Gelation for Marcus-Lushnikov Process

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Abstract

Marcus-Lushnikov Process is a simple mean field model of coagulating particles that converges to the homogeneous Smoluchowski equation in the large mass limit. If the coagulation rates grow sufficiently fast as the size of particles get large, giant particles emerge in finite time. This is known as gelation and such particles are known as gels. Gelation comes in different flavors; simple, instantaneous and complete. In the case of an instantaneous gelation, giant particles are formed in a very short time. If all particles coagulate to form a single particle in a time interval that stays bounded as total mass gets large, then we have a complete gelation. In this article, we describe conditions which guarantee any of the three possible gelations with explicit bounds on the size of gels and the time of their creations.

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1 Introduction

The Smoluchowski equation is a coupled system of differential equations that describes the evolving densities (or concentrations) of a system of particles (or clusters) that are prone to coagulate in pairs. A sequence of functions $f_n : [0,\infty) \to [0,\infty)$, $n \in \mathbb{N}$, is a solution of the (discrete and homogeneous) Smoluchowski equation (SE) if it satisfies

$$\frac{d}{dt} f_n(t) = Q_n(f)(t), \tag{1.1}$$

with $Q_n = Q_n^+ - Q_n^-$, where

$$Q_n^+(f)(t) = \frac{1}{2}\sum_{m=1}^{n-1} \alpha(m, n-m) f_m(t) f_{n-m}(t), \quad Q_n^-(f)(t) = \sum_{m=1}^{\infty} \alpha(n,m) f_n(t) f_m(t).$$

The function f_n represents the density of particles of size n and the symmetric function $\alpha : \mathbb{N} \times \mathbb{N} \to (0,\infty)$ denotes the coagulation rate. Formally we have

$$\frac{d}{dt}\sum_n \psi(n) f_n = \frac{1}{2}\sum_{m,n} \alpha(m,n) f_m(t) f_n(t)(\psi(m+n) - \psi(m) - \psi(n)), \tag{1.2}$$

for any function ψ. An important choice for ψ is $\psi(n) = n$ with the sum $\sum_n n f_n$ interpreted as the total mass of particles. For such a choice the right hand side of (1.2) is 0 and this is consistent with our intuition; the total mass for coagulating particles is conserved. In reality the equation (1.2) is not valid and in the case of $\psi(n) = n$ we only have

$$\frac{d}{dt}\sum_n n f_n \leq 0. \tag{1.3}$$

Analytically speaking, we cannot interchange the differentiation with the summation in (1.2) and such an interchange can take place only if some suitable restrictions on the size of the coagulation rate $\alpha(m,n)$ is imposed as m and n get large. The strict inequality in (1.3) does not contradict the conservation of mass; for the sufficiently fast growing α, particles of infinite size - the so-called gels- are formed and the sum $\sum_n n f_n$ no longer represents the total mass. More precisely, if we write $g_n = n f_n$ for the total mass of particles of size n, then what we really have is

$$\frac{d}{dt}\left(\sum_{n=1}^{\infty} g_n + g_\infty\right) = 0. \tag{1.4}$$

Marcus-Lushnikov Process (MLP) is formulated as a simple microscopic model to study coagulation and gelation phenomena. MLP is a Markov process which is defined on a finite state space E_N given by

$$E_N = \left\{ \mathbf{L} = (L_1, L_2, \dots, L_n, \dots) : \sum_n nL_n = N, 0 \le L_n \in \mathbb{Z} \text{ for each } n \right\}.$$

What we have in mind is that L_n is the total number of particles of size n and the condition $\sum_n nL_n = N$ means that N is indeed the total mass of particles. The process $(\mathbf{L}^{(N)}(t) = \mathbf{L}(t) : t \in [0, \infty))$ is a Markov process with infinitesimal generator $\mathcal{A} = \sum_{m,n=1}^{\infty} \mathcal{A}_{m,n}$, where

$$\mathcal{A}_{m,n} F(\mathbf{L}) = \frac{1}{2N} \alpha(m,n) \left(L_m L_n - \mathbb{1}(m = n) L_m \right) \left(F(\mathbf{L}^{m,n}) - F(\mathbf{L}) \right).$$

When $m \ne n$, $\mathbf{L}^{m,n}$ is obtained from $\mathbf{L} = (L_1, L_2, \dots)$ by replacing L_n, L_m and L_{n+m} with $L_n - 1$, $L_m - 1$ and $L_{n+m} + 1$, respectively; when $m = n$, $\mathbf{L}^{m,n}$ is obtained from $\mathbf{L} = (L_1, L_2, \dots)$ by replacing L_n and L_{2n} with $L_n - 2$ and $L_{2n} + 1$, respectively. In words, with rate $\alpha(m,n)/N$, a pair of particles of sizes m and n is replaced with a single particle of size $m + n$. Note that the number of such pairs is $L_m L_n$ if $n \ne m$ and this number becomes $L_n(L_n - 1)$ if $m = n$. Also note that we intentionally have chosen a coagulation rate proportional to N^{-1}. The reason for this has to do with the fact that all pairs of particles are prone to coagulate and as a result, a typical particle undergoes a huge number of coagulations in one unit of time as N gets large. Our rescaling of α guarantees that in average a single particle experiences only a finite number of coagulations. The probability measure and the expectation associated with the Markov process $\mathbf{L}(t)$ are denoted by $\mathbb{P}_N$ and $\mathbb{E}_N$ respectively.

The connection between MLP and SE is that the large N limit $f_n := \lim_N L_n^{(N)}/N$ is expected to exist and satisfy SE. For this however suitable assumptions on α are needed. Before stating these conditions and a precise theorem relating MLP to SE, let us make some preparations. Set,

$$E = \left\{ \mathbf{f} = (f_1, f_2, \dots, f_n, \dots) : \sum_n n f_n \le 1, f_n \ge 0 \text{ for each } n \right\} \subset E' = [0, \infty)^{\mathbb{N}}.$$

We equip E' with the product topology. Evidently, E is a compact subset of E'. Let us write $\mathcal{D} = \mathcal{D}([0,T]; E)$ for the Skorohod space of functions from the interval $[0,T]$ into E. The space $\mathcal{D}$ is equipped with Skorohod topology. The Markov process $(\mathbf{L}(t) : t \in [0,T])$ induces a probability measure $\mathcal{P}_N$ on $\mathcal{D}$ via the transformation $\mathbf{L} \mapsto \mathbf{f}$, where $\mathbf{f} = (f_n : n \in \mathbb{N})$, with $f_n = L_n/N$. We are now ready to state our first result.

Theorem 1.1 *Assume*

$$\sup_{n,m} \frac{\alpha(m,n)}{m+n} < \infty, \tag{1.5}$$

and that initially

$$\lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{E}_N\frac{1}{N}\sum_{n\geq k}nL_n(0)=0,\qquad \lim_{N\to\infty}\mathbb{E}_N\left|\frac{L_n(0)}{N}-f_n^0\right|=0. \tag{1.6}$$

Then the sequence of probability measures $\{\mathcal{P}_N\}$ is tight and if $\mathcal{P}$ is a limit point of $\{\mathcal{P}_N\}$, then $\mathcal{P}$ is concentrated on the unique solution to SE subject to the initial condition $\mathbf{f}(0)=\mathbf{f}^0$.

Remark 1.1 The existence of a unique solution to SE under (1.5) has been established in Ball and Carr [BC]. Even though we have not been able to find a proof of Theorem 1.1 in the literature, we skip the proof because a straight forward adaption of [BC] can be used to prove Theorem 1.1. □

We now turn to the question of gelation which is the primary purpose of this article. We first recall a result of Escobedo at al. [EMP] on solutions to SE. We set $M(t)=M(\mathbf{f},t)=\sum_n nf_n(t)$.

Theorem 1.2 *Assume that $\alpha(m,n)\geq(mn)^a$, for some $a>\frac{1}{2}$. Then there exists a constant $C_0(a)$ such that for any solution $\mathbf{f}$ of SE,*

$$\int_0^\infty M(t)^2dt\leq C_0(a)M(0). \tag{1.7}$$

In particular, gelation occurs sometime before $T_0=C_0(a)/M(0)$. That is, for $t>T_0$, we have $M(t)<M(0)$.

We now discuss the microscopic analog of Theorem 1.2 for MLP. For this, let us define stopping times

$$\tau^{(N)}(b,c,\delta)=\tau(b,c,\delta)=\inf\left\{t:N^{-1}\sum_{n\geq cN^b}nL_n(t)\geq\delta\right\}. \tag{1.8}$$

The following was established by Jeon [J1].

Theorem 1.3 *Assume that $\alpha(m,n)\geq(mn)^a$, for some $a>\frac{1}{2}$. Then for every b and $\delta\in(0,1)$ and $c>0$,*

$$\sup_N\mathbb{E}_N\tau(b,c,\delta)<\infty. \tag{1.9}$$

Remark 1.2

- (i) Section 2 is devoted to the proof of Theorem 1.3. Even though we are not introducing any new idea and employing the same approach as in [J1], our proof is shorter, more straight forward and simpler.
- (ii) A weaker form of Theorem 1.3 was established by Aldous [A2] for a special class of coagulation rates α.

$\square$

Note that if the assumption of Theorem 1.3 holds, then the condition (1.5) is no longer true and in fact we need to modify SE if the sol-gel interaction is significant. It turns out that if

$$\lim_{m\to\infty} \frac{\alpha(m,n)}{m} =: \bar{\alpha}(n), \tag{1.10}$$

exists for every n, then it is not hard to figure out what the corrected SE looks like. Under (1.10), we still have (1.1), but now with a modified loss term. More precisely, $Q_n = Q_n^+ - \hat{Q}_n^-$, where the modified loss term $\hat{Q}_n^-$ reads as

$$\hat{Q}_n^-(f)(t) = \sum_{m=1}^{\infty} \beta(m,n) g_m(t) g_n(t) + \beta(n,\infty) g_n(t) g_\infty(t), \tag{1.11}$$

with $g_n = nf_n$, $\beta(n,m) = \alpha(n,m)/(mn)$ and $\beta(n,\infty)$ measures the amount of coagulation between particles of size n and gels. When the condition of Theorem 1.2 or 1.3 holds, we have that $g_\infty(t) > 0$ for $t > T_{gel}$. In fact, if (1.10) holds, then $\beta(n,\infty)$ is simply given by

$$\beta(n,\infty) = \frac{\bar{\alpha}(n)}{n}. \tag{1.12}$$

$\square$

The analog of Theorem 1.1 in this case is Theorem 1.4.

Theorem 1.4 *Assume* (1.10)*. Then the sequence of probability measures* $\{\mathcal{P}_N\}$ *is tight. Moreover, if* $\mathcal{P}$ *is a limit point of* $\{\mathcal{P}_N\}$*, then* $\mathcal{P}$ *is concentrated on the space of solutions to the modified SE with the loss term given by* (1.11) *and* (1.12) *and* $g_\infty = 1 - \sum_n g_n$.

Remark 1.3

- (i) Theorem 1.1 under the stronger condition $\bar{\alpha}(n) = \beta(n,\infty) = 0$ was established in [J1]. This condition does not exclude gelation. However, even though a fraction of the density comes from gels (i.e. $g_\infty > 0$) after the gelation time, the sol-gel interaction is sufficiently weak that can be ignored in the macroscopic description of the model.

- (ii) The continuous analog of ML model has been studied in Norris [N] and Fournier-Giet [FoG]. In the continuous variant of ML the cluster sizes take values in $(0, \infty)$ and all m summations in SE (1.1) and modified SE are replaced with dm integrations. In the continuous case, Theorem 1.1 under the stronger condition $\bar{\alpha}(n) = 0$ was established in [N] and under the assumption (1.10) in [FoG]. As is stated in [FoG], the modified SE has already been predicted by Flory [Fl]. See also Fournier and Laurencot [FoL] where a variant of continuous ML with cutoff has been studied.

