}} \frac{1}{\sqrt{d\nu t_1}} p(z_2 - z_1, 4(t_2 - t_1)).
\end{aligned}$$

This and Lemma 2.3 imply that the term I_{21} is bounded above by a constant multiple of

$$\begin{aligned}
&\nu^{-1}\int_{\Delta}\int_{\mathbb{R}^{2d}} |f_1(z_1,t_1)f_2(z_2,t_2)|t_1^{\beta-\frac{1}{2}}p(z_1,2t_1)|(t_2-t_1)^{-\frac{1}{2}}p(z_2-z_1,4(t_2-t_1))\\
&\qquad\qquad\qquad\qquad \times (t-t_2)^{\alpha}\ dz_1dz_2dt_1dt_2\\
&\quad\le c_3\nu^{-\left(\frac{d}{r}+1\right)}\int_{\Delta} f_1'(t_1)f_2'(t_2)t_1^{\beta-\frac{d}{2r}-\frac{1}{2}}(t_2-t_1)^{-\frac{d}{2r}-\frac{1}{2}}(t-t_2)^{\alpha}\ dt_1dt_2\\
&\quad\le c_3\nu^{-\left(\frac{d}{r}+1\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ \left(\int_{\Delta} t_1^{\left(\beta-\frac{d}{2r}-\frac{1}{2}\right)q'}(t_2-t_1)^{-(\frac{d}{2r}+\frac{1}{2})q'}(t-t_2)^{\alpha q'}\ dt_1dt_2\right)^{\frac{1}{q'}}\\
&\quad= c_3\nu^{-\left(\frac{d}{r}+1\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ t^{2\delta_1+\alpha+\beta}\eta(\alpha,\beta;q).
\end{aligned}$$

In summary,

$$|I_{21}|\le c_4\nu^{-\left(\frac{d}{r}+1\right)}\eta(\alpha,\beta;q)\|f_1\|_{r,q}\|f_2\|_{r,q}\ t^{2\delta_1+\alpha+\beta}, \tag{4.5}$$

It remains to bound I_{22}.

Step 3. Let us write

$$\begin{aligned}
f_{1k}(z_1,t_1)&=f_1(z_1,t_1)1\!\!1(z_1\in B_k(t_1))p(z_1,2t_1),\\
f_{2\ell}(z_2,t_2)&=f_2(z_2,t_2)1\!\!1(z_2\in B_\ell(t_2)),
\end{aligned}$$

so that $I_2(k,\ell)$ can be expressed as

$$\int_{\Delta_2}\int_{\mathbb{R}^{2d}} f_{1k}(z_1,t_1)f_{2\ell}(z_2,t_2)p^{(2)}(z_2-z_1,t_2-t_1)t_1^{\beta}(t-t_2)^{\alpha}\ dz_1dz_2dt_1dt_2.$$

Recall that $p^{(2)}$ is a function of type 2. This means that $p^{(2)}(z,s)=p_{z^iz^j}(z,t)$ is a second derivative of p. By Plancheral's formula we learn that $I_2(k,\ell)$ equals to

$$-(2\pi)^2\int_{\Delta_2}\int_{\mathbb{R}^d}\xi^i\xi^j\hat f_{1k}(\xi,t_1)\check f_{2\ell}(\xi,t_2)e^{-4\pi^2\nu(t_2-t_1)|\xi|^2}t_1^{\beta}(t-t_2)^{\alpha}\ d\xi dt_1dt_2,$$

where

$$\hat h(\xi,s)=\int e^{-2i\pi x\cdot\xi}h(x,s)\ dx,\quad \check h(\xi,s)=\int e^{2i\pi x\cdot\xi}h(x,s)\ dx.$$

As a result, the term $|I_2(k,\ell)|$ is bounded above by

$$\begin{aligned}
2\pi^2\int_{\Delta_2}\int_{\mathbb{R}^d}&\left(\delta^{-1}(\nu t_1)^{\frac{d}{2}}|\hat f_{1k}(\xi,t_1)|^2+\delta(\nu t_1)^{-\frac{d}{2}}|\check f_{2\ell}(\xi,t_2)|^2\right)\ |\xi|^2\\
&\qquad\times e^{-4\pi^2\nu(t_2-t_1)|\xi|^2}t_1^{\beta}(t-t_2)^{\alpha}\ d\xi dt_1dt_2=:I_2^1(k,\ell)+I_2^2(k,\ell).
\end{aligned}$$

for any $\delta > 0$. Further, $I_2^1(k,\ell)$ equals to

$$\begin{aligned}
&\frac{2\pi^2}{\delta}\int_{\Delta_2}\int_{\mathbb{R}^d}(\nu t_1)^{\frac{d}{2}}|\hat{f}_{1k}(\xi,t_1)|^2|\xi|^2e^{-4\pi^2\nu(t_2-t_1)|\xi|^2}t_1^{\beta}(t-t_2)^{\alpha}\ d\xi dt_1dt_2\\
&\quad\le\frac{2\pi^2}{\delta}\int_{\Delta}\int_{\mathbb{R}^d}(\nu t_1)^{\frac{d}{2}}|\hat{f}_{1k}(\xi,t_1)|^2|\xi|^2e^{-4\pi^2\nu(t_2-t_1)|\xi|^2}t_1^{\beta}(t-t_1)^{\alpha}\ d\xi dt_1dt_2\\
&\quad\le\frac{1}{2\nu\delta}\int_0^t\int_{\mathbb{R}^d}(\nu t_1)^{\frac{d}{2}}|\hat{f}_{1k}(\xi,t_1)|^2t_1^{\beta}(t-t_1)^{\alpha}\ d\xi dt_1\\
&\quad=\frac{1}{2\nu\delta}\int_0^t\int_{\mathbb{R}^d}(\nu t_1)^{\frac{d}{2}}|f_{1k}(x,t_1)|^2t_1^{\beta}(t-t_1)^{\alpha}\ dxdt_1\\
&\quad\le c_5e^{-\frac{1}{4}(|k|-\sqrt{d})^{+2}}(\nu\delta)^{-1}\int_0^t(\nu t_1)^{-\frac{d}{2}}\int_{B_k(t_1)}|f(x,t_1)|^2t_1^{\beta}(t-t_1)^{\alpha}\ dxdt_1,
\end{aligned}$$

