Equilibrium Fluctuations for a Model of Coagulating-Fragmenting planar Brownian Particles

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1 Introduction

One of the main purposes of statistical mechanics is to explain the macroscopic behavior of various phenomena in terms of the statistics of their microscopic structures. Macroscopically we often have a PDE involving a small number of parameters. The microscopic description however involves a large number of components that are evolving by either deterministic or stochastic rules. Let us name three reasons to justify our interest in understanding the connection between the microscopic and macroscopic descriptions.

As our first reason, we remark that historically the macroscopic equation is formally derived from the microscopic description of the phenomenon under study. It is an important task of statistical mechanics to justify such a derivation rigorously and verify the validity of the macroscopic PDE.

For our second reason, we note that we often have simple dynamical rules for the microscopic model and the main challenging feature of the model has to do with its large size. On the other hand, the macroscopic evolution involve only a few variables but dictated by a rather sophisticated nonlinear rules. It is the case for many examples that the macroscopic equation is not fully understood. Hopefully by exploring its relation with its microscopic counterpart we may discover new tools and techniques for the macroscopic equation.

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As our third reason, we should mention that even though the macroscopic equation is preferred because of its dependence on a small number of variables, it is only a reduced description of the microscopic phenomenon at hand and we would like to find practical ways of recovering some of the lost information as we switch to the macroscopic world. Since the passage from the microscopic details to macroscopic parameters can be recast as a law of large numbers for some conserved quantities in many scenarios, probability theory suggests some standard routes for going beyond law of large numbers and gain new information. The celebrated central limit theorem and large deviations for classical examples are guidelines for producing some vital information for the microscopic model under study.

It is the latter reason which is the chief motivation for the present article. Our microscopic model is a system of coagulating-fragmenting Brownain particles which is macroscopically described by an inhomogeneous Smoluchowski's equation. This equation is derived as a law of large numbers. The main contribution of this article is a central limit theorem for the aforementioned law of large numbers when the system is in equilibrium.

In our model , we start with N particles with each particle traveling in $\mathbb{R}^d$ as a Brownain motion. Each particle has a size $m \in \mathbb{Z}$ and a radius $r \in (0, \infty)$. In fact our interpretation of the location x of a particle is that x is the center of a ball of radius r and in some sense only a small fraction of the ball is occupied by the true particle. It turns out that in reality each particle is a cluster of smaller objects and the cluster is a complex fractal like entity that is too complicated to be treated with the existing techniques. That is why we simplify the model by replacing the cluster with a ball of radius $r(m) = m^\chi$ so that when $\chi > \frac{1}{d}$, we are taking into account the fact that the mass of the particle comes from a small portion of the ball which is occupied by the cluster. This may appear somewhat native and not too realistic from physical point of view. Nevertheless, as was explained in [HR1–3] and [R2], the model does exhibit some expected features of the underlying physics. For example, the condition $\chi < \frac{1}{d-2}$ guarantees that no ***gelation*** occurs in finite time. That is, no particle of size infinity is formed in finite time at macroscopic level. We also conjecture that an instantaneous gelation would occur if $\chi > \frac{1}{d-2}$.

In fact the true radius of a particle is εr with ε very small. A calculation involving Wiener sausages reveals that if $N = \varepsilon^{2-d}$ when $d \geq 3$ and $N =\mid \log \varepsilon \mid$ when $d = 2$, then the expected value of the number of times a particle coagulates with other particles in one unit of time stays positive and finite as $\epsilon \to 0$. This property allows us to obtain the Smoluchowski's equation for the evolution of cluster densities in low ε limit. To further simplify the involving mathematical technicalities, we forget about balls presenting each particle and regard each particle as a point. Now the coagulation occurs stochastically only when particles of positions x and y and masses m and n, satisfy

$$\mid x - y \mid \leq c_0 \varepsilon (m^\chi + n^\chi),$$

for a constant c_0. In the preceding works [HR1–2], [R2] and [HRY], we were able to derive the macroscopic equation as a law of large numbers; if we label the locations and masses of

particles as $(x_i, m_i), i \in I$, then our law of large numbers asserts

$$\frac{1}{K_\varepsilon} \sum_{i \in I} \delta_{x_i(t)}(dx) \mathbb{1}(m_i(t) = n) \to f_n(x,t)dx$$

with f_n solving the Smoluchowski's equation (2.6) of Section 2. Here

$$K_\varepsilon = \begin{cases} \varepsilon^{2-d} & \text{if } d \geq 3, \\ |\log \varepsilon| & \text{if } d = 2. \end{cases} \tag{1.1}$$