- (iii) It is not hard to understand why a condition like (1.10) facilitates the derivation of the modified Smoluchowski's equation. The main idea is that even though the function $\mathbf{f} \mapsto \sum_m \alpha(m,n) f_m$ is not a continuous function with respect to the product topology whenever $\bar{\alpha}(n) \neq 0$, the function $\mathbf{f} \mapsto \sum_m (\alpha(m,n) - m\bar{\alpha}(n)) f_m$ is continuous. This can be easily used to establish Theorem 1.4 by standard arguments, providing a rather more direct proof of Theorem 1.4 than the one appeared in [FoG].

- (iv) If the condition (1.10) fails and instead we have the weaker property

$$\sup_m \alpha(m,n)/m < \infty,$$

it is not clear what macroscopic equation, if any describes the evolution of densities.

□

We next address the question of instantaneous gelation. We first recall a result of Carr and da Costa [CdC].

Theorem 1.5 *Assume that for some $q > 1$, we have that $\alpha(m,n) \geq m^q + n^q$. Then $M(t) < M(0)$ for every solution of SE and every $t > 0$. In words, gelation occurs instantaneously.*

We now state a theorem that is the microscopic analog of Theorem 1.5. To this end, let us define

$$T_k(\delta) = \inf \left\{ t : N^{-1} \sum_{n \geq k} n L_n(t) \geq \delta \right\}, \quad \hat{T}_A^{(N)}(\delta) = \hat{T}_A(\delta) = T_{A \log N / \log\log N}(\delta).$$

Theorem 1.6 *Assume that $\alpha(m,n) \geq m^q + n^q$, for some $q \in (1,2)$. Then for every positive $\delta < 1$, $A < q(2-q)^{-1}(6-q)^{-1}$, and $\theta < \bar{\eta}$, there exists a constant $C_2 = C_2(q, \theta, A)$, such that*

$$\mathbb{E}_N \hat{T}_A(\delta) \leq C_2 (1-\delta)^{-1} (\log N)^{-\theta}. \tag{1.13}$$

Here $\bar{\eta} = \bar{\eta}(q, A) = \min((q-1)/4, \bar{s} + q - 2)$ with $\bar{s}$ given by (3.5) *below.*

Remark 1.4

- (i) Note that the condition of Theorem 1.6 is stronger than what we assume in Theorem 1.3 because $m^q + n^q \geq 2(mn)^{q/2}$.
- (ii) Theorem 1.6 is more satisfactory than Theorem 1.5 for three reasons. On one hand in Theorem 1.5 we only claim that if there exists a solution to SE, then such a solution experiences an instantaneous gelation. In other words, we are only showing that there is no mass conserving solution. However it is not known that under the assumption of Theorem 1.5 a solution exists. On the other hand, the macroscopic densities coming from MLP cannot satisfy (1.1) and (1.11) because $\beta(n, \infty) = \infty$ and presumably a suitable modification of SE would be necessary. Finally, in Theorem 1.6 we are giving a bound on the time of the formation of a large particle. That is, we are giving more information about how instantaneous the gelation is. We should mention though that our Lemma 3.2 in Section 3 is partly inspired by the proof of Carr and da Costa in [CdC].
- (iii) We note that under the assumption of Theorem 1.3, the quickest way for gelation is to wait first for the creation of several large particles and then large particles coagulate among themselves to produce even larger particles very quickly. After all if both m and n are of order ℓ, then α is at least of order ℓ^{2a} with $2a > 1$. However, under the assumption of Theorem 1.6, gelation is the result of the coagulations of a large particle with any other particle. Note that for a particle of size ℓ to coagulate with another particle, it takes a short time of order $O(\ell^{-q})$ and $\sum_{\ell > \ell_0} \ell^{-q}$ is small if ℓ_0 is large. This explains why in Theorem 1.6 we have instantaneous gelation; once a single large particle is formed, this large particle coagulates almost immediately with the others to grow even larger.
- (iv) For instantaneous gelation, we only need $\alpha(n, m) \geq \eta(m) + \eta(n)$ with η satisfying $\sum_n \eta(n)^{-1} < \infty$. A similar comment applies to Theorem 1.7 below.
- (v) For simplicity, we avoided the case $q \geq 2$. In fact when $q = 2$, (1.13) is valid with no restriction on A and $\bar{\eta} = 1/2$ (see Remark 3.1 in Section 3). The condition $q > 2$ leads to instantaneous complete gelation that will be discussed in Theorem 1.7 below.

□

We finally turn to the question of complete gelation. Define

$$\tilde{\tau}^{(N)} = \tilde{\tau} = \inf \{ t : L_N(t) = 1 \} .$$

Theorem 1.7 *Assume that $\alpha(m, n) \geq m^q n + n^q m$, for some $q > 1$. Then there exists a constant $C_3 = C_3(q)$ such that*

$$\mathbb{E}_N \tilde{\tau} \leq C_3 \left(\frac{\log \log N}{\log N} \right)^{q-1} . \tag{1.14}$$

Remark 1.5 In Jeon [J2] it has been shown that a complete instantaneous gelation occurs if the requirement of Theorem 1.7 is satisfied. No bound on the time of complete gelation is provided in [J2] and we believe that our proof is simpler. □

Even though our assumption on α as it appears in Theorem 1.3 is the most commonly used condition to guarantee gelation, we now argue that it is the assumption of Theorem 1.6 that is more physically relevant. In a more realistic model for the coagulation phenomenon we would allow spatial dependence for particles. We are now interested on the evolution of particle density $\mathbf{f}(x,t) = (f_n(x,t) : n \in \mathbb{N})$ where $x \in \mathbb{R}^d$ represents the spatial position. The homogeneous SE is now replaced with the inhomogeneous SE,

$$\frac{\partial}{\partial t} f_n(x,t) = \frac{1}{2} d(n) \Delta_x f_n(x,t) + Q_n(f)(x,t),$$

where $d(n)$ denotes the diffusion coefficient of particles of size n, the operator Δ_x denotes the Laplace operator in x variable, and $Q(f)$ has the same form as in the homogeneous SE. Microscopically, particles have positions, masses and radii. Each particle travels as a Brownian motion with diffusion coefficient $d(m)$ where m denotes the mass of the particle. Particles may coagulate only when they are sufficiently close. For example the coagulation occurs between particles of positions x and x' only when $|x - x'|$ is of order $\varepsilon(r + r')$ where r and r' are the radii of particles and ε is a small parameter. When the dimension d is 3 or more, the initial number of particles is of order $O(N)$ with $N = \varepsilon^{2-d}$. When particles are close, they coagulate randomly with a rate that is proportional to $\alpha(m,n)$. This microscopic coagulation rate α is not the macroscopic coagulation rate that appears in SE. One can calculate the macroscopic coagulation rate $\hat{\alpha}$ from the microscopic coagulation rate α and the diffusion coefficient $d(\cdot)$ after some potential theory. We refer the reader to [HR1-2] and [R] for more details on this model and a precise formula of $\hat{\alpha}$. In this model of coagulating Brownian particles, a large microscopic coagulation rate would not lead to gelation. Instead, the radii of particles are what matter when it comes to the issue of gelation. Indeed, if the relationship between the mass m of a particle and its radius r is given by $r = m^{\chi}$, then for a gelation we need a condition of the form $\chi > (d-2)^{-1}$. This is quite understandable in view of Theorem 1.5 because for a uniformly positive α, the macroscopic coagulation rate $\hat{\alpha}(m,n)$ behaves like $(d(m) + d(m))(m^{\chi} + n^{\chi})^{2-d}$ as m and n get large (see [R]). As a result, if the diffusion coefficients $(d(n), n \in \mathbb{N})$ are uniformly positive and $\chi > (d-2)^{-1}$, then $\hat{\alpha}$ has a super-linear growth as the size of particles get large. Based on this we conjecture that an instantaneous gelation would occur if $\chi > (d-2)^{-1}$.

We end this introduction with the outline of the paper; Section 2 is devoted to the proof of Theorems 1.3. Theorem 1.6 will be established in Section 3. Section 4 is devoted to the proof of Theorem 1.7.

2 Simple Gelation

Proof of Theorem 1.3. For (1.9). it suffices to show that for every $b \in ((2a)^{-1}, 1)$ and positive δ, there exist constants $C_0(a, b, \delta)$ and $C_0'(a, b, \delta)$ such that

$$\sup_N \mathbb{E}_N \tau(b, C_0(a, b, \delta), \delta) \leq C_0'(a, b, \delta). \tag{2.1}$$

Explicit expressions for the constants C_0 and C_0' are given in (2.5) below.

Pick $\beta > 0$ and set $\delta_i = \delta + c2^{-i\beta}$ with the constant $c \in (0, 1-\delta]$ so that we always have $\delta_i \leq 1$. Define the stopping time

$$T_k = \inf \left\{ t : \sum_{n \geq 2^i} nL_n(t) \geq \delta_i N \text{ for } i = 0, 1, \ldots, k \right\}$$

for each $k \in \mathbb{N}$. Evidently $T_k \leq T_{k+1}$. We also define

$$F_k(\mathbf{L}) = \frac{1}{N} \sum_{n \geq 2^{k+1}} nL_n.$$

By strong Markov property,

$$\mathbb{E}_N F_k(\mathbf{L}(T_{k+1})) = \mathbb{E}_N F_k(\mathbf{L}(T_k)) + \mathbb{E}_N \int_{T_k}^{T_{k+1}} \mathcal{A} F_k(\mathbf{L}(t)) dt. \tag{2.2}$$

Note that if $T_k \leq t < T_{k+1}$, then

$$\sum_{n \geq 2^k} nL_n(t) \geq \delta_k N, \quad \sum_{n \geq 2^{k+1}} nL_n(t) < \delta_{k+1} N,$$
$$\frac{1}{N} \sum_{n=2^k}^{2^{k+1}-1} nL_n(t) \geq \delta_k - \delta_{k+1}.$$

Let us simply write $\mathbf{L} = \mathbf{L}(t)$ with t satisfying $T_k \leq t < T_{k+1}$. For such a configuration $\mathbf{L}$ we have that $\mathcal{A}F_k(\mathbf{L})$ equals

$$
\begin{aligned}
&\frac{1}{2N^2}\sum_{m,n}\alpha(m,n)L_m(L_n-\mathbb{1}(m=n))\left[(m+n)\mathbb{1}(m+n\geq 2^{k+1})-m\mathbb{1}(m\geq 2^{k+1})-n\mathbb{1}(n\geq 2^{k+1})\right]\\
&\geq \frac{1}{2N^2}\sum_{m,n}(mn)^aL_m(L_n-\mathbb{1}(m=n))\left[(m+n)\mathbb{1}(m+n\geq 2^{k+1})-m\mathbb{1}(m\geq 2^{k+1})-n\mathbb{1}(n\geq 2^{k+1})\right]\\
&= \frac{1}{2N^2}\sum_{m,n}(mn)^aL_mL_n\left[(m+n)\mathbb{1}(m+n\geq 2^{k+1})-m\mathbb{1}(m\geq 2^{k+1})-n\mathbb{1}(n\geq 2^{k+1})\right]\\
&\qquad -\frac{1}{2N^2}\sum_{m}m^{2a}L_m\left[(2m)\mathbb{1}(2m\geq 2^{k+1})-2m\mathbb{1}(m\geq 2^{k+1})\right]\\
&\geq \frac{1}{2N^2}\sum_{m,n}(mn)^aL_mL_n(m+n)\mathbb{1}(m+n\geq 2^{k+1}>m,n)-\frac{1}{2N^2}\sum_m m^{2a}L_m(2m)\mathbb{1}(2m\geq 2^{k+1}>m)\\
&= \frac{1}{2N^2}\sum_{m,n}(mn)^aL_mL_n(m+n)\mathbb{1}(m+n\geq 2^{k+1}>m,n)-\frac{1}{N^2}\sum_{m=2^k}^{2^{k+1}-1}m^{2a+1}L_m\\
&\geq \frac{1}{N^2}\left(\sum_{m=2^k}^{2^{k+1}-1}m^{a+1}L_m\right)\left(\sum_{n=2^k}^{2^{k+1}-1}n^aL_n\right)-\frac{1}{N^2}\sum_{m=2^k}^{2^{k+1}-1}m^{2a+1}L_m\\
&\geq \frac{1}{N^2}2^{ka}2^{k(a-1)}2^{-(a-1)^-}\left(\sum_{m=2^k}^{2^{k+1}-1}mL_m\right)^2-\frac{1}{N^2}2^{2(k+1)a}\left(\sum_{m=2^k}^{2^{k+1}-1}mL_m\right)\\
&\geq 2^{k(2a-1)}2^{-(a-1)^-}(\delta_k-\delta_{k+1})^2-\frac{1}{N}2^{2a+2ak}=c^2(1-2^{-\beta})^2(2^k)^{2a-1-2\beta}2^{-(a-1)^-}-\frac{1}{N}2^{2a}(2^k)^{2a}.
\end{aligned}
$$