where we used $\alpha\ge 0$ and $t_1\le t_2$ for the first inequality. We now apply Hölder's inequality to assert that $I_2^1(k,\ell)$ is bounded above by

$$\begin{aligned}
&c_6e^{-\frac{1}{4}(|k|-\sqrt{d})^{+2}}(\nu\delta)^{-1}\int_0^t\left((\nu t_1)^{-\frac{d}{2}}\int_{B_k(t_1)}|f(x,t_1)|^r\ dx\right)^{\frac{2}{r}}t_1^{\beta}(t-t_1)^{\alpha}\ dt_1\\
&\quad\le c_7e^{-\frac{1}{5}|k|^2}(\nu\delta)^{-1}\int_0^t f'(t_1)^2\ (\nu t_1)^{-\frac{d}{r}}\ t_1^{\beta}(t-t_1)^{\alpha}\ dt_1\\
&\quad\le c_7\nu^{-\left(\frac{d}{r}+1\right)}e^{-\frac{1}{5}|k|^2}\delta^{-1}\|f_1\|_{r,q}^2\left(\int_0^t t_1^{-\frac{dq}{r(q-2)}}t_1^{\frac{\beta q}{q-2}}(t-t_1)^{\frac{\alpha q}{q-2}}\ dt_1\right)^{\frac{q-2}{q}}\\
&\quad=c_7\nu^{-\left(\frac{d}{r}+1\right)}e^{-\frac{1}{5}|k|^2}\delta^{-1}\|f_1\|_{r,q}^2\ \eta'(\alpha,\beta)\ t^{2\delta_1+\alpha+\beta},
\end{aligned}$$

where

$$\eta'(\alpha,\beta;q)=\left(\frac{\Gamma((2\delta_1+\beta)q'')\Gamma(\alpha q''+1)}{\Gamma((2\delta_1+\alpha+\beta)q''+1)}\right)^{\frac{1}{q''}},$$

for $q''=q/(q-2)$. Note that for the equality, we have used the fact that $dq/\big(r(q-2)\big)<1$, which is equivalent to (1.5). In summary,

$$|I_2^1(k,\ell)|\le c_7\nu^{-\left(\frac{d}{r}+1\right)}e^{-\frac{1}{5}|k|^2}\delta^{-1}\eta'(\alpha,\beta;q)\|f_1\|_{r,q}^2\ t^{2\delta_1+\alpha+\beta}. \tag{4.6}$$

On the other hand, $I_2^2(k,\ell)$ is bounded above by

$$\begin{aligned}
&2\pi^2\delta\int_{\Delta_2}\int_{\mathbb{R}^d}(\nu t_1)^{-\frac{d}{2}}|\check f_{2\ell}(\xi,t_2)|^2|\xi|^2e^{-4\pi^2\nu(t_2-t_1)|\xi|^2}t_1^\beta(t-t_2)^\alpha\ d\xi dt_1dt_2\\
&\le 2\pi^2\delta 2^{d/2}\int_{\Delta_2}\int_{\mathbb{R}^d}(\nu t_2)^{-\frac{d}{2}}|\check f_{2\ell}(\xi,t_2)|^2|\xi|^2e^{-4\pi^2\nu(t_2-t_1)|\xi|^2}t_2^\beta(t-t_2)^\alpha\ d\xi dt_1dt_2\\
&\le \delta\nu^{-1}2^{d/2-1}\int_0^t\int_{\mathbb{R}^d}(\nu t_2)^{-\frac{d}{2}}|\check f_{2\ell}(\xi,t_2)|^2\ t_2^\beta(t-t_2)^\alpha\ d\xi dt_2\\
&= \delta\nu^{-1}2^{d/2-1}\int_0^t\int_{\mathbb{R}^d}(\nu t_2)^{-\frac{d}{2}}|f_{2\ell}(x,t_2)|^2\ t_2^\beta(t-t_2)^\alpha\ dx dt_2\\
&= \delta\nu^{-1}2^{d/2-1}\int_0^t\int_{B_\ell(t_2)}(\nu t_2)^{-\frac{d}{2}}|f_2(x,t_2)|^2 t_2^\beta(t-t_2)^\alpha\ dx dt_2\\
&\le c_8\delta\nu^{-1}\int_0^t(\nu t_2)^{-\frac{d}{r}}f_2'(t_2)^2t_2^\beta(t-t_2)^\alpha dt_2\\
&\le c_8\delta\nu^{-\left(\frac{d}{r}+1\right)}\|f_2\|_{r,q}^2\left(\int_0^t t_2^{\left(\beta-\frac{d}{r}\right)q''}(t-t_2)^{\alpha q''}dt_2\right)^{\frac{1}{q''}}\\
&= c_8\delta\nu^{-\left(\frac{d}{r}+1\right)}\|f_2\|_{r,q}^2\eta'(\alpha,\beta;q)t^{2\delta_1+\alpha+\beta},
\end{aligned}$$

where we used $t_1\le t_2\le 2t_1$ for the first inequality. In summary,

$$|I_2^2(k,\ell)|\le c_8\delta\nu^{-\left(\frac{d}{r}+1\right)}\eta'(\alpha,\beta;q)\|f_2\|_{r,q}^2\ t^{2\delta_1+\alpha+\beta}. \tag{4.7}$$