As a central limit theorem, we are interested in the limit of the ***fluctuation fields***

$$\xi_n^\varepsilon(dx,t) = \sqrt{K_\varepsilon} \left({K_\varepsilon}^{-1} \sum_i \delta_{x_i(t)}(dx)\ \mathbb{1}(m_i(t) = n) - f_n(x,t)dx \right) \tag{1.2}$$

as $\varepsilon \to 0$. In Section 2, we state a conjecture regarding the evolution of ξ_n^ε in low ε limit. According to this conjecture, the limit ξ solves an Ornstein-Uhlenbeck stochastic differential equation in an infinite dimensional setting with ξ living in a negative Sobolev space. The conjecture is formulated using the so-called ***fluctuation–dissipation principle*** of non-equilibrium statistical mechanics. In this article, we establish the conjecture only when the dimension is 2 and the model satisfies a detailed-balance condition. Some steps of our proof does not apply to higher dimensions. The case $d \geq 3$ is more challenging and is left for a future investigation.

We continue this introduction by mentioning some previous work related to our model. Smoluchowski's equation was introduced by Smoluchowski in the seminal work [Sm]. The first mathematically rigorous derivation of Smoluchowski's equation from a model of coagulating Brownian particles was carried out by Lang–Nyugen [LN] when $d = 3$ and all particles have the same diffusion coefficient. A related problem has been studied by Sznitman [Sz] when $d = 2$. A completely different approach has been employed in [HR1-2] and [YRH] to treat the model in general. A thorough survey on related models and their applications can be found in Aldous [A]. In fact Open Problem 16 in [A] is exactly our central limit theorem when there is no spacial dependence. We refer to the monograph [Sp] for an introduction to related questions in statistical mechanics and a discussion of the fluctuation–dissipation principle. An equilibrium fluctuation result has been studied in [R1] for a model of colliding particle associated with discrete Boltzmann equation.

We end this section with an outline of the paper. In Section 2, we state a conjecture for the macroscopic evolution of the fluctuation fields. In Section 3, a family of reversible invariant measures for the microscopic model is constructed. In this section, the conjecture is restated as the main result of this paper under the assumption that the model starts from one of the reversible measures and that the dimension is 2. In Section 4, the strategy of the proof is described. The first step of the proof is a regularity of the coagulation term and is carried out in Sections 5–7. The proof of the main result is completed in Sections 8 and 9.

2 A Conjecture

We start with the description of our model. The configuration space Ω consists of pairs $\omega = (\mathbf{x}, \mathbf{m})$ with $\mathbf{x}$ a subset of $\mathbb{R}^d$ with no accumulation point and $\mathbf{m} : \mathbf{x} \to \mathbb{N} = \{1, 2, 3, \dots\}$ is a map that assigns a positive integer to each element of $\mathbf{x}$. Throughout this section we assume that $d \geq 2$. It is often convenient to write $\omega = (x_i, m_i)_{i \in I}$ with $x_i \in \mathbb{R}^d$ and $m_i \in \mathbb{N}$, where $I = I(\omega)$ is a countable index set. We regard $\mathbf{x}$ as a collection of cluster positions in $\mathbb{R}^d$ with no accumulation point and $\mathbf{m}$ assigns a size to each such a position. We may also identify $\omega = (\mathbf{x}, \mathbf{m})$ as a discrete measure $\hat{\omega} = \sum_{i \in I} \delta_{(x_i, m_i)}$ on $\mathbb{R}^d \times \mathbb{N}$. Using this identification we equip the space Ω with the topology of vague convergence.