First we want to make sure that the negative term does not cancel the positive term. For example, we may try to have

$$\frac{c^2}{2}(1-2^{-\beta})^2(2^k)^{2a-1-2\beta}2^{-(a-1)^-}\geq \frac{1}{N}2^{2a}(2^k)^{2a}.$$

For this it is suffices to assume

$$2^k\leq\left(c^2(1-2^{-\beta})^22^{-2a-(a-1)^--1}\right)^{\frac{1}{1+2\beta}}N^{\frac{1}{1+2\beta}}.$$

For such integer k we use (2.2) to deduce

$$1\geq \mathbb{E}_NF_k(\mathbf{L}(T_{k+1}))\geq\frac{c^2}{2}(1-2^{-\beta})^2(2^k)^{2a-1-2\beta}2^{-(a-1)^-}\ \mathbb{E}_N(T_{k+1}-T_k).$$

Hence,

$$\mathbb{E}_N(T_{k+1}-T_k) \le 2c^{-2}(1-2^{-\beta})^{-2}2^{(a-1)^-}(2^k)^{-(2a-1-2\beta)}.$$

Summing these inequalities over k yields,

$$\begin{aligned}\mathbb{E}_N T_\ell &\le 2c^{-2}(1-2^{-\beta})^{-2}2^{(a-1)^-}\sum_{k=0}^{\ell-1}(2^k)^{-(2a-1-2\beta)}\\ &\le 2c^{-2}(1-2^{-\beta})^{-2}2^{(a-1)^-}(1-2^{-(2a-1-2\beta)})^{-1},\end{aligned}$$

provided that $\beta < a-\frac{1}{2}$ and

$$2^\ell \le \left(c^2(1-2^{-\beta})^2 2^{-(a-1)^-}2^{-2a-1}\right)^{\frac{1}{1+2\beta}} N^{\frac{1}{1+2\beta}}. \tag{2.3}$$

If ℓ is the largest integer for which (2.3) holds, then

$$2^\ell \ge 2^{-1}\left(c^2(1-2^{-\beta})^2 2^{-(a-1)^-}2^{-2a-1}\right)^{\frac{1}{1+2\beta}} N^{\frac{1}{1+2\beta}} =: C(c,a,\beta)N^{\frac{1}{1+2\beta}}.$$

From this we deduce

$$\mathbb{E}_N\tau'_\beta \le 2c^{-2}(1-2^{-\beta})^{-2}2^{(a-1)^-}(1-2^{-(2a-1-2\beta)})^{-1} =: C'(c,a,\beta), \tag{2.4}$$

where τ'_β is the first time

$$N^{-1}\sum_{n\ge k} nL_n(t) \ge \delta,$$

with $k = C(c,a,\beta)N^{\frac{1}{1+2\beta}}$. Since $\beta \in (0, a-\frac{1}{2})$ is arbitrary, $b=(1+2\beta)^{-1}$ can take any value in the interval $((2a)^{-1},1)$. Finally we choose $c = 1-\delta$ to derive (2.1) from (2.4) with

$$C_0(a,b,\delta) = C\left(1-\delta, a, (b^{-1}-1)/2\right), \quad C_0'(a,b,\delta) = C'\left(1-\delta, a, (b^{-1}-1)/2\right). \tag{2.5}$$

□

3 Instantaneous Gelation

This section is devoted to the proof of Theorem 1.6. The main ingredient for the proof of Theorem 1.6 is Theorem 3.1.

Theorem 3.1 *Assume that $\alpha(m,n) \ge m^q + n^q$, for some $q \in (1,2)$. There exist positive constants $C_1 = C_1(q,s,\eta,\nu)$ and $k_0 = k_0(q,s,\eta)$ such that if $s > 2-q$, $\eta \in (0,(q-1)/4)$, $\delta \in (0,1)$, and $\nu > 1$, then*

$$\begin{aligned}\mathbb{E}_N T_k(\delta) \le& 4(2-q)^{-1}k^{-s+2-q} + 8k^{s(k-1)+3-q}N^{-1}\\ &+ C_1(1-\delta)^{-1}k^{-\eta}(\log k)^{1-\eta} + C_1(1-\delta)^{-1}k^{3-q/2}(\log k)^3 N^{\frac{-q}{2s(k-1)}},\end{aligned} \tag{3.1}$$

for every k satisfying $k > k_0$ and

$$k^{(k-1)s+2} \leq N \leq e^{k^\nu}, \quad 2k^{-s} \leq 1. \tag{3.2}$$

Remark 3.1. For simplicity, we avoided the case $q = 2$. In fact when $q = 2$, (3.1) is valid if we replace the first term on the right-hand side with $4k^{-s} \log k$ (see (3.18) below). □

We first demonstrate how Theorem 3.1 implies Theorem 1.6.

Proof of Theorem 1.6. Set $k = A \log N / \log\log N$ in Theorem 3.1. We note that (3.2) is satisfied for large N if $sA < 1$. Let us first look at the second term on the right-hand side of (3.1). In fact the second term decays like a negative power of N if $sA < 1$. This is because

$$k^{s(k-1)+3-q} N^{-1} \leq c_0 N^{sA-1} (\log N)^{c_1}, \tag{3.3}$$

for some constants c_0 and c_1. To see this, take the logarithm of both sides to write,

$$sk \log k + (3 - q - s) \log k \leq \log c_0 + sA \log N + c_1 \log\log N.$$

First select c_1 large enough so that

$$(3 - q - s) \log k \leq (3 - q - s)(\log A + \log\log N) \leq c_1 \log\log N.$$

Then observe that if N satisfies $\log\log N \geq A$, then

$$sk \log k \leq sk \log\log N = sA \log N.$$

This completes the proof of (3.3) with $c_0 = 1$, provided that N satisfies $\log\log N \geq A$. Finally we adjust the constant c_0 to have the inequality (3.3) even when N satisfies $\log\log N < A$.

We now turn to the last term on the right-hand side of (3.1). By taking the logarithm of the last term, it is not hard to show that for a positive constant c_2,

$$k^{3-\frac{q}{2}} (\log k)^3 N^{\frac{-q}{2s(k-1)}} \leq c_2 (\log N)^{3-\frac{q}{2}-\frac{q}{2sA}} (\log\log N)^{\frac{q}{2}}.$$

The right-hand side of (3.1) goes to 0 as $N \to \infty$, if

$$s > 2 - q, \quad sA < \left(\frac{q}{6-q} \right) \wedge 1.$$

Now (3.1) implies

$$\mathbb{E}_N T_k(\delta) \leq c_3 (1-\delta)^{-1} \left[(\log N)^{-\eta} + (\log N)^{-\eta'} + (\log N)^{-\eta''} \right] (\log\log N)^{\gamma}, \tag{3.4}$$

with

$$\eta' = s + q - 2, \quad \eta'' = \frac{q}{2sA} + \frac{q}{2} - 3, \quad \gamma = \max(1, s + q - 2).$$

We now try to optimize (3.4) over s. By our assumption on A, we know that $(2-q)(6-q) < q/A$. Choose $s = \bar{s}$, where

$$\eta' = \bar{s} + q - 2 = \frac{q}{2\bar{s}A} + \frac{q}{2} - 3 = \eta''.$$

Hence,

$$\begin{aligned} \bar{s} &= \left(\sqrt{(1+q/2)^2 + 2q/A} - 1 - q/2\right)/2, \\ (\bar{s} - 2 + q)&(2\bar{s} + 6 - q) = q/A + (2-q)(q-6) > 0. \end{aligned} \tag{3.5}$$

As a result $\bar{s} > 2 - q$, $\eta' = \eta'' > 0$ and we can easily see

$$\bar{s}A = \frac{q}{2(\bar{s} + 1 + q/2)} < \frac{q}{6-q} \wedge 1,$$

is also valid. In summary,

$$\mathbb{E}_N T_k(\delta) \le 3c_3(1-\delta)^{-1}(\log N)^{-(\eta\wedge\eta')}(\log\log N)^{\gamma}, \tag{3.6}$$

where $\eta' = \bar{s} + q - 2$ with $\bar{s}$ as in (3.5). Finally observe that $\eta \wedge \eta'$ in (3.6) can be chosen to be any positive number $\theta < \bar{\eta}$. By decreasing θ a little bit, we can forget about the double logarithm and deduce (1.13). □

It remains to establish (3.1). The main ingredients for the proof of Theorem 3.1 are Lemmas 3.1 and 3.2. Before stating these lemmas and explaining that how they imply Theorem 3.1, let us provide some heuristics. Perhaps the best way to motivate our strategy is by taking a solution **f** of (1.1) and establish an instantaneous gelation for it. This is exactly what Carr and da Costa proved in [CdC]. However, we offer an alternative proof that is flexible enough to be carried out microscopically. The bottom line is that we would like to show that very quickly a good fraction of particles are large. We may start with the worst case scenario initially, namely when all particles are of size 1. That is, $f_1(0) = 1$ and $f_n(0) = 0$ for $n > 1$. We then use (1.2) to show that if $M_k(t) = \sum_{n\ge k} n f_n(t)$, then

$$\frac{dM_{k+1}(t)}{dt} \ge k^{q-1} M_k(t)(1 - M_{k+1}(t)).$$

(See the proof of Lemma 3.1 below.) Note that if $\theta(\delta)$ is the first time $M_2(t) \ge \delta$, then for $t < \theta(\delta)$,

$$\frac{dM_{k+1}(t)}{dt} \ge k^{q-1} M_k(t)(1-\delta).$$

The point is that staring from $M_1(t) = 1$ and $M_k(0) = 0$ for $k > 1$, we can use induction to deduce

$$M_{k+1}(t) \geq (k!)^{q-2}((1-\delta)t)^k := \bar{\delta}_{k+1}(t), \tag{3.7}$$

provided that $t < \theta(\delta)$. What we learn from this is that it takes a short time to have $\bar{\delta}_k$ fraction of mass constituting of particles of sizes at least k, provided that we choose δ_k positive but super-exponentially small as k gets large. As we try to carry out this argument for **L**, we encounter two difficulties: the discrete nature of the ML model introduces an additional error coming from coagulations between two particles of the same size (a microscopic coagulation rate $L_n^2 - L_n$ instead of L_n^2), and the noise in the system. However the inductive nature of the above argument allows us to handle these difficulties and establish a variant of (3.7) in Lemma 3.1.