We choose $\delta=e^{-|k|^2/10}$ and use (4.6) and (4.7) to deduce

$$|I_2(k,\ell)|=|I_1^2(k,\ell)+I_2^2(k,\ell)|\le c_9e^{-\frac{|k|^2}{10}}\nu^{-\left(\frac{d}{r}+1\right)}\eta'(\alpha,\beta;q)\left[\|f_1\|_{r,q}^2+\|f_2\|_{r,q}^2\right]\ t^{2\delta_1+\alpha+\beta}.$$

From this and the definition of I_{22} we learn

$$\begin{aligned}
|I_{22}|&\le\sum_{(k,\ell)\in\Lambda_2}|I_2(k,\ell)|\\
&\le c_{10}\nu^{-\left(\frac{d}{r}+1\right)}\eta'(\alpha,\beta;q)\left[\|f_1\|_{r,q}^2+\|f_2\|_{r,q}^2\right]\ t^{2\delta_1+\alpha+\beta}\sum_k|k|^de^{-\frac{|k|^2}{10}} \qquad (4.8)\\
&= c_{11}\nu^{-\left(\frac{d}{r}+1\right)}\eta'(\alpha,\beta;q)\left[\|f_1\|_{r,q}^2+\|f_2\|_{r,q}^2\right]\ t^{2\delta_1+\alpha+\beta}.
\end{aligned}$$

Final Step. From (4.4), (4.5) and (4.8) we learn that there exists a constant c_{12} such that if $\|f_1\|_{r,q},\|f_2\|_{r,q}\le 1$, then

$$|I'(f_1,f_2)|\le c_{12}\nu^{-\left(\frac{d}{r}+1\right)}\left[\eta+\eta'\right](\alpha,\beta;q)\ t^{2\delta_1+\alpha+\beta}.$$

From this and a scaling argument we deduce that for every f_1 and f_2,

$$|I'(f_1, f_2)| \leq c_{12}\nu^{-\left(\frac{d}{r}+1\right)} \left[\eta + \eta'\right](\alpha, \beta; q)\ t^{2\delta_1+\alpha+\beta}\|f_1\|_{r,q}\|f_2\|_{r,q}. \tag{4.9}$$

First assume that $\beta \geq 1$. Using Stirling's formula, we can readily show that there exist constants c_{13} and c_{14} such that

$$\eta(\alpha, \beta; q) \leq c_{13}\frac{\alpha^{\alpha+\frac{1}{2q'}}(\beta+\delta_1-1+q^{-1})^{\beta+\delta_1-\frac{1}{2q'}}}{(\alpha+\beta+2\delta_1)^{\alpha+\beta+2\delta_1+\frac{1}{2q'}}} \leq c_{14}\frac{\alpha^\alpha\beta^\beta}{(\alpha+\beta+2\delta_1)^{\alpha+\beta+2\delta_1}},$$

$$\eta'(\alpha, \beta; q) \leq c_{13}\frac{\alpha^{\alpha+\frac{1}{2q''}}(\beta+2\delta_1-1+2q^{-1})^{\beta+2\delta_1-\frac{1}{2q''}}}{(\alpha+\beta+2\delta_1)^{\alpha+\beta+2\delta_1+\frac{1}{2q''}}} \leq c_{14}\zeta(\alpha, \beta),$$

because

$$\delta_1 - (2q')^{-1}, \delta_1 - 1 - q^{-1}, 2\delta_1 - 1 + 2q^{-1} < 0,$$

This and (4.9) imply (4.1) when $\beta \geq 1$. The case $\beta \in [0, 1)$ can be treated likewise. $\square$

Proof of Lemma 4.2. As before, we define r' and q' by $r' = r/(r-1)$ and $q' = q/(q-1)$. By Hölder's inequality we have

$$|J(f_1, \ldots, f_\ell; z_{\ell+1}, t_{\ell+1})| \leq \prod_{i=1}^{\ell}\|f_i\|_{r,q}\left(\int_{\Delta^\ell(t_{\ell+1})} A(t_1, \ldots, t_\ell; z_{\ell+1}, t_{\ell+1})^{\frac{q'}{r'}}\prod_{i=1}^{\ell} dt_i\right)^{\frac{1}{q'}},$$

where $A(t_1, \ldots, t_\ell; z_{\ell+1}, t_{\ell+1})$ equals

$$\int_{\mathbb{R}^{d\ell}} p(z_1, t_1)^{r'}\prod_{i=2}^{\ell+1}\left|p_i^{(1)}(z_i - z_{i-1}, t_i - t_{i-1})\right|^{r'}\prod_{i=1}^{\ell} dz_i.$$

On the other hand, by completing squares (or Markov Property) we know

$$\int_{\mathbb{R}^d} p(z, s)p(a-z, t)\ dz = p(a, s+t)\int_{\mathbb{R}^d} p\left(z - \frac{t}{s+t}a, \frac{st}{s+t}\right)\ dz = p(a, s+t).$$

From this and (3.1) we deduce that $A(t_1,\dots,t_\ell;z_{\ell+1},t_{\ell+1})$ is bounded above by