We now describe the evolution of coagulating and fragmenting Brownian clusters as a Markov process on the configuration space Ω. For this, functions $d : \mathbb{N} \to (0, \infty)$, $\alpha : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ and $\beta : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ are given which represent the diffusion coefficient, the coagulation rate and the fragmentation rate respectively. We assume that both α and β are symmetric. Also a parameter $\chi \in [0, \infty)$ and a continuously differentiable function $V : \mathbb{R}^d \to [0, \infty)$ are given for our model. We then define a Markov process $\omega(t)$ with infinitesimal generator $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_c + \mathcal{A}_f$ where $\mathcal{A}_0$ represents the "Brownian motion" part of the dynamics, and $\mathcal{A}_c$ and $\mathcal{A}_f$ represent the coagulation and fragmentation parts of the evolution. For the "Brownian motion" part, we use the representation $\omega = (x_i, m_i)_{i \in I}$ to write

$$\mathcal{A}_0 F(\omega) = \sum_{i \in I} d(m_i) \Delta_{x_i} F(\omega), \tag{2.1}$$

for any C^2 function F. Here Δ_{x_i} represents the Lapalce operator which acts on the x_i variable.

As for the coagulation part, we write $\mathcal{A}_c F(\omega) = \mathcal{A}_c^+ F(\omega) - \mathcal{A}_c^- F(\omega)$ with $\mathcal{A}_c^+$ and $\mathcal{A}_c^-$ are given by

$$\mathcal{A}_c^+ F(\omega) = \frac{1}{2} \sum_{\substack{i,j \in I \\ i \neq j}} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \left[\frac{m_i}{m_i + m_j} F(S_{ij}^1 \omega) + \frac{m_j}{m_i + m_j} F(S_{ij}^2 \omega) \right],$$

$$\mathcal{A}_c^- F(\omega) = \frac{1}{2} \sum_{\substack{i,j \in I \\ i \neq j}} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j) F(\omega).$$

Here,

(i) $\varepsilon > 0$ is a small parameter that represents the range of interaction.

(ii) the function $V_\varepsilon(x; m, n) = \lambda(\varepsilon) V\left(\frac{x}{\varepsilon}; m, n\right)$ where

$$\lambda(\varepsilon) = \begin{cases} |\log \varepsilon|^{-1} \varepsilon^{-2} & \text{if } d = 2, \\ \varepsilon^{-2} & \text{if } d \geq 3, \end{cases} \tag{2.2}$$

and

$$V(x;m,n) = r(m,n)^{-2} V\left(\frac{x}{r(m,n)}\right), \tag{2.3}$$

with $r(m,n) = r(m) + r(n)$, for $r(n) = n^\chi$, and V is a symmetric Hölder continuous function of compact support and total integral 1.

(iii) we denote by $S^1_{ij}\omega$ the configuration formed from ω by removing x_j from $\mathbf{x}$ and assigning the size $m_i + m_j$ to the surviving cluster at x_i. The configuration $S^2_{ij}\omega$ is defined in the same way, except that we remove x_i from $\mathbf{x}$ and assign the size $m_i + m_j$ to the cluster at x_j. We note that the cluster at x_i survives the coagulation with probability $\frac{m_i}{m_i+m_j}$.

Before describing the fragmentation part of the dynamics, let us explain the form of the function V_ε. Note that $V_\varepsilon(x_i - x_j; m_i, m_j) \neq 0$ only if $x_i - x_j$ is of order $\varepsilon(r(m_i) + r(m_j))$ with $r(m) = m^\chi$. This means that we regard a particle of size m to be roughly a ball of radius $\varepsilon r(m)$ so that a pair of clusters of centers x_i and x_j coagulate when their corresponding balls overlap. If we assume that the mass of the ith cluster is distributed evenly in the ball $B_{r(m_i)}(x_i)$, then we expect to have $\chi = \frac{1}{d}$. However, in reality a cluster is far from being a round ball and expected to be a fractal like object. By allowing $\chi \in (0, \infty)$ we are hoping to have a more physically relevant model. In particular, the case $\chi < \frac{1}{d}$ represents a scenario in which the ball $B_{r(m_i)}(x_i)$ contains the true cluster and only a fraction of the ball is occupied with the cluster. We believe that the case $\chi > \frac{1}{d-2}$ corresponds to the occurrence of "gelation". We refer to [HR1], [HR3] and [R2] for more discussions. (Note that no finite χ can cause gelation when $d = 2$; we guess that the radius must grow exponentially with the mass in order to have a gel in this case.)