Lemma 3.1 gives us a weak lower bound on the total mass of large particles because $\bar{\delta}_k$ in (3.7) is very small for large k. To see how such a weak lower bound can be improved, let us recall that as in [CdC] we may look at moments $R_p = \sum_n n^p f_n$ and show that in fact

$$\frac{dR_p(t)}{dt} \geq pR_p(t)^{1+\beta} M_1(t),$$

with $\beta = (q-1)/(p-1)$. If t is before the gelation time, then $M_1(t) = 1$ and we learn that $R_p(t)$ blows up at a finite time t_p which is very small if p is very large. Because of the randomness in our ML model, we do not know how to work out a microscopic variant of [CdC] argument. Instead we switch to the moments of large particles $M_{p,\ell} = \sum_{n \geq \ell} n^p f_n$ and observe that now

$$\frac{dM_{p,\ell+1}(t)}{dt} \geq pM_{p,\ell}(t)^{1+\beta}(1 - M_\ell(t))M_\ell(t)^{-\beta},$$

and if $t < \theta(\delta, \ell - 1)$, then

$$\frac{dM_{p,\ell+1}(t)}{dt} \geq pM_{p,\ell}(t)^{1+\beta}(1-\delta)\delta^{-\beta}. \tag{3.8}$$

The point is that now the right-hand side of (3.8) depends on the previous $M_{p,\ell}$ and therefore an inductive argument can be used to show that $M_{p,\ell}(t)$ can get very large for a time t that is small and p that is large. In other words, instead of showing that R_p becomes infinite at a time t_p that is small, we would rather show that $M_{p,\ell}(t)$ gets extremely large very quickly. The inductive nature of (3.8) makes it very useful in its microscopic form. More precisely, in the case of ML process we can show that a variant of (3.8) is true for the **L** process provided that we take the expectation of both sides. Then by induction on ℓ we can show that $M_{p,\ell}(t)$ gets very large very quickly. This is exactly the role of Lemma 3.2 below. In fact the induction starts from $\ell = k$ and we use Lemma 3.1 to argue that $M_{p,k}(t)$ is already large for some small t provided that p is sufficiently large. With the aid of Lemma 3.2, we show that if we wait for another short period of time, either a good fraction of particles are large, or else the high moments of density become super-exponentially large in k. Then a crude bound on moments of particle density demonstrates that the second alternative cannot occur and hence gels have already been formed.

To prepare for the statement of the first lemma, we take a sequence $(\delta_\ell : \ell = 1, \dots, k)$, and define

$$\sigma_\ell = \inf \left\{ t : \frac{1}{N} \sum_{n \geq r} n L_n(t) \geq \delta_r \text{ for } r = 1, 2, \dots, \ell \right\}.$$

Lemma 3.1 *For every decreasing sequence $(\delta_\ell : \ell = 1, \dots, k)$ which satisfies*

$$\delta_1 = 1, \quad 2\delta_2 \leq 1, \text{ and } \quad \frac{8k}{N} \leq \delta_k, \tag{3.9}$$

we have

$$\mathbb{E}_N \sigma_k \leq 4 \sum_{\ell=1}^{k-1} \ell^{1-q} \frac{\delta'_{\ell+1}}{\delta_\ell}. \tag{3.10}$$

where $\delta'_{\ell+1} = \delta_{\ell+1} + 2\ell N^{-1}$.

Define

$$T_{p,r}(A) = \inf \left\{ t : \frac{1}{N} \sum_{n \geq r} n^p L_n(t) \geq A \right\}.$$

Recall that we simply write $T_r(A)$ for $T_{p,r}(A)$ when $p = 1$.

Lemma 3.2 *Let $\{m_\ell : k \leq \ell \leq h\}$ be an increasing sequence and pick $p \geq 2$, $\delta > 0$. Assume that $N m_{\ell+1} \geq p \ell^p$ for every ℓ and write τ_r for $T_{p,r}(m_r) \wedge T_k(\delta)$. Then for $h > k$,*

$$\mathbb{E}_N(\tau_h - \tau_k) \leq \frac{2}{1-\delta} \sum_{\ell=k}^{h-1} \left[\frac{\delta^\beta m_{\ell+1}}{p m_\ell^{\beta+1}} + \ell^2 \left(\frac{p\ell}{N m_{\ell+1}} \right)^{\frac{q}{p-1}} \right], \tag{3.11}$$

where $\beta = (q-1)/(p-1)$ with q as in the statement of Theorem 1.6.

Proof of Lemma 3.1. To bound the stopping time σ_ℓ, we use the strong Markov property to write

$$\mathbb{E}_N G'_{\ell+1}(\mathbf{L}(\sigma_{\ell+1})) = \mathbb{E}_N G'_{\ell+1}(\mathbf{L}(\sigma_\ell)) + \mathbb{E}_N \int_{\sigma_\ell}^{\sigma_{\ell+1}} \mathcal{A} G'_{\ell+1}(\mathbf{L}(t)) dt, \tag{3.12}$$

where $G'_{\ell+1}(\mathbf{L}) = G_{\ell+1}(\mathbf{L}) \wedge \delta'_{\ell+1}$, with

$$G_k(\mathbf{L}) = \frac{1}{N}\sum_{n\ge k} nL_n.$$

Assume that $\sigma_\ell < \sigma_{\ell+1}$ and set

$$\Delta_{m,n} = (m+n)1\!\!1(m+n \ge \ell+1) - n1\!\!1(n \ge \ell+1) - m1\!\!1(m \ge \ell+1).$$

We certainly have that $\mathcal{A}G'_{\ell+1}(\mathbf{L})$ is bounded below by

$$\frac{1}{2N^2}\sum_{m,n}(m^q+n^q)L_n\left(L_m - 1\!\!1(m=n)\right)1\!\!1\left(G_{\ell+1}(\mathbf{L}) + N^{-1}\Delta_{m,n} \le \delta'_{\ell+1}\right)\Delta_{m,n}$$
$$\ge \frac{1}{N^2}\sum_{m,n}(m^q+n^q)L_n\left(L_m - 1\!\!1(m=n)\right)1\!\!1\left(G_{\ell+1}(\mathbf{L}) + (m+\ell)/N \le \delta'_{\ell+1}\right)m1\!\!1(n \ge \ell \ge m).$$

Here we restricted the summation to the cases $n \ge \ell \ge m$ and $m \ge \ell \ge n$ and used symmetry to consider the former case only. We note that if $n \ge \ell \ge m$, then either $\Delta_{m,n} = m$ or $m+\ell$. Also note that if $\mathbf{L} = \mathbf{L}(t)$ for some $t \in (\sigma_\ell, \sigma_{\ell+1})$ and $n \ge \ell \ge m$, then $G_{\ell+1}(\mathbf{L}) \le \delta_{\ell+1}$ and $G_{\ell+1}(\mathbf{L}) + (m+\ell)/N \le \delta'_{\ell+1}$. Hence for such a configuration $\mathbf{L}$,

$$\begin{aligned}\mathcal{A}G'_{\ell+1}(\mathbf{L}) &\ge \frac{1}{N^2}\sum_{m,n}(m^q+n^q)m1\!\!1(n \ge \ell \ge m)L_mL_n - \frac{2}{N^2}\ell^{q+1}L_\ell \\ &\ge \frac{1}{N^2}\left(\sum_{n\ge\ell} n^qL_n\right)\left(\sum_{m<\ell+1} mL_m\right) - \frac{2\ell^q}{N^2}\sum_m mL_m \\ &\ge \ell^{q-1}G_\ell(\mathbf{L})(1-G_{\ell+1}(\mathbf{L})) - \frac{2\ell^q}{N}.\end{aligned}$$

If $\sigma_\ell \le t < \sigma_{\ell+1}$, then $G_\ell(\mathbf{L}(t)) \ge \delta_\ell$, and $1 - G_{\ell+1}(\mathbf{L}(t)) \ge 1 - \delta_{\ell+1} \ge 1/2$ for $\ell \ge 1$, because by our assumption (3.9), $\delta_{\ell+1} \le 1/2$. Hence

$$\mathcal{A}G'_{\ell+1}(\mathbf{L}) \ge \frac{1}{2}\ell^{q-1}\delta_\ell - 2\ell^qN^{-1} \ge \frac{1}{4}\ell^{q-1}\delta_\ell,$$

where we have used the assumption (3.9) for the second inequality. From this and (3.12) we deduce

$$\frac{1}{4}\ell^{q-1}\delta_\ell\ \mathbb{E}_N(\sigma_{\ell+1} - \sigma_\ell) \le \mathbb{E}_N\left[G'_{\ell+1}(\mathbf{L}(\sigma_{\ell+1})) - G'_{\ell+1}(\mathbf{L}(\sigma_\ell))\right] \le \delta'_{\ell+1}.$$

As a result,

$$\frac{1}{4}\ell^{q-1}\delta_\ell\ \mathbb{E}_N(\sigma_{\ell+1} - \sigma_\ell) \le \delta'_{\ell+1},$$

Hence

$$\mathbb{E}_N(\sigma_{\ell+1}-\sigma_\ell) \le 4\ell^{1-q}\frac{\delta'_{\ell+1}}{\delta_\ell}.$$

Summing this inequality over ℓ and remembering that $\sigma_1 = 0$, leads to (3.10). □

Proof of Lemma 3.2. Step 1. We note that since $m_\ell < m_{\ell+1}$, we have that $\tau_\ell \le \tau_{\ell+1}$. Fix some positive $n_0 \in \mathbb{R}$ and write θ for the first time $L_n \ne 0$ for some $n \ge n_0$. We also set $\tau'_{\ell+1} = \tau_{\ell+1} \wedge (\theta \vee \tau_\ell)$. We use the strong Markov property to write

$$\mathbb{E}_N M'_{p,\ell+1}(\mathbf{L}(\tau'_{\ell+1})) = \mathbb{E}_N M'_{p,\ell+1}(\mathbf{L}(\tau_\ell)) + \mathbb{E}_N \int_{\tau_\ell}^{\tau'_{\ell+1}} \mathcal{A}M'_{p,\ell+1}(\mathbf{L}(t))dt, \tag{3.13}$$

where

$$M_{p,r}(\mathbf{L}) = \frac{1}{N}\sum_{n\ge r} n^p L_n, \quad M'_{p,r}(\mathbf{L}) = M_{p,r}(\mathbf{L}) \wedge (2m_r).$$

Write

$$\Delta_{m,n} := N^{-1}[(m+n)^p - n^p - m^p] \ge N^{-1}pn^{p-1}m =: \Delta'_{m,n}.$$

(Here we have used our assumption $p \ge 2$.) We certainly have that the expression $\mathcal{A}M'_{p,\ell+1}(\mathbf{L})$ is bounded below by

$$\begin{aligned}
&\frac{1}{2N}\sum_{m,n}(m^q+n^q)L_n\left(L_m - 1\!\!1(m=n)\right)1\!\!1(m\ge\ell>n \text{ or } n\ge\ell>m)\\
&\qquad\cdot\left[(M_{p,\ell+1}+\Delta_{m,n})\wedge(2m_{\ell+1}) - M_{p,\ell+1}\wedge(2m_{\ell+1})\right]\\
&\ge \frac{1}{N}\sum_{m,n}(m^q+n^q)L_nL_m1\!\!1(n\ge\ell>m)\left[\left(M_{p,\ell+1}+\Delta'_{m,n}\right)\wedge(2m_{\ell+1}) - M_{p,\ell+1}\wedge(2m_{\ell+1})\right]\\
&\ge \frac{1}{N^2}\sum_{m,n}(m^q+n^q)pn^{p-1}mL_mL_n1\!\!1(n\ge\ell>m)1\!\!1\left(M_{p,\ell+1}+\Delta'_{m,n}\le 2m_{\ell+1}\right).
\end{aligned}$$

We now assume that $m < \ell$ and that $\mathbf{L} = \mathbf{L}(t)$ for some $\tau_\ell < t < \tau'_{\ell+1}$. For such m and $\mathbf{L}$, we have

$$M_{p,\ell+1}(\mathbf{L}) + \Delta'_{m,n} \le m_{\ell+1} + N^{-1}pn_0^{p-1}\ell \le 2m_{\ell+1},$$

provided that we choose

$$n_0 = \left(\frac{Nm_{\ell+1}}{p\ell}\right)^{\frac{1}{p-1}}.$$

For such choices of $\mathbf{L}$ and n_0, we deduce

$$\mathcal{A}M'_{p,\ell+1}(\mathbf{L}) \ge \frac{p}{N^2}\left(\sum_{n\ge\ell} n^{p+q-1}L_n\right)\left(\sum_{m<\ell} mL_m\right) \ge pM_{p+q-1,\ell}(\mathbf{L})(1-G_\ell(\mathbf{L})).$$