$$
\begin{aligned}
&C_6^{\ell-1}\prod_{i=2}^{\ell+1}(\nu(t_i-t_{i-1}))^{-\frac{r'}{2}}\int_{\mathbb{R}^{d\ell}}p(z_1,t_1)^{r'}\prod_{i=2}^{\ell+1}p(z_i-z_{i-1},2(t_i-t_{i-1}))^{r'}\ \prod_{i=1}^{\ell}dz_i\\
&\le 2^{\frac{dr'}{2}}C_6^{\ell-1}\prod_{i=2}^{\ell+1}(\nu(t_i-t_{i-1}))^{-\frac{r'}{2}}\int_{\mathbb{R}^{d\ell}}p(z_1,2t_1)^{r'}\prod_{i=2}^{\ell+1}p(z_i-z_{i-1},2(t_i-t_{i-1}))^{r'}\ \prod_{i=1}^{\ell}dz_i\\
&\le c_1^\ell(\nu t_1)^{-\frac{dr'}{2}+\frac{d}{2}}\prod_{i=1}^{\ell}(\nu(t_i-t_{i-1}))^{-r'\left(\frac{d}{2}+\frac{1}{2}\right)+\frac{d}{2}}\\
&\qquad\times\int_{\mathbb{R}^{d\ell}}p(z_1,2t_1/r')\prod_{i=2}^{\ell+1}p(z_i-z_{i-1},2(t_i-t_{i-1})/r')\ \prod_{i=1}^{\ell}dz_i\\
&=c_1^\ell p\left(z_{\ell+1},\frac{2t_{\ell+1}}{r'}\right)(\nu t_1)^{-\frac{dr'}{2}+\frac{dr'}{2}}\prod_{i=1}^{\ell}(\nu(t_{i+1}-t_i))^{-r'\left(\frac{d}{2}+\frac{1}{2}\right)+\frac{d}{2}}.
\end{aligned}
$$

Hence $|J(f_1,\dots,f_\ell;z_{\ell+1},t_{\ell+1})|$ is bounded above by

$$
\begin{aligned}
&c_1^\ell\ \nu^{-\left(1+\frac{d}{r}\right)\frac{\ell}{2}}\ t_{\ell+1}^{\frac{d}{2r}}\ p\left(z_{\ell+1},2t_{\ell+1}\right)\ \prod_{i=1}^{\ell}\|f_i\|_{r,q}\\
&\qquad\times\left(\int_{\Delta^\ell(t_{\ell+1})}t_1^{-\frac{dq'}{2}+\frac{dq'}{2r'}}\prod_{i=1}^{\ell}(t_{i+1}-t_i)^{-q'\left(\frac{d}{2}+\frac{1}{2}\right)+\frac{dq'}{2r'}}\ dt_i\right)^{\frac{1}{q'}}\\
&=c_1^\ell\ \nu^{-\left(1+\frac{d}{r}\right)\frac{\ell}{2}}\ t_{\ell+1}^{\frac{d}{2r}}\ p\left(z_{\ell+1},2t_{\ell+1}\right)t_{\ell+1}^{-\frac{d}{2r}+\ell\delta_1}\eta_\ell\ \prod_{i=1}^{\ell}\|f_i\|_{r,q}\\
&=c_1^\ell\ \nu^{-\left(1+\frac{d}{r}\right)\frac{\ell}{2}}\ p\left(z_{\ell+1},2t_{\ell+1}\right)t_{\ell+1}^{\ell\delta_1}\ \eta_\ell\ \prod_{i=1}^{\ell}\|f_i\|_{r,q},
\end{aligned}
$$

where

$$
\eta_\ell=\left(\frac{\Gamma(\delta_1q')^\ell\Gamma((\delta_1+1/2)q')}{\Gamma\left((1/2+(\ell+1))\delta_1)\,q'\right)}\right)^{\frac{1}{q'}}\le c_2^\ell(\ell\delta_1)^{-\ell\delta_1},
$$

by Stirling's formula. This completes the proof of (4.2).

□

Proof of Theorem 3.1 when $k=1$. Without loss of generality, we may assume that $t_0=0$. In this case, the poof of (3.2) is an immediate consequence of Lemma 2.3, Hölder's inequality and (4.3):

$$
\begin{aligned}
|I_1(f_1)| &\le c_0 \int_0^t f_1'(t_1)(\nu t_1)^{-\left(\frac{d}{2r}+\frac{1}{2}\right)}(t-t_1)^\alpha \, dt_1 \\
&\le c_0 \nu^{-\left(\frac{d}{2r}+\frac{1}{2}\right)} \|f_1\|_{r,q} \left(\int_0^t t_1^{-\left(\frac{d}{2r}+\frac{1}{2}\right)q'} (t-t_1)^{\alpha q'} \, dt_1 \right)^{\frac{1}{q'}} \\
&= c_1 \nu^{-\left(\frac{d}{2r}+\frac{1}{2}\right)} \eta(\alpha) \|f_1\|_{r,q} \, t^{\alpha+\delta_1},
\end{aligned}
$$

where

$$
\eta(\alpha) := \left(\frac{\Gamma(\delta_1 q')\Gamma(\alpha q'+1)}{\Gamma(\delta_1 q' + \alpha q' + 1)} \right)^{\frac{1}{q'}},
$$

Using Stirling's formula, we can readily show that $\eta(\alpha) \le c_2 \gamma_1(\alpha)$, for a constant c_2. This completes the proof when $k=1$. □

Proof of Theorem 3.1 when $k \ge 2$. Without loss of generality, we may assume that $t_0 = 0$. Evidently (4.1) in the case of $\beta = 0$ implies (3.2) when $k=2$ because $p(z_1,t_1) \le 2^{\frac{d}{2}} p(z_1, 2t_1)$ (or we can readily show that Lemma 4.1 is true if $p(z_1,t_1)$ is replaced with $p(z_1,t_1)$). Let us assume that $k > 2$. We may express $I^k(f_1,\dots,f_k)$ as

$$
\int_0^t \int_0^{t_k} \int_{\mathbb{R}^{2d}} A(z_{k-1},t_{k-1}) f_k(z_k,t_k) p^{(2)}(z_k - z_{k-1}, t_k - t_{k-1})(t-t_k)^\alpha \, dz_{k-1} dz_k dt_{k-1} dt_k,
$$

where $A(z_{k-1},t_{k-1})$ is given by

$$
\int_{\Delta^{k-2}(t_{k-1})} \int_{\mathbb{R}^{(k-2)d}} f_1(z_1,t_1) p(z_1,t_1) \prod_{i=2}^{k-1} f_i(z_i,t_i) p_i^{(1)}(z_i - z_{i-1}, t_i - t_{i-1}) \prod_{i=1}^{k-2} dz_i dt_i.
$$