The occurrence of the factor $\lambda(\varepsilon)$ in the definition of V_ε is to guarantee that when two clusters collide, then they coagulate with a probability that stays away from 0 as $\varepsilon \to 0$. Indeed $B = x_i - x_j$ is a Brownian motion that spends a time of order $\varepsilon^2 r(m_i, m_j)^2$ (respectively $\varepsilon^2 r(m_i, m_j)^2 |\log \varepsilon|$) in the support of V_ε when $d \geq 3$ (respectively $d = 2$). We also multiply the sum in the definition $\mathcal{A}_c$ by $1/2$ to ensure that the summation is over unordered pairs $\{i, j\}$.

As for the fragmentation part, $\mathcal{A}_f F(\omega) = \mathcal{A}_f^+ F(\omega) - \mathcal{A}_f^- F(\omega)$, is given by

$$\frac{1}{2} \sum_i \sum_{m=1}^{m_i - 1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m)(F(S_i^{y,m}\omega) - F(\omega))dy,$$

with

$$\mathcal{A}_f^+ F(\omega) = \frac{1}{2} \sum_i \sum_{m=1}^{m_i - 1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m) F(S_i^{y,m}\omega)dy.$$

Here,

$$V^\varepsilon(a;m,n) = \varepsilon^{-d} V\left(\frac{a}{\varepsilon};m,n\right), \tag{2.4}$$

and $S_i^{y,m}$ is that configuration formed from ω by replacing (x_i, m_i) with a pair of clusters of positions x_i and y and sizes m and $m_i - m$.

The central object to study is the cluster density of a given size. Microscopically we are interested in the empirical measures

$$g_n^\varepsilon(dx,t) = K_\varepsilon^{-1} \sum_i \delta_{x_i(t)}(dx) \mathbb{1}(m_i(t) = n),$$

where K_ε was defined by (1.1). If for example we select $(x_1(0), m_1(0)), \ldots, (x_N(0), m_N(0))$ randomly and independently with the law

$$\mathbb{P}(x_i(0) \in A, m_i(0) = n) = \frac{1}{Z}\int_A f_n^0(x)dx, \tag{2.5}$$

with $Z = \sum_n \int f_n^0 dx$, then by the law of large numbers, $g_n^\varepsilon(dx,0)$ converges weakly to $f_n^0(x)dx$ provided that $N = K_\varepsilon Z$ and $\varepsilon \to 0$. Note that such a choice of initial condition implies that on average there are $K_\varepsilon \int f_n^0(x)dx$ many particles of size n. A Wiener Sausage calculation reveals that in average, each particle in our model experiences finitely many coagulations per unit time. This explains our reason for choosing K_ε as above.

The main result of [HR1]–[HR2], and [R1] states that if $\chi < \frac{1}{d-2}$, there is no fragmentation, and α satisfies some technical conditions, then the empirical density $g_n^\varepsilon(dx,t)$ converges to $f_n(x,t)dx$ where f_n is a solution to the Smoluchowski's equation, subject to the initial condition $f_n(x,0) = f_n^0(x)$. It is shown in [HR3] that this solution is unique.