If $t < T_k(\delta)$ and $k \le \ell$, then $G_\ell(\mathbf{L}(t)) \le G_k(\mathbf{L}(t)) < \delta$, and $1 - G_\ell(\mathbf{L}(t)) \ge 1 - \delta$. Hence

$$\mathcal{A}M'_{p,\ell+1}(\mathbf{L}) \ge p(1-\delta)M_{p+q-1,\ell}(\mathbf{L}), \tag{3.14}$$

whenever $\mathbf{L} = \mathbf{L}(t)$ for some $t < T_k(\delta)$. On the other hand, by Hölder Inequality,

$$M_{p+q-1,\ell}(\mathbf{L}) = \frac{1}{N}\sum_{n\ge\ell} n^{p+q-2} n L_n \ge G_\ell(\mathbf{L})^{-\beta} M_{p,\ell}^{1+\beta}$$

where $\beta = (q-1)/(p-1)$. From this and (3.14) we deduce that if $\mathbf{L} = \mathbf{L}(t)$ for some t satisfying $t \in (\tau_\ell, \tau'_{\ell+1})$, then

$$\mathcal{A}M'_{p,\ell+1}(\mathbf{L}) \ge p(1-\delta)\delta^{-\beta}M_{p,\ell}^{\beta+1}(\mathbf{L}) \ge p(1-\delta)\delta^{-\beta}m_\ell^{\beta+1}.$$

Here we have used the fact that if $\tau_\ell < \tau'_{\ell+1}$, then $\tau_\ell = T_{p,\ell}(m_\ell)$. (Simply because if $\tau_\ell \ne T_{p,\ell}(m_\ell)$, then we must have that $T_k(\delta) < T_{p,\ell}(m_\ell)$, which implies that $\tau'_{\ell+1} = \tau_\ell = T_k(\delta)$, $\tau'_{\ell+1} - \tau_\ell = 0$.) This and (3.13) imply

$$p(1-\delta)\delta^{-\beta}m_\ell^{\beta+1}\ \mathbb{E}_N(\tau'_{\ell+1} - \tau_\ell) \le \mathbb{E}_N\left[M'_{p,\ell+1}(\mathbf{L}(\tau'_{\ell+1})) - M'_{p,\ell+1}(\mathbf{L}(\tau_\ell))\right] \le 2m_{\ell+1}.$$

Therefore,

$$\mathbb{E}_N(\tau'_{\ell+1} - \tau_\ell) \le \frac{2\delta^\beta}{p(1-\delta)}\frac{m_{\ell+1}}{m_\ell^{\beta+1}},$$

Hence for (3.11) it suffices to establish

$$\mathbb{E}_N\left(\tau_{\ell+1} - \tau'_{\ell+1}\right) \le \frac{2\ell^2}{1-\delta}\left(\frac{Nm_{\ell+1}}{p\ell}\right)^{\frac{-q}{p-1}}. \tag{3.15}$$

Step 2. To establish (3.15), observe that if $\tau_{\ell+1} > \tau'_{\ell+1}$, then the configuration $\mathbf{L}(\tau'_{\ell+1})$ has at least one particle of size $n \ge n_0$. Let us mark one such particle and follow its interaction with other particles for $t \ge \tau'_{\ell+1}$. When this particle coagulates with any other particle of size a, then we increase its size $n(t)$ by a and remove the other particle from the system. We write $\beta_1 < \beta_2 < \dots$ for the consecutive coagulation times of the marked particle with particles of sizes $m < \ell$. Let us define an auxiliary process $(Z(t), K(t))$ that is defined for $t \ge \tau'_{\ell+1}$ with $Z(\tau'_{\ell+1}) = K(\tau'_{\ell+1}) = 0$ and each time our marked particle coagulates with a particle of size $m < \ell$, the value of K increases by 1 and the value of Z increases by $pn_0^{p-1}N^{-1}$. So, the process K simply counts the number of such coagulations and $Z(t) = pn_0^{p-1}N^{-1}K(t)$. Since at such a coagulation, the expression $M_{p,\ell+1}$ increases by $\Delta_{m,n(t)} \ge pn_0^{p-1}mN^{-1} \ge pn_0^{p-1}N^{-1}$, with n denoting the size of the marked particle, we have

$$M_{p,\ell+1}(\mathbf{L}(\beta_j)) \ge jpn_0^{p-1}N^{-1} = jm_{\ell+1}\ell^{-1}.$$

The right hand side is $m_{\ell+1}$ if $j=\ell$. As a result, $\beta_\ell \ge T_{p,\ell+1}(m_{\ell+1})$ and (3.15) would follow if we can show

$$\mathbb{E}_N\left(\tau_{\ell+1}-\tau'_{\ell+1}\right) \le \mathbb{E}_N\left(\beta_\ell \wedge T_k(\delta)-\tau'_{\ell+1}\right) \le \frac{2\ell^2}{1-\delta}\left(\frac{Nm_{\ell+1}}{p\ell}\right)^{\frac{-q}{p-1}}. \tag{3.16}$$

For this, use Markov property to write

$$\begin{aligned}
\ell &= \mathbb{E}_N(K(\beta_\ell)-K(\tau'_{\ell+1})) \ge \mathbb{E}_N(K(\beta_\ell\wedge T_k(\delta))-K(\tau'_{\ell+1}))\\
&= \mathbb{E}_N\int_{\tau'_{\ell+1}}^{\beta_\ell\wedge T_k(\delta)} \frac{1}{2N}\sum_{m<\ell}\alpha(m,n(t))(L_m(t)-1\!\!1(n(t)=m))dt\\
&= \mathbb{E}_N\int_{\tau'_{\ell+1}}^{\beta_\ell\wedge T_k(\delta)} \frac{1}{2N}\sum_{m<\ell}\alpha(m,n(t))L_m(t)dt \ge \mathbb{E}_N\int_{\tau'_{\ell+1}}^{\beta_\ell\wedge T_k(\delta)} \frac{n_0^q}{2\ell N}\sum_{m<\ell} mL_m(t)dt\\
&\ge \frac{n_0^q}{2\ell}\int_{\tau'_{\ell+1}}^{\beta_\ell\wedge T_k(\delta)}(1-G_\ell(\mathbf{L}(t)))dt \ge \frac{(1-\delta)n_0^q}{2\ell}\mathbb{E}_N\left(\beta_\ell\wedge T_k(\delta)-\tau'_{\ell+1}\right),
\end{aligned}$$

where $n(t)$ denotes the size of the marked particle. Here the third equality requires an explanation: Recall that by our assumption $Nm_{\ell+1}\ge p\ell^p$, which implies that $n(t)\ge n_0\ge \ell$ and $1\!\!1(n(t)=m)=0$ for $m<\ell$. Hence (3.16) is true and this completes the proof of (3.11). □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Step 1. There are various parameters in Lemmas 3.1 and 3.2 that we need to choose to serve our goal. We start from specifying the sequence $\{\delta_\ell : \ell=1,\dots,k\}$. We are going to choose $\delta_\ell = k^{-s(\ell-1)}$. Note that the conditions in (3.9) hold if

$$8k^{sk-s+1}\le N,\quad 2k^{-s}\le 1. \tag{3.17}$$

By (3.10),

$$\mathbb{E}_N\ \sigma_k \le 4k^{-s}\sum_{\ell=1}^{k-1}\left(\ell^{1-q}+2N^{-1}\ell^{2-q}k^{s\ell}\right)\le 4(2-q)^{-1}k^{-s+2-q}+8k^{s(k-2)+3-q}N^{-1}, \tag{3.18}$$

because $\delta_{\ell+1}/\delta_\ell = k^{-s}$.

Step 2. We now would like to apply Lemma 3.2. For this we first specify p to be $2s(k-1)+1$. We note that $p>2$ because $k>1+(2s)^{-1}$ follows from the condition $2k^{-s}\le 1$ of (3.17). Also note that if $\mathbf{L}=\mathbf{L}(\sigma_k)$, then

$$M_{p,k}(\mathbf{L}) = \frac{1}{N}\sum_{n\ge k} n^pL_n \ge k^{p-1}\delta_k = k^{p-1-s(k-1)} = k^{s(k-1)}. \tag{3.19}$$

Because of this, we are going to set $m_k = k^{s(k-1)}$, so that $T_{p,k}(m_k) \le \sigma_k$. We then specify m_ℓ for $\ell > k$. We require that $m_{\ell+1} m_\ell^{-\beta-1} = \ell^{-\eta}$ for some $\eta \in (0,1)$. This requirement leads to the formula

$$m_\ell = m_k^{(\beta+1)^{\ell-k}} \prod_{r=k}^{\ell-1} r^{-\eta(\beta+1)^{\ell-r-1}}. \tag{3.20}$$

In order to apply Lemma 3.2, we need to check that $\{m_\ell : k \le \ell \le h\}$ is an increasing sequence and that $N m_{\ell+1} \ge p\ell^p$. We establish this assuming that $h = Ak \log k$ and k is sufficiently large. Since $m_{\ell+1}/m_\ell = m_\ell^\beta \ell^{-\eta}$, we only need to show that $m_\ell^\beta > \ell^\eta$ for the monotonicity of m_ℓ. Note that for $m_k^\beta > k^\eta$, we need to assume that $\eta < (q-1)/2$. As we will see shortly, for $m_\ell^\beta > \ell^\eta$ for $k \le \ell \le h$, with $h = Ak\log k$ we need to assume more; it suffices to have $\eta < (q-1)/4$.

Observe

$$\log m_\ell = (\beta+1)^{\ell-k} \left[\log m_k - \eta \sum_{r=k}^{\ell-1} (\beta+1)^{k-r-1} \log r \right].$$

Let us write $a = \log(\beta+1)$. Note that for sufficiently large $k > k_1((q-1)/s)$, the function $r \mapsto e^{-(r+1)a} \log r$ is decreasing over the interval $[k, \infty)$. As a result,

$$\begin{aligned}
\sum_{r=k}^{\ell-1} (\beta+1)^{k-r-1} \log r &= \sum_{r=k}^{\ell-1} e^{(k-r-1)a} \log r \le e^{ka} \int_k^\infty e^{-ra} \log(r-1) dr \\
&\le e^{ka} \int_k^\infty e^{-ra} \log r \; dr = a^{-1} \log k + a^{-1} e^{ka} \int_k^\infty e^{-ra} r^{-1} dr \\
&= a^{-1} \log k + a^{-1} e^{ka} \int_{ak}^\infty e^{-r} r^{-1} dr \\
&\le a^{-1} \log k + a^{-1} e^{ka} (\log^+ (ak)^{-1} + c_2),
\end{aligned} \tag{3.21}$$

where $c_2 = \int_1^\infty e^{-r} r^{-1} dr$. Here we integrated by parts for the second equality. Recall that $a = \log(\beta+1)$ with $\beta = (q-1)/(p-1)$ and $p - 1 = 2s(k-1)$. As a result, ak is bounded and bounded away from 0, and

$$\lim_{k\to\infty} \frac{a^{-1} \log k + a^{-1} e^{ka} (\log^+ (ak)^{-1} + c_2)}{k \log k} = \frac{2s}{q-1}.$$

From all this we learn

$$\liminf_{k\to\infty} (\beta+1)^{-(\ell-k)} \frac{\log m_\ell}{k \log k} \ge s - \frac{2\eta s}{q-1}. \tag{3.22}$$

We choose $\eta \in (0,(q-1)/2)$ so that the left hand side is positive. For such η, choose γ such that

$$s\left(1-\frac{2\eta}{q-1}\right) > \gamma > 0.$$

Hence, for sufficiently large $k > k_2(q,s,\gamma)$ and every $\ell > k$,

$$\log m_\ell \geq \gamma(\beta+1)^{\ell-k} k \log k, \tag{3.23}$$

which implies

$$\beta \log m_\ell \geq \frac{\gamma(q-1)}{2s}(\beta+1)^{\ell-k}\log k. \tag{3.24}$$

Note that k_2 is independent of ℓ because (3.22) follows from (3.21) and the right-hand side of (3.21) is independent of ℓ. For the monotonicity of the sequence $\{m_\ell : k \leq \ell \leq h\}$, we need to show that $\beta \log m_\ell > \eta \log \ell$. By (3.24), it suffices to have

$$\beta \log m_\ell \geq \frac{\gamma(q-1)}{2s}(\beta+1)^{\ell-k}\log k > \eta \log h \geq \eta \log \ell. \tag{3.25}$$

Since $h = Ak\log k$, it suffices to have

$$\frac{\gamma(q-1)}{2s}(\beta+1)^{\ell}\log k > \eta(\beta+1)^k[\log k + \log\log k + \log A],$$

for $\ell \geq k$. This is true if $k > k_3(q,s,\gamma,\eta,A)$ for a suitable k_3 and $\gamma(q-1)/(2s) > \eta$. As a result, we need to select γ such that

$$s\left(1-\frac{2\eta}{q-1}\right) > \gamma > \frac{2s\eta}{q-1}. \tag{3.26}$$

Such a number γ exists if $\eta \in (0,(q-1)/4)$. So, let us assume that $\eta \in (0,(q-1)/4)$ and choose $\gamma = s/2$. In summary, there exists a constant $k_4 = k_4(q,s,\eta,A)$ such that if $k > k_4$ and $h = Ak\log k$, then the sequence $(m_\ell : \ell = k,\ldots,h)$ is increasing.