By Lemma 4.2 the expression $A(z_{k-1},t_{k-1})$ is bounded above by

$$
\le C_9^{k-2} \, \nu^{-\left(1+\frac{d}{r}\right)\frac{k-2}{2}} \, f_{k-1}(z_{k-1},t_{k-1}) \, p\left(z_{k-1}, 2t_{k-1}\right) \left((k-2)\delta_1\right)^{-(k-2)\delta_1} t_{k-1}^{(k-2)\delta_1} \prod_{i=1}^{k-2} \|f_i\|_{r,q}.
$$

Hence,

$$
\begin{aligned}
I^k(f_1,\dots,f_k) = \int_0^t \int_0^{t_k} \int_{\mathbb{R}^{2d}} & g(z_{k-1},t_{k-1}) f_k(z_k,t_k) p(z_{k-1}, 2t_{k-1}) p^{(2)}(z_k - z_{k-1}, t_k - t_{k-1}) \\
& \times t_{k-1}^{(k-2)\delta_1} (t-t_k)^\alpha \, dz_{k-1} dz_k dt_{k-1} dt_k,
\end{aligned}
$$

where $g = f_{k-1} B$ with

$$
|B(z_{k-1},t_{k-1})| \le C_9^{k-2} \, \nu^{-\left(1+\frac{d}{r}\right)\frac{k-2}{2}} \, \left((k-2)\delta_1\right)^{-(k-2)\delta_1} \prod_{i=1}^{k-2} \|f_i\|_{r,q}. \tag{4.10}
$$

Since I^k can be written as $I'(g, f_k)$, we can use Lemma 4.1 and (4.10) to assert that the expression $|I^k(f_1, \ldots, f_k)|$ is bounded above by

$$C_8 C_9^{k-2}\, \nu^{-\left(1+\frac{d}{r}\right)\frac{k}{2}}\ ((k-2)\delta_1)^{-(k-2)\delta_1}\, \frac{\alpha^\alpha\, ((k-2)\delta_1)^{(k-2)\delta_1}}{(\alpha + k\delta_1)^{\alpha+k\delta_1}}\, ((k-2)\delta_1+1)^{\delta_1 - \frac{d}{2r}}\ t^{\alpha+k\delta_1} \prod_{i=1}^{k} \|f_i\|_{r,q}.$$

From this we can readily deduce (3.2). □

4.2 Bounding Double Integrals

The main reason that we were able to bound the block integrals $I(\beta_1, \ldots, \beta_n)$ that appeared in (3.3) has to do with the fact that $\beta_1 + \cdots + \beta_n = n$. This means that any second derivative of p much be matched with a 0-th derivative so that the singular integral associated with a second derivative can be controlled. However, if in place of (1.5) we assume the stronger condition (1.17), then bounding the block integrals of type I_k becomes easier because we can bound double integrals involving first and second derivatives of p. To explain this, let us define $K(f_1, f_2)$ as

$$\int_{\Delta^2} \int_{\mathbb{R}^{2d}} f_1(z_1, t_1) f_2(z_2, t_2) p^{(1)}(z_1, t_1 - t_0) p^{(2)}(z_2 - z_1, t_2 - t_1)(t - t_2)^\alpha\ dz_1 dz_2 dt_1 dt_2,$$

where $p^{(1)} = p_{z^i}$ and $p^{(2)} = p_{z^j z^k}$ for some $i, j, k \in \{1, \ldots, d\}$.

Theorem 4.1 *Assume* (1.17). *There exists a constant* $C_{10} = C_{10}(r, q)$ *such that*

$$|K(f_1, f_2)| \leq C_{10} \nu^{-\left(\frac{d}{2r}+\frac{1}{2}\right)} \hat{\zeta}(\alpha)(t - t_0)^{\alpha+2\delta_2} \|f_1\|_{r,q} \|f_2\|_{r,q}, \tag{4.11}$$

where

$$\hat{\zeta}(\alpha) = \frac{\alpha^\alpha}{(\alpha + 2\delta_2)^{\alpha+2\delta_2}}.$$

Proof. The proof is only sketched because it is very similar to the proof of Lemma 4.1 when $k = 2$. Without loss of generality, assume that $t_0 = 0$, and define f_i' as in the proof of Lemma 4.1. We decompose $K = K_1 + K_2$ where K_i is obtained from K by replacing the domain of integration Δ^2 with Δ_i, and Δ_1 and Δ_2 are defined as in the proof of Lemma 4.1.

The expression $|K_1|$ is bounded above by

$$
\begin{aligned}
&c_0\nu^{-\left(\frac{d}{r}+\frac{3}{2}\right)}\int_{\Delta_1} f_1'(t_1)f_2'(t_2)t_1^{-\frac{d}{2r}-\frac{1}{2}}(t_2-t_1)^{-\frac{d}{2r}-1}(t-t_2)^{\alpha}\ dt_1dt_2\\
&\quad\le c_0\nu^{-\left(\frac{d}{r}+\frac{3}{2}\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ \left(\int_{\Delta_1} t_1^{-\left(\frac{d}{2r}+\frac{1}{2}\right)q'}(t_2-t_1)^{-\left(\frac{d}{2r}+1\right)q'}(t-t_2)^{\alpha q'}\ dt_1dt_2\right)^{\frac{1}{q'}}\\
&\quad\le c_0\nu^{-\left(\frac{d}{r}+\frac{3}{2}\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ \left(\int_{\Delta_1} t_1^{-\left(\frac{d}{2r}+\frac{3}{4}\right)q'}(t_2-t_1)^{-\left(\frac{d}{2r}+\frac{3}{4}\right)q'}(t-t_2)^{\alpha q'}\ dt_1dt_2\right)^{\frac{1}{q'}}\\
&\quad= c_0\nu^{-\left(\frac{d}{r}+\frac{3}{2}\right)}\|f_1\|_{r,q}\|f_2\|_{r,q}\ t^{2\delta_1+\alpha}\eta(\alpha),
\end{aligned}
$$

where η is defined by

$$
\eta(\alpha):=\left(\frac{\Gamma(\delta_2q')^2\Gamma(\alpha q'+1)}{\Gamma(2\delta_2q'+\alpha q'+1)}\right)^{\frac{1}{q'}},
$$

with $q'=q/(q-1)$. As a result,

$$
|K_1|\le c_0\nu^{-\left(\frac{d}{r}+\frac{3}{2}\right)}\eta(\alpha)\|f_1\|_{r,q}\|f_2\|_{r,q}\ t^{2\delta_2+\alpha}, \tag{4.12}
$$