Step 3. So far we know that m_ℓ is increasing. In order to apply Lemma 3.2, we still need to check that $Nm_{\ell+1} \geq p\ell^p$ for ℓ satisfying $k \leq \ell \leq h$. We establish this by induction on ℓ. If $\ell = k$, then what we need is

$$Nm_{k+1} = Nm_k m_k^\beta k^{-\eta} = Nk^{s(k-1)}k^{(q-1)/2}k^{-\eta} \geq pk^p = (2s(k-1)+1)k^{2s(k-1)+1}.$$

Since $(q-1)/2 > \eta$, it suffices to have

$$N \geq k^{s(k-1)+2}, \tag{3.27}$$

and $k \geq k_5(q, s, \eta)$.

We now assume that $Nm_\ell \geq p(\ell-1)^p$ is valid and try to deduce $Nm_{\ell+1} \geq p\ell^p$. Indeed

$$Nm_{\ell+1} = Nm_\ell m_\ell^\beta \ell^{-\eta} \geq p(\ell-1)^p m_\ell^\beta \ell^{-\eta},$$

by induction hypothesis, and this is greater than $p\ell^p$, if

$$m_\ell^\beta \geq \ell^\eta \left(1 + \frac{1}{\ell-1}\right)^p \quad \text{or} \quad \beta \log m_\ell \geq \eta \log \ell + p \log\left(1 + \frac{1}{\ell-1}\right).$$

Since $p = 2s(k-1)+1$, the second term on the right-hand side is bounded and we only need to verify

$$\beta \log m_\ell \geq \eta \log h + c_3 \geq \eta \log \ell + c_3, \tag{3.28}$$

for a constant c_3. Except for the extra constant c_3, this is identical to (3.25) and we can readily see that the condition (3.26) would guarantee (3.28) if $k \geq k_6(q, s, \gamma, \eta, A)$. In summary, $Nm_{\ell+1} \geq p\ell^p$ is valid for ℓ satisfying $k \leq \ell \leq h$, provided that k is sufficiently large and (3.27) is satisfied. We observe that (3.27) implies the first inequality in (3.17) for $k \geq 8$.

Step 4. We assume that $\eta \in (0, (q-1)/4)$ and that $\gamma = s/2$. As before, we set $\tau_r = T_{p,r}(m_r) \wedge T_k(\delta)$. Since $\tau_k \leq T_{p,k}(m_k) \leq \sigma_k$, we may apply Lemma 3.2 to assert,

$$\begin{aligned}
\mathbb{E}_N\,(\tau_h - \sigma_k) \leq & \frac{2\delta^\beta}{p(1-\delta)} \sum_{\ell=k}^{h-1} \ell^{-\eta} + \frac{2N^{\frac{-q}{p-1}}}{(1-\delta)} \sum_{\ell=k}^{h-1} \ell^2 \left(\frac{p\ell}{m_{\ell+1}}\right)^{\frac{q}{p-1}} \\
\leq & \frac{2\delta^\beta}{p(1-\delta)(1-\eta)} h^{1-\eta} + \frac{2(m_k N)^{\frac{-q}{p-1}}}{(1-\delta)} \sum_{\ell=k}^{h-1} \ell^2 (p\ell)^{\frac{q}{p-1}} \\
\leq & \frac{2}{p(1-\delta)(1-\eta)} h^{1-\eta} + \frac{2(m_k N)^{\frac{-q}{p-1}}}{(1-\delta)} h^3 (ph)^{\frac{q}{p-1}}.
\end{aligned}$$

Hence

$$\mathbb{E}_N\,(\tau_h - \sigma_k) \leq 2(p(1-\delta)(1-\eta))^{-1} h^{1-\eta} + 2(1-\delta)^{-1} k^{-\frac{q}{2}} N^{\frac{-q}{2s(k-1)}} h^3 (ph)^{\frac{q}{p-1}}. \tag{3.29}$$

Our strategy is to choose h sufficiently large so that $\tau_h = T_k(\delta)$, because we are interested in bounding $T_k(\delta)$. We have the trivial bound $M_{p,h} \leq N^{p-1}$ because $N^{-1} \sum_n nL_n = 1$. Hence if h is sufficiently large so that $m_h > N^{p-1}$, then $T_{p,h}(m_h) = \infty$ and as a result $\tau_h = T_k(\delta)$. For $m_h > N^{p-1}$, we need

$$\log m_h > (p-1) \log N = 2s(k-1) \log N. \tag{3.30}$$

By (3.23),

$$\log m_h \geq \gamma(\beta+1)^{h-k} k \log k,$$

for $k > k_2(q,s,\gamma)$ and $\gamma = s/2$. As a result, the condition $m_h > N^{p-1}$ is realized if

$$\gamma(\beta+1)^{h-k} k \log k \geq 2sk \log N,$$

or equivalently

$$\log\gamma + Ak\log(\beta+1)\log k - k\log(\beta+1) + \log\log k \geq \log(2s) + \log\log N.$$

Since $\lim_k k\log(\beta+1) = (q-1)/(2s)$, as k goes to infinity, we pick $\mu \in (0,(q-1)/2s)$ and choose $k_7((q-1)/s)$ so that if $k > k_7$, then $k\log(\beta+1) > \mu$. For such k, we only need to have

$$\mu(A\log k - 1) + \log\log k \geq \log\frac{2s}{\gamma} + \log\log N = \log 4 + \log\log N,$$

to guarantee (3.30). Again for large $k > k_8(\mu)$, we have $\mu + \log 4 \leq \log\log k$ and we only need to have

$$k^{\mu A} \geq \log N. \tag{3.31}$$

In summary, there exists a constant $k_9 = k_9(q,s,\mu,A)$ such that (3.29) is valid with $\tau_h = T_k(\delta)$ if $k > k_9$, $h = Ak\log k$ and k satisfies (3.17), (3.27) and (3.31) with $\mu \in (0,(q-1)/2s)$.

Final Step. From (3.29) and (3.18) we learn

$$\begin{aligned}\mathbb{E}_N T_k(\delta) \leq& 4(2-q)^{-1}k^{-s+2-q} + 8k^{s(k-1)+3-q}N^{-1}\\ &+ 2(p(1-\delta)(1-\eta))^{-1}h^{1-\eta} + 2(1-\delta)^{-1}k^{-\frac{q}{2}}N^{\frac{-q}{2s(k-1)}}h^3(ph)^{\frac{q}{p-1}},\end{aligned} \tag{3.32}$$

because $\tau_h = T_k(\delta)$. The condition (3.31) combined with (3.17) and (3.27) yield

$$k^{\mu A} \geq \log N \geq (sk-s+2)\log k, \quad 2k^{-s} \leq 1. \tag{3.33}$$

For this to be plausible for large k, it suffices to have $\nu := \mu A > 1$. Since $\mu \in (0,(q-1)/(2s))$, we pick some

$$A > \frac{2s}{q-1},$$

and select $\mu \in (A^{-1},(q-1)/(2s))$. Since $h = Ak\log k$ and $p = 2s(k-1)+1$, the bound (3.32) implies

$$\begin{aligned}\mathbb{E}_N T_k(\delta) \leq& 4(2-q)^{-1}k^{-s+2-q} + 8k^{s(k-1)+3-q}N^{-1}\\ &+ c_4(1-\delta)^{-1}k^{-\eta}(\log k)^{1-\eta} + c_4(1-\delta)^{-1}k^{3-q/2}(\log k)^3 N^{\frac{-q}{2s(k-1)}},\end{aligned}$$

because $(ph)^{\frac{q}{p-1}}$ is uniformly bounded in k. This completes the proof of (3.1) because (3.33) is exactly (3.2). □

4 Complete Gelation

This section is devoted to the proof of Theorem 1.7. Lemma 4.1 is the main ingredient for the proof of Theorems 1.7. In this lemma we see how the stronger condition of Theorem 1.7 leads to a stronger variant of Lemma 3.1.

Lemma 4.1 *Assume that* $\alpha(m,n) \geq m^q n + n^q m$ *for some* $q > 1$. *Let* $(\sigma_\ell : \ell = 1, 2, \dots, h)$ *be as in Lemma 3.1 with* $(\delta_\ell : \ell = 1, 2, \dots, h)$ *a decreasing sequence satisfying* $\delta_1 = 1$, $2\delta_2 \leq 1$, *and* $4h/N \leq \delta_h$. *Then*

$$\mathbb{E}_N \sigma_h \leq 8 \sum_{\ell=1}^{h-1} \ell^{-q} \left(\frac{\delta_{\ell+1}}{\delta_\ell} + \frac{\ell}{N\delta_\ell} \right). \tag{4.1}$$

Proof. We wish to bound $\sigma_{\ell+1} - \sigma_\ell$. Given ℓ, we define an auxiliary process $(K(t) : t \geq \sigma_\ell)$ by the following rules: $K(\sigma_\ell) = 0$ and K increases by 1 each time a particle of size $n \geq \ell$ coagulates with a particle of size $m \leq \ell$; otherwise K stays constant. For simplicity we write $\mathcal{A}$ for the generator of the augmented process $\hat{\mathbf{L}}(t) = (\mathbf{L}(t), K(t))$. The successive instances at which K increases are denoted by $\theta_1 < \theta_2 < \dots$. We also abuse the notation to write K for the function that maps $\hat{\mathbf{L}}$ to its second component K. By strong Markov property,

$$\mathbb{E}_N K(\sigma_{\ell+1} \wedge \theta_r) = \mathbb{E}_N \int_{\sigma_\ell}^{\sigma_{\ell+1} \wedge \theta_r} \mathcal{A} K(\hat{\mathbf{L}}(t)) dt. \tag{4.2}$$

If $\hat{\mathbf{L}} = \hat{\mathbf{L}}(t)$ for some $t \in (\sigma_\ell, \sigma_{\ell+1})$, then

$$\begin{aligned}
\mathcal{A}K(\hat{\mathbf{L}}) &= \frac{1}{2N} \sum_{m,n} \alpha(m,n) L_m \left(L_n - 1\!\!1(m=n)\right) 1\!\!1(n \geq \ell \geq m) \\
&\geq \frac{1}{2N} \sum_{m,n} n^q m L_m \left(L_n - 1\!\!1(m=n)\right) 1\!\!1(n \geq \ell \geq m) \\
&\geq \frac{1}{2N} \sum_{m,n} n^q m L_m L_n 1\!\!1(n \geq \ell \geq m) - \frac{1}{2N} \ell^{q+1} L_\ell \\
&= \frac{1}{2N} \left(\sum_{n \geq \ell} n^q L_n \right) \left(\sum_{m \leq \ell} m L_m \right) - \frac{1}{2N} \ell^{q+1} L_\ell \\
&\geq \frac{1}{2} \ell^{q-1} N \left(\frac{1}{N} \sum_{n \geq \ell} n L_n \right) \left(\frac{1}{N} \sum_{m \leq \ell} m L_m \right) - \frac{1}{2} \ell^q \\
&= \frac{1}{2} \ell^{q-1} N \left(\frac{1}{N} \sum_{n \geq \ell} n L_n \right) \left(1 - \frac{1}{N} \sum_{m \geq \ell+1} m L_m \right) - \frac{1}{2} \ell^q \\
&\geq \frac{1}{2} \ell^{q-1} N \delta_\ell (1 - \delta_{\ell+1}) - \frac{1}{2} \ell^q \geq \frac{1}{8} \ell^{q-1} N \delta_\ell,
\end{aligned}$$

because by assumption we know that $\delta_{\ell+1} \le 1/2$ and $\ell \le N\delta_\ell/4$. From this and (4.2) we deduce

$$r \ge \mathbb{E}_N K(\sigma_{\ell+1} \wedge \theta_r) \ge \frac{1}{8}\ell^{q-1} N\delta_\ell \, \mathbb{E}_N \left(\sigma_{\ell+1} \wedge \theta_r - \sigma_\ell\right).$$

As a result

$$\mathbb{E}_N \left(\sigma_{\ell+1} \wedge \theta_r - \sigma_\ell\right) \le 8\ell^{1-q} \frac{r}{N\delta_\ell}. \tag{4.3}$$

Observe that each time K increases by 1, the expression

$$G_{\ell+1} = N^{-1} \sum_{n \ge \ell+1} nL_n,$$

increases by at least $(\ell+1)/N$. Hence $G_{\ell+1}(\theta_r) \ge r(\ell+1)N^{-1}$. If we choose

$$r = \left[\frac{N\delta_{\ell+1}}{\ell+1}\right] + 1 \le \frac{N\delta_{\ell+1}}{\ell+1} + 1,$$

then for such r, we have that $G_{\ell+1}(\theta_r) \ge \delta_{\ell+1}$ and as a result $\theta_r \ge \sigma_{\ell+1}$. (Here $[\cdot]$ denotes the integer part.) Using this r in (4.3) yields

$$\mathbb{E}_N \left(\sigma_{\ell+1} - \sigma_\ell\right) \le 8\ell^{-q} \left(\frac{\delta_{\ell+1}}{\delta_\ell} + \frac{\ell}{N\delta_\ell}\right).$$

Adding this over ℓ yields the desired result.

□

Proof of Theorem 1.7. Given k and h with $h > k$, we apply Lemma 4.1 to the sequence $\delta_\ell = k^{(1-q)(\ell \wedge k-1)}$ for $\ell = 1, \dots, h$, to assert

$$\begin{aligned}\mathbb{E}_N \sigma_h &\le 8k^{1-q} \sum_1^{k-1} \ell^{-q} + 8 \sum_k^{h-1} \ell^{-q} + 8N^{-1} \sum_1^{k-1} \ell^{1-q} k^{(q-1)(\ell-1)} + 8N^{-1} k^{(q-1)(k-1)} \sum_k^{h-1} \ell^{1-q} \\ &\le c_1 k^{1-q} + c_1 N^{-1} k^{(q-1)(k-2)+2-q} + c_1 N^{-1} k^{(q-1)(k-1)} h^{2-q}, \end{aligned} \tag{4.4}$$

provided that

$$4N^{-1} h k^{(q-1)(k-1)} \le 1, \quad k \ge 2^{\frac{1}{q-1}}. \tag{4.5}$$

These requirements come from our conditions on δ_ℓ in Lemma 4.1. We may use (4.5) to simplify the bound (4.4) to

$$\mathbb{E}_N \sigma_h \le c_1 k^{1-q} + \frac{1}{2} c_1 h^{1-q} \le \frac{3}{2} c_1 k^{1-q}, \tag{4.6}$$

where in view of (4.5), h can be chosen to be as large as

$$h = \left[4^{-1} N k^{(1-q)(k-1)}\right], \tag{4.7}$$

with $[\cdot]$ denoting the integer part.

Note that by definition, at time σ_h we already have a particle of size at least h. We mark one such particle and keep track of its size $\bar{N}(t)$ at later times $t \geq \sigma_h$. We also define an auxiliary process $(K(t) : t \geq \sigma_\ell)$ by the following rules: $K(\sigma_h) = 0$ and K increases by 1 each time the marked particle coagulates with another particle. We would like to use this marked particle to produce a complete gelation. Define the stopping time

$$S_r = \inf\left\{t : \bar{N}(t) \geq r\right\}.$$

Our goal is bounding $S_{r+1} - S_r$. Note that if $S_{r+1} - S_r \neq 0$, then $\bar{N}(t) = r$ for every $t \in (S_r, S_{r+1})$, and

$$K(S_{r+1}) - K(S_r) = 1,$$

because any coagulation of the marked particle results in $\bar{N} \geq r+1$. As before we write $\mathcal{A}$ for the generator of the augmented process $\hat{\mathbf{L}}(t) = (\mathbf{L}(t), K(t))$ and abuse the notation to write K for the function that maps $\hat{\mathbf{L}}$ to its second component K. Note that if $\hat{\mathbf{L}} = \hat{\mathbf{L}}(t)$ for some $t \in (S_r, S_{r+1})$, then

$$\begin{aligned}\mathcal{A}K(\hat{\mathbf{L}}) &= \frac{1}{N}\sum_m \alpha(r,m)[L_m - 1\!\!1(m=r)] \geq \frac{1}{N}\sum_m (r^q m + m^q r)[L_m - 1\!\!1(m=r)] \\ &\geq \frac{1}{N}\sum_m m r^q [L_m - 1\!\!1(m=r)] = \left(1 - \frac{r}{N}\right) r^q.\end{aligned}$$

From this and strong Markov property

$$1 \geq \mathbb{E}_N\left(K(S_{r+1}) - K(S_r)\right) = \mathbb{E}_N \int_{S_r}^{S_{r+1}} \mathcal{A}K(\hat{\mathbf{L}}(t))dt,$$

we deduce

$$\mathbb{E}_N(S_{r+1} - S_r) \leq \left(1 - \frac{r}{N}\right)^{-1} r^{-q}.$$

Summing this over r yields

$$\begin{aligned}\mathbb{E}_N(S_N - S_h) &\leq \sum_{r=h}^{N-1}\left(1 - \frac{r}{N}\right)^{-1} r^{-q} \leq \frac{N}{h^q}\sum_{r=h}^{N-1}(N-r)^{-1} \\ &\leq \frac{N}{h^q}\left[\log(N-h) + 1\right] \leq N h^{-q}(1 + \log N).\end{aligned}$$

From this and (4.6) we learn that if $\tilde{\tau}$ denotes the time of the complete gelation, then

$$\mathbb{E}_N\tilde{\tau} \le \frac{3}{2}c_1k^{1-q} + Nh^{-q}(1+\log N) \le \frac{3}{2}c_1k^{1-q} + c_2k^{(k-1)(q-1)q}N^{1-q}\log N, \tag{4.8}$$

because h is given by (4.7).

We now specify k. As in Theorem 1.6, we try $k = A\log N/\log\log N$. Since

$$k^{(k-1)(q-1)q}N^{1-q}\log N \le N^{(q-1)(Aq-1)}\log N,$$

whenever $A \le \log\log N$, we learn that if we choose $A < q^{-1}$, then the second term on the right-hand side of (4.8) dies out like a negative power of N and hence smaller than the first term for large N. This completes the proof of (1.14). □

5 Proof of Theorems 1.1 and 1.4

We first settle the issue of tightness.

Lemma 5.1 *Assume*

$$\alpha'(n) := \sup_m \frac{\alpha(m,n)}{m} < \infty, \tag{5.1}$$

for every $n \in \mathbb{N}$. Then the sequence $\{\mathcal{P}_N : N \in \mathbb{N}\}$ is tight.

Proof. Since $\mathbf{L}(t)$ is a Markov process, we know that for any bounded continuous function F, the processes

$$M(t) := F(\mathbf{L}(t)) - F(\mathbf{L}(0)) - \int_0^t \mathcal{A}F(\mathbf{L}(s))ds,$$
$$R(t) := M(t)^2 - \int_0^t \left(\mathcal{A}F^2 - 2F\mathcal{A}F\right)(\mathbf{L}(s))ds, \tag{5.2}$$

are martigales. Also, applying Doob's inequality to the martinagale $M(t)$ yields

$$\mathbb{E}_N \sup_{0\le t\le T} M(t)^2 \le 4\mathbb{E}_N M(T)^2 = 4\mathbb{E}_N \int_0^T \left(\mathcal{A}F^2 - 2F\mathcal{A}F\right)(\mathbf{L}(s))ds. \tag{5.3}$$

For the function F we choose

$$F(\mathbf{L}) = \frac{1}{N}\sum_n J_nL_n \tag{5.4}$$

where $(J_n : n \in \mathbb{N})$ is a sequence in $\mathbb{R}$, with $J_n = 0$ for $n > n_0$. We use (5.3) to assert

$$\begin{aligned}
\mathbb{E}_N \sup_{0\le t\le T} M(t)^2 &\le 4\int_0^T \frac{1}{2N^3}\sum_{m,n}\alpha(m,n)(L_m(s)L_n(s) - 1\!\!1(m=n)L_m(s))(J_{m+n}-J_m-J_n)^2 ds \\
&\le \int_0^T \frac{2}{N^3}\sum_{m,n}\alpha(m,n)L_m(s)L_n(s)(J_{m+n}-J_m-J_n)^2 ds \\
&\le \int_0^T \frac{c_0}{N^3}\sum_{m,n}\alpha(m,n)L_m(s)L_n(s)1\!\!1(m \text{ or } n\le n_0) ds \\
&\le 2\max_{n\le n_0}\alpha'(n)\int_0^T \frac{c_0}{N^3}\sum_{m,n} mnL_m(s)L_n(s)ds \le \frac{c_1T}{N},
\end{aligned}$$

for constants c_0 and c_1. As a result,

$$\mathbb{E}_N \sup_{0\le t\le T} M(t)^2 \le \frac{c_1T}{N}. \tag{5.5}$$

On the other hand,

$$\begin{aligned}
\mathcal{A}F(\mathbf{L}) &= \frac{1}{2N^2}\sum_{m,n}\alpha(m,n)(L_mL_n - 1\!\!1(m=n)L_n)(J_{m+n}-J_m-J_n) \\
&= \frac{1}{2N^2}\sum_{m,n}\alpha(m,n)L_mL_n(J_{m+n}-J_m-J_n) - \frac{1}{2N^2}\sum_{n\le n_0}\alpha(n,n)L_n(J_{2n}-2J_n).
\end{aligned}$$

As a consequence,

$$\mathcal{A}F(\mathbf{L}) = \frac{1}{2N^2}\sum_{m,n}\alpha(m,n)L_mL_n(J_{m+n}-J_m-J_n) + Error, \tag{5.6}$$

with $Error$ satisfying

$$|Error| \le \frac{c_2}{N}, \tag{5.7}$$

where c_2 is a constant that depends on J only. on the other hand, we may use (5.1) to show

$$\left|\frac{1}{N^2}\sum_{m,n}\alpha(m,n)L_mL_n(J_{m+n}-J_m-J_n)\right| \le c_3,$$

for a constant c_3 that depends on J only. From this, (5.7) and (5.6), we can readily deduce

$$\sup_{s\le t\le t+\delta}\left|\int_s^t \mathcal{A}F(\mathbf{L}(\theta))d\theta\right| \le c_4\delta,$$

for a constant c_4. This (5.2) and (5.5) imply

$$\mathbb{E}_N \sup_{s\le t\le t+\delta\le T} |F(\mathbf{L}(t)) - F(\mathbf{L}(s))| \le c_5(\delta + N^{-1}).$$