It remains to bound K_2.

Step 2. We next decompose K_2 as $K_{21}+K_{22}$, where K_{21} and K_{22} are defined as in Step 2 of the proof of Lemma 4.1. This time the corresponding $I_2(k,\ell)$ is defined by

$$
\int_{\Delta_2}\int_{B_{k\ell}(t_1,t_2)} f_1(z_1,t_1)f_2(z_2,t_2)p^{(1)}(z_1,t_1)p^{(2)}(z_2-z_1,t_2-t_1)(t-t_2)^{\alpha}\ dz_1dz_2dt_1dt_2.
$$

Again if $(z_1,z_2)\in B_{k\ell}(t_1,t_2)$, with $(t_1,t_2)\in\Delta_2$ and some $(k,\ell)\in\Lambda_1$, then

$$
|z_2-z_1|^2\ge d\nu t_1.
$$

As a result,

$$
\begin{aligned}
|p^{(2)}(z_2-z_1,t_2-t_1)| &\le \frac{c_1}{\nu(t_2-t_1)}p(z_2-z_1,2(t_2-t_1))\\
&= \frac{c_1}{(\nu(t_2-t_1))^{\frac{3}{4}}}\frac{|z_2-z_1|^{\frac{1}{2}}}{(\nu(t_2-t_1))^{\frac{1}{4}}}\frac{1}{|z_2-z_1|^{\frac{1}{2}}}p(z_2-z_1,2(t_2-t_1))\\
&\le \frac{c_2}{(\nu(t_2-t_1))^{\frac{3}{4}}}\frac{1}{|z_2-z_1|^{\frac{1}{2}}}p(z_2-z_1,4(t_2-t_1))\\
&\le \frac{c_2}{(\nu(t_2-t_1))^{\frac{3}{4}}}\frac{1}{(\nu t_1)^{\frac{1}{4}}}p(z_2-z_1,4(t_2-t_1)).
\end{aligned}
$$

This in turn implies that the term K_{21} is bounded above by a constant multiple of

$$\begin{aligned}
\nu^{-1} & \int_\Delta \int_{\mathbb{R}^{2d}} |f_1(z_1,t_1) f_2(z_2,t_2)| t_1^{-\frac{1}{4}} p^{(1)}(z_1, 2t_1)|(t_2-t_1)^{-\frac{3}{4}} p(z_2-z_1, 4(t_2-t_1)) \\
& \qquad\qquad \times (t-t_2)^\alpha \; dz_1 dz_2 dt_1 dt_2 \\
& \leq c_3 \nu^{-\left(\frac{d}{r}+\frac{3}{2}\right)} \int_\Delta f_1'(t_1) f_2'(t_2) t_1^{-\frac{d}{2r}-\frac{3}{4}} (t_2-t_1)^{-\frac{d}{2r}-\frac{3}{4}} (t-t_2)^\alpha \; dt_1 dt_2 \\
& \leq c_3 \nu^{-\left(\frac{d}{r}+\frac{3}{4}\right)} \|f_1\|_{r,q} \|f_2\|_{r,q} \; t^{2\delta_2+\alpha} \eta(\alpha).
\end{aligned}$$

As a result,

$$|I_{21}| \leq c_4 \nu^{-\left(\frac{d}{r}+\frac{3}{4}\right)} \eta(\alpha) \|f_1\|_{r,q} \|f_2\|_{r,q} \; t^{2\delta_2+\alpha}, \tag{4.13}$$

It remains to bound K_{22}.

Step 3. Define f_{1k} and $f_{2\ell}$ as in Step 3 of the proof of Lemma 4.1 and use Plancheral's formula, to assert that for any $\delta > 0$, the term $|K_2(k,\ell)|$ is bounded above by

$$\begin{aligned}
2\pi^2 & \int_{\Delta_2} \int_{\mathbb{R}^d} \left(\delta^{-1} (\nu t_1)^{\frac{d+1}{2}} |\hat{f}_{1k}(\xi,t_1)|^2 + \delta (\nu t_1)^{-\frac{d+1}{2}} |\check{f}_{2\ell}(\xi,t_2)|^2 \right) |\xi|^2 \\
& \qquad\qquad \times e^{-4\pi^2 \nu (t_2-t_1)|\xi|^2} (t-t_2)^\alpha \; d\xi dt_1 dt_2 =: K_2^1(k,\ell) + K_2^2(k,\ell).
\end{aligned}$$