From this we can readily deduce the tightness of the sequence $\{\mathcal{P}_N\}$ by standard arguments. □

Proof of Theorem 1.4. We use the martigale $M(t)$ and the function F as in Lemma 2.1. From (5.2), (5.5), (5.6) and (5.7) we deduce

$$\lim_{N\to\infty} \int \Big| \sum_n J_n f_n(t) - \sum_n J_n f_n(0) - \frac{1}{2}\int_0^t \sum_{m,n} \alpha(m,n) f_m(s) f_n(s)(J_{m+n} - J_n - J_m)ds \Big| \mathcal{P}_N(d\mathbf{f}(\cdot)) = 0. \tag{5.8}$$

To finish the proof, we need to use (5.8) to deduce that if $\mathcal{P}$ is a limit point of $\{\mathcal{P}_N\}$, then

$$\int \Big| \sum_n J_n f_n(t) - \sum_n J_n f_n(0) - \int_0^t \big[\frac{1}{2}\sum_{m,n} \alpha(m,n) f_m(s) f_n(s)(J_{m+n} - J_n - J_m) - \sum_n \bar{\alpha}(n) J_n f_n(s) g_\infty(s)\big]ds \Big| \mathcal{P}(d\mathbf{f}(\cdot)) = 0, \tag{5.9}$$

where $g_\infty = 1 - \sum_n n f_n$. To achieve this, we need to study the continuity of the integrand with respect to the topology of the space $\mathcal{D}$. Define

$$G(\mathbf{f}) = \frac{1}{2}\sum_{m,n} \alpha(m,n) f_m f_n (J_{m+n} - J_n - J_m),$$
$$G^+(\mathbf{f}) = \frac{1}{2}\sum_{m,n} \alpha(m,n) f_m f_n J_{m+n}, \quad G^-(\mathbf{f}) = \sum_{m,n} \alpha(m,n) f_m f_n J_n,$$
$$\hat{G}(\mathbf{f}) = \sum_{m,n} (\alpha(m,n) - m\bar{\alpha}(n)) f_m f_n J_n. \tag{5.10}$$

Note that

$$G(\mathbf{f}) = G^+(\mathbf{f}) - G^-(\mathbf{f}).$$

Evidently $G^+ : E \to \mathbb{R}$ is a continuous function because it involves a finite sum. (Recall that $J_n = 0$ if $n > n_0$.) In fact the function $G : E \to \mathbb{R}$ is not continuous, because G^- involves an infinite sum. It turns out that G coincides with a continuous function G' in the support of of $\mathcal{P}_N$. The point is that in the support of $\mathcal{P}$, we may have $\sum_n n f_n(t) < 1$ for sufficiently large t when gelation occurs. This is not the case for functions in the support of

$\mathcal{P}_N$. That is $\sum_n nf_n(t) = 1$ for every t, with probability 1 with respect to $\mathcal{P}_N$. There is no contradiction with the fact that $\mathcal{P}$ is a limit point of $\mathcal{P}_N$ because the function $\mathbf{f} \to \sum_n nf_n$ is not a continuous function! Indeed

$$G^-(\mathbf{f}) = \hat{G}(\mathbf{f}) + \sum_n \bar{\alpha}(n) f_n J_n \sum_m m f_m = \hat{G}(\mathbf{f}) + \sum_n \bar{\alpha}(n) f_n J_n,$$

with the second equality valid only in the support of $\mathcal{P}_N$. Hence $G = G'$ in the support of $\mathcal{P}_N$, where

$$G'(\mathbf{f}) = G^+(\mathbf{f}) - \hat{G}(\mathbf{f}) - \sum_n \bar{\alpha}(n) f_n J_n.$$

We now claim that G' is a continuous function. For this it suffices to show that $\hat{G}$ is continuous because $G' + \hat{G}$ involves a finite sum of f_n's. To prove the continuity of $\hat{G}$, define

$$\hat{G}_\ell(\mathbf{f}) = \sum_{m,n} (\alpha(m,n) - m\bar{\alpha}(n)) f_m f_n J_n 1\!\!1(m \leq \ell).$$

Evidently G_ℓ is continuous. On the other hand

$$\begin{aligned} |\hat{G}_\ell(\mathbf{f}) - \hat{G}(\mathbf{f})| &\leq \sum_{n \leq n_0} f_n |J_n| \max_{m > \ell} \left| \frac{\alpha(m,n)}{m} - \bar{\alpha}(n) \right| \sum_m m f_m \\ &\leq \sum_{n \leq n_0} f_n |J_n| \max_{m > \ell} \left| \frac{\alpha(m,n)}{m} - \bar{\alpha}(n) \right|, \end{aligned}$$

and this goes to 0 as $\ell \to \infty$ by our assumption (1.10). From this we deduce that any limit point $\mathcal{P}$ is concentrated on functions $\mathbf{f}$ with

$$f_n(t) - f_n(0) = \int_0^t (Q_n^+(\mathbf{f}(s)) - \hat{Q}_n^-(\mathbf{f}(s)))ds,$$

where

$$\begin{aligned} \hat{Q}_n^-(\mathbf{f}) &= \sum_m (\alpha(m,n) - m\bar{\alpha}(n)) f_m f_n + \bar{\alpha}(n) f_n \\ &= \sum_m \alpha(m,n) f_m f_n + \bar{\alpha}(n) f_n (1 - \sum_m m f_m) \\ &= \sum_m \alpha(m,n) f_m f_n + \bar{\alpha}(n) f_n g_\infty. \end{aligned}$$

This completes the proof. □

It remains to establish Theorem 1.1.

Proof of Theorem 1.1. As in the proof of Theorem 1.4, we use the martigale $M(t)$ and the function F as in Lemma 5.1 to deduce (5.8). To finish the proof, we need to use (5.8) to deduce that if $\mathcal{P}$ is a limit point of $\{\mathcal{P}_N\}$, then

$$\int \Big| \sum_n J_n f_n(t) - \sum_n J_n f_n(0) - \frac{1}{2}\int_0^t \sum_{m,n} \alpha(m,n) f_m(s) f_n(s)(J_{m+n} - J_n - J_m) ds \Big| \, \mathcal{P}(d\mathbf{f}(\cdot)) = 0. \tag{5.11}$$

To ease the notation, we write $\{\mathcal{P}_N\}$ for the subsequence that converges to $\mathcal{P}$. We will achieve (5.11) by establishing some kind of approximate continuity of the integrand with respect to the topology of the space $\mathcal{D}$. Again, since J_n's are 0 for large n, all we need to do is replacing the infinite sum $\sum_n \alpha(m,n) f_n$ with a finite sum in (5.11) for a small error. For this, it suffices to show

$$\lim_{k\to\infty} \limsup_{N\to\infty} \int \left[\int_0^t \sum_{n\geq k} n f_n(s) \; ds \right] \; \mathcal{P}_N(d\mathbf{f}(\cdot)) = 0, \tag{5.12}$$

because by our assumption (1.5) we have that $\alpha(m,n) \leq c_0(m+n)$. After such a replacement, the integrand in (5.11) becomes continuous and we can pass to the limit by replacing $\mathcal{P}_N$ with the limit point $\mathcal{P}$. To replace back the finite sum with the infinite sum, we need to show

$$\lim_{k\to\infty} \int \left[\int_0^t \sum_{n\geq k} n f_n(s) \; ds \right] \; \mathcal{P}(d\mathbf{f}(\cdot)) = 0. \tag{5.13}$$

In summary, we need to establish (5.12) and (5.13) to complete the proof. In fact (5.13) is straightforward because for every $\mathbf{f} \in E$, $\lim_{k\to\infty} \sum_n \mathbb{1}(k \leq n) n f_n = 0$ and this implies (5.13) by Bounded Convergence Theorem.

It remains to establish (5.12). For this, we use Markov property to assert

$$\frac{\partial}{\partial t} \mathbb{E}_N F_k(\mathbf{L}(t)) = \mathbb{E}_N \mathcal{A} F_k(\mathbf{L}(t)), \tag{5.14}$$

with $F_k(\mathbf{L}) = \sum_n (n-k)^+ L_n/N$. We certainly have

$$\begin{aligned}
\mathcal{A}F_k(\mathbf{L}) &= \frac{1}{2N^2}\sum_{m,n}\alpha(m,n)L_m[L_n - 1\!\!1(m=n)]\Big[1\!\!1(m,n<k\le m+n)(m+n-k)\\
&\quad + 1\!\!1(n<k\le m)n + 1\!\!1(m<k\le n)m + 1\!\!1(k\le m,n)k\Big]\\
&\le \frac{c_0}{2N^2}\sum_{m,n}(m+n)L_mL_n\Big[1\!\!1(m,n<k\le m+n)(m+n-k)\\
&\quad + 1\!\!1(n<k\le m)n + 1\!\!1(m<k\le n)m + 1\!\!1(k\le m,n)k\Big]\\
&\le \frac{c_0}{2N^2}\sum_{m,n}(m+n)(m\wedge n)L_mL_n\\
&\quad \cdot\Big[1\!\!1(m,n<k\le m+n) + 1\!\!1(n<k\le m) + 1\!\!1(m<k\le n) + 1\!\!1(k\le m,n)\Big]\\
&\le \frac{c_0}{2N^2}\sum_{m,n}L_mL_n(m\wedge n)(m+n)1\!\!1(m \text{ or } n\ge k/2)\\
&\le \frac{2c_0}{N^2}\sum_{m,n}L_mL_nmn1\!\!1(m\ge k/2) = \frac{2c_0}{N}\sum_{m\ge k/2}mL_m\\
&= \frac{2c_0}{N}\sum_{m\ge k}mL_m + \frac{2c_0}{N}\sum_{k>m\ge k/2}mL_m \le \frac{2c_0}{N}\sum_{m\ge k}mL_m + \frac{2c_0k}{N}\sum_{m>k/2}L_m\\
&\le 2c_0F_k(\mathbf{L}) + \frac{4c_0k}{N}\sum_{m\ge k/2}L_m,
\end{aligned}$$

where we used the assumption (1.5) for the first inequality. From this, (5.14) and Grownwall's inequality we deduce,

$$\mathbb{E}_N F_k(\mathbf{L}(t)) \le e^{2c_0t}\mathbb{E}_N F_k(\mathbf{L}(0)) + 4c_0e^{2c_0t}\mathbb{E}_N\int_0^t \frac{k}{N}\sum_{m\ge k/2}L_m(s)ds. \tag{5.15}$$

Note that since $\sum_n nf_n \le 1$ in E, the function $\mathbf{f}\mapsto \sum_{n\ge k} f_n$ is continuous. As a consequence of this and (5.13),

$$\begin{aligned}
\lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{E}_N\int_0^t \frac{k}{N}\sum_{m\ge k/2}L_m(s)ds &= \lim_{k\to\infty}\limsup_{N\to\infty}\int\left[\int_0^t k\sum_{n\ge k/2}f_n(s)ds\right]\mathcal{P}_N(d\mathbf{f}(\cdot))\\
&= \lim_{k\to\infty}\int\left[\int_0^t\sum_{n\ge k/2}kf_n(s)ds\right]\mathcal{P}(d\mathbf{f}(\cdot)) = 0.
\end{aligned}$$

From this, our assumption (1.6), and (5.15) we deduce

$$\lim_{k\to\infty}\limsup_{N\to\infty}\int\left[\int_0^t\sum_{n\geq k}(n-k)f_n(s)\ ds\right]\ \mathcal{P}_N(d\mathbf{f}(\cdot))=0,$$

This implies (5.12) because

$$\lim_{k\to\infty}\int\left[\int_0^t\sum_{n\geq k}kf_n(s)ds\right]\ \mathcal{P}(d\mathbf{f})=0,$$

by (5.13). We are done. □

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