Further, $K_2^1(k,\ell)$ bounded above by

$$\begin{aligned}
& \frac{2\pi^2}{\delta} \int_\Delta \int_{\mathbb{R}^d} (\nu t_1)^{\frac{d+1}{2}} |\hat{f}_{1k}(\xi,t_1)|^2 |\xi|^2 e^{-4\pi^2 \nu (t_2-t_1)|\xi|^2} (t-t_1)^\alpha \; d\xi dt_1 dt_2 \\
& \leq \frac{1}{2\nu\delta} \int_0^t \int_{\mathbb{R}^d} (\nu t_1)^{\frac{d+1}{2}} |\hat{f}_{1k}(\xi,t_1)|^2 (t-t_1)^\alpha \; d\xi dt_1 \\
& = \frac{1}{2\nu\delta} \int_0^t \int_{\mathbb{R}^d} (\nu t_1)^{\frac{d+1}{2}} |f_{1k}(x,t_1)|^2 (t-t_1)^\alpha \; dx dt_1 \\
& \leq c_5 e^{-\frac{1}{4}(|k|-\sqrt{d})^{+2}} (\nu\delta)^{-1} \int_0^t (\nu t_1)^{-\left(\frac{d}{2}+\frac{1}{2}\right)} \int_{B_k(t_1)} |f(x,t_1)|^2 (t-t_1)^\alpha \; dx dt_1.
\end{aligned}$$

We now apply Hölder's inequality to assert that $K_2^1(k,\ell)$ bounded above by

$$|K_2^1(k,\ell)| \leq c_6 \nu^{-\left(\frac{d}{r}+\frac{3}{4}\right)} e^{-\frac{1}{5}|k|^2} \eta'(\alpha) \|f_1\|_{r,q}^2 \; t^{2\delta_2+\alpha}. \tag{4.14}$$

where

$$\eta'(\alpha) = \left(\frac{\Gamma(2\delta_2 q/(q-2)) \Gamma(\alpha q/(q-2)+1)}{\Gamma((2\delta_2+\alpha)q/(q-2)+1)} \right)^{\frac{q-2}{q}}.$$

In the same fashion we show

$$|K_2^2(k,\ell)| \leq c_7 \delta \nu^{-\left(\frac{d}{r}+1\right)} \eta'(\alpha) \|f_2\|_{r,q}^2 \; t^{2\delta_2+\alpha}. \tag{4.15}$$

The rest of the proof is as in the proof of Lemma 4.1. $\square$

5 Symplectic Diffusions and Navier-Stokes Equation

Proof of Theorem 1.2. Step 1. For $u \in L^{r,q}$, with r and q satisfying (1.5), choose a sequence of smooth functions u_N such that $\|u_N - u\|_{r,q} \to 0$ as $N \to \infty$. Write Ω for the space of pair of continuous functions $X : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ and $B : [0,T] \to \mathbb{R}^d$ such that X is weakly differentiable with respect to the spatial variables and D_aX is locally in L^p for every $p \geq 1$. We equip Ω with a topology of L^∞_{loc} for X and weak topology for D_aX. Consider the SDE

$$dX = u_N(X,t)dt + \sigma dB, \tag{5.1}$$

where B is a standard Brownian motion. The law of the pair (X, B) is a probability measure $\mathcal{P}^N$ on the space Ω such that the B component is a standard Brownian motion. Using the equations (5.1), (2.2), (2.10) and Girsanov's formula we can readily show

$$\int \sup_{t\in[0,T]} |X(a,t) - a|^2 \; d\mathcal{P}^N \leq c_0 T + c_0 \int \int_0^T |u_N(X(a,t)|^2 dt \; d\mathcal{P}^N \leq c_1 T, \tag{5.2}$$

for a constant c_1 independent of N. We may use (5.2), Theorem 1.1 and Corollary 1.2 to assert that the family $\{\mathcal{P}_N\}_{N=1}^\infty$ is tight. Let $\mathcal{P}$ be a limit point of the family $\{\mathcal{P}_N\}_{N=1}^\infty$ as $N \to \infty$. Let $J : \mathbb{R}^d \to [0,\infty)$ be a continuous function of compact support. Use (5.1) to assert

$$\begin{aligned}
&\lim_{N\to\infty} \int \left\{ \sup_{t\in[0,t]} \int_{\mathbb{R}^d} \left| X(a,t) - a - B(t) - \int_0^t u(X(a,s),s)\; ds \right| \; J(a)\; da \right\} \; d\mathcal{P}_N \\
&\quad = \lim_{N\to\infty} \int \left[\sup_{t\in[0,t]} \int_{\mathbb{R}^d} \left| \int_0^t (u_N - u)(X(a,s),s)\; ds \right| \; J(a)\; da \right] \; d\mathcal{P}_N \\
&\quad = \lim_{N\to\infty} \int \left[\sup_{t\in[0,t]} \int_{\mathbb{R}^d} \left| \int_0^t (u_N - u)(a,s)\; ds \right| \; J\left(X^{-1}(a,t)\right) \left|\det D_a X^{-1}(a,t)\right| \; da \right] \; d\mathcal{P}_N = 0.
\end{aligned}$$

Note that the expression inside the curly brackets is a continuous functional. As a result, we may use our bounds on D_aX to show

$$\int \left\{ \int_{\mathbb{R}^d} \left| X(a,t) - a - B(t) - \int_0^t u(X(a,s),s)\; ds \right| \; J(a)\; da \right\} \; d\mathcal{P} = 0,$$

which in turn implies that the equation

$$X(a,t) = a + \int_0^t u(X(a,s),s)\; ds + B(t),$$

is valid $\mathcal{P}$-almost surely for almost all $a \in \mathbb{R}^d$, and hence for all a by continuity.

Step 2. We now verify (1.10). Since u_N is smooth, we apply Proposition 3.1 of [R] to assert

$$\int \left(X_t^*\beta^t - \beta^0\right)(V)\ dx - \int_0^t \left(\int X_s^*\left[\dot{\beta}^s + \mathcal{A}_{u_N}\beta^s\right](V)\ dx\right)\ ds$$
$$= \int_0^t \int \left(\sum_i X_s^*\gamma_i^s\right)(V)\ dx\ dB^i(s), \tag{5.3}$$

$\mathcal{P}_N$-almost surely. Replacing u_N with u results in an error that is bounded above by a constant multiple of

$$\begin{aligned} Err_t(X) :=& \int_0^t \int |D_a X(a,s)|\ |(u_N - u)(X(a,s),s)|\ |V(a)|\ dads \\ &+ \int_0^t \int |(u_N - u)(X(a,s),s)|\ |\nabla \cdot V(a)|\ dads. \end{aligned}$$

Finally we use (1.6) and (5.2) to show

$$\lim_{N\to\infty} \int \sup_{t\in[0,T]} Err_t\ d\mathcal{P}_N = 0.$$

This allows us to pass to the limit in (5.3) and deduce (1.10). □

Proof of Corollary 1.3 Let us write $x = (q,p)$ and set $\lambda = p \cdot dq$. We certainly have

$$\mathcal{A}_u\lambda = -\hat{d}(H - p \cdot H_p), \quad w_{q^i} = 0, \quad w_{p^i} \cdot dx = dp^i,$$

where $w = [p, 0]$. As a result the forms $\mathcal{A}_u\lambda$ and $w_{x^i} \cdot dx$ are exact for $i = 1, \dots, n$. From this and Theorem 1.2 we learn that $X_t^*\lambda$ is exact. This in turn implies that $X_t^*\hat{d}\lambda = 0$, as desired. □

Given a classical solution $u(\cdot,t)$ of (1.1), let us write $\alpha^t = u(\cdot,t) \cdot dx$ for the 1-form associated with u. In terms of α, the equation (1.1) may be written as

$$\dot{\alpha}^t + i_u\hat{d}\alpha^t + \hat{d}H^t = 0, \tag{5.4}$$

where $H^t(x) = \frac{1}{2}|u(x,t)|^2 + P(x,t)$. Here i_u denotes the contraction operator and we are simply using the identity

$$\sum_j u^i_{x^j}u^j = \sum_j (u^i_{x^j} - u^j_{x^i})u^j + \left(\frac{1}{2}|u|^2\right)_{x^i}.$$

Further, if we use Cartan's formula and write $\mathcal{L}_u = \hat{d} \circ i_u + i_u \circ \hat{d}$ for the Lie derivative in the direction of u, we may rewrite (5.4) as

$$\dot{\alpha} + \mathcal{L}_u \alpha^t - \hat{d} L^t = 0, \tag{5.5}$$

where $L = \frac{1}{2}|u|^2 - P$. Equation (5.5) can be used to give a geometric description of the Euler Equation (1.1): If we write $X(\cdot, t) = X_t(\cdot)$ for the flow of u as in (1.2), then (5.3) really means

$$\frac{d}{dt} X_t^* \alpha^t = X_t^* \hat{d} L^t,$$

or equivalently

$$X_t^* \alpha^t - \alpha^0 = \hat{d} K^t, \tag{5.6}$$

for $K^t = \int_0^t L^s \circ X_s \, ds$. The identity (5.6) is the celebrated Kelvin's circulation formula and coupled with the incompressibility condition $\nabla \cdot u = 0$ is equivalent to Euler Equation.

In the case of viscid fluid, the fluid velocity satisfies Navier-Stokes Equation. For our purposes, it is more convenient to use backward Navier-Stokes Equation

$$u_t + (Du)u + \nabla P + \nu \Delta u = 0. \tag{5.7}$$

For a classical solution of (5.7), we may write

$$\dot{\alpha}^t + \mathcal{A}_u \alpha^t - \hat{d} L = 0. \tag{5.8}$$

On the other hand, if X_t denotes the flow of SDE (1.4) and $\beta^t = X_t^* \alpha^t$, then

$$M^t := \beta^t - \beta^0 - \int_0^t X_s^* (\dot{\alpha} + \mathcal{A}_u \alpha) \, ds = \beta^t - \beta^0 - \hat{d} K^t,$$

is a martingale. In summary

$$X_t^* \alpha^t = \alpha^0 + M^t + \hat{d} K^t. \tag{5.9}$$

By taking the exterior derivative, we obtain

$$X_t^* \hat{d} \alpha^t = \hat{d} \alpha^0 + \hat{d} M^t. \tag{5.10}$$

For both (5.9) and (5.10) we are assuming that u is a classical solution of Navier-Stokes Equation. For a weak solution of (5.7) we wish to show that M^t is still a martingale. This is exactly the content of Theorem 1.3 provided that the solution u can be approximated by suitable regular functions.

Proof of Theorem 1.3. Assume that $(v, w) = (v^\varepsilon, w^\varepsilon)$ solves Camassa-Holm Equation with $v = w - \varepsilon\Delta w$. Set $\bar{\alpha}^t = v(\cdot, t) \cdot dx$ and write Y for the flow of the SDE

$$dY = w(Y, t)dt + \sigma dB.$$

As in (5.6), the equation (1.14) can be rewritten as

$$\frac{d}{dt}\bar{\alpha}^t + \mathcal{A}_w\bar{\alpha}^t - \hat{d}\bar{L}^t = 0.$$

This in turn implies

$$Y_t^*\bar{\alpha}^t = \bar{\alpha}^0 + \bar{M}^t + \hat{d}\bar{K}^t, \quad Y_t^*\hat{d}\bar{\alpha}^t = \hat{d}\bar{\alpha}^0 + \hat{d}\bar{M}^t, \tag{5.11}$$

where $\bar{M}^t$ is a martingale and $\bar{K}^t = \int_0^t \bar{L}^s \circ Y_s \, ds$. We now choose a subsequence of $w = w^\varepsilon$ so that $w^\varepsilon \to u$. From our assumption (1.15), Theorem 1.1 and Corollary 1.2 we can choose a further subsequence such that $Y = Y^\varepsilon \to X$ in L^∞_{loc}, and $D_aY^\varepsilon \to D_aX$ weakly in any L^p space. This allows us to pass to the limit in (5.11) to assert that the process (1.16) is a martingale. □

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