Smoluchowski's equation has the form

$$\frac{\partial f_n}{\partial t}(x,t) = d(n)\Delta_x f_n(x,t) + Q_n^{+,c}(\mathbf{f}) - Q_n^{-,c}(\mathbf{f}) + Q_n^{+,f}(\mathbf{f}) - Q_n^{-,f}(\mathbf{f}), \tag{2.6}$$

where $\mathbf{f} = (f_n : n \in \mathbb{N})$, and

$$Q_n^{+,c}(\mathbf{f}) = \frac{1}{2}\sum_{m=1}^{n} \hat{\alpha}(m,n-m) f_m f_{n-m}, \quad Q_n^{-,c}(\mathbf{f}) = \sum_{m=1}^{\infty} \hat{\alpha}(m,n) f_m f_n,$$

$$Q_n^{+,f}(\mathbf{f}) = \sum_{m=1}^{\infty} \hat{\beta}(m,n) f_{n+m}, \qquad Q_n^{-,f}(\mathbf{f}) = \frac{1}{2}\sum_{m=1}^{n-1} \hat{\beta}(m,n-m) f_n,$$

with

$$\hat{\alpha}(m,n) = \eta(m,n)\alpha(m,n), \quad \hat{\beta}(m,n) = \eta(m,n)\beta(m,n). \tag{2.7}$$

The function $\eta(m,n)$ is calculated in terms of the microscopic details of the model. We start with the case $d=2$. In this case η is independent of the function V and the parameter χ, and is simply given by

$$\eta(m,n) = \frac{2\pi(d(m)+d(n))}{2\pi(d(m)+d(n)) + \alpha(m,n)}. \tag{2.8}$$

The formula for $\eta(m,n)$ is more complicated when $d \geq 3$ and does depend on both V and χ. Here is the recipe for η: First we find the unique solution to the equation

$$(d(m)+d(n))\Delta u_{m,n}(x) = \alpha(m,n)V(x;m,n)(1+u_{m,n}(x)) \tag{2.9}$$

with $u(x;m,n) = u_{m,n}(x)$ satisfying $u_{m,n}(x) \to 0$, as $|x| \to \infty$. Then we set

$$\eta(m,n) = \int V(x;m,n)(1+u_{m,n}(x))dx. \tag{2.10}$$

Remark 2.1.

- For the purposes of this section, we have assumed that $\sum_n \int f_n^0 dx < \infty$, which implies that there are finitely many particles almost surely. However in Section 3 when the main result of this article is discussed, the density f_n^0 is constant and the system involves infinitely many particles. The existence of such a particle system is no longer obvious, and in Remark 3.5 we will explain how such a particle system is constructed.

- Note that we deliberately choose a mechanism for the fragmentation that is, in some sense, dual to the coagulation mechanism. This allows us to easily construct reversible invariant measures for the process $\omega(t)$. In other words the fragmentation is defined in such a way that if we reverse time after a coagulation, we obtain a fragmentation. For the kinetic limit however, we can use a kernel W for the fragmentation that is different from V, or even choose two new locations y_1 and y_2 near x_i for the locations of new clusters of a fragmented cluster. However, for this fragmentation mechanism, the macroscopic coagulation and fragmentation rates read $\hat{\alpha} = \alpha\eta$, $\hat{\beta} = \beta\eta'$ with possibly $\eta \neq \eta'$.

- Let us write $Q_n = Q_n^{+,c} - Q_n^{-,c} + Q_n^{+,f} - Q_n^{-,f}$. We then have the following useful formula: For any sequence $(J_n : n \in \mathbb{N})$,

$$\sum_n J_n Q_n = \frac{1}{2}\sum_{n,m}(\hat{\alpha}(m,n)f_m f_n - \hat{\beta}(m,n)f_{m+n})(J_{m+n} - J_m - J_n).$$

□

The main goal of this article is to derive an equation for the evolution of the density fluctuations about the solution to the Smoluchowski's equation.

To this end, recall the fluctuation fields $\xi_n^\varepsilon(dx,t)$ that was defined by (1.2). Let us assume that $\chi < (d-2)^{-1}$ and that the total mass $\int \sum_n n f_n^0 \, dx$ is finite.

Conjecture 2.1 *As $\varepsilon \to 0$, the process ξ_n^ε converges to ξ_n where ξ_n is the unique solution to the Uhlenbeck–Ornstein equation*

$$\begin{aligned} \frac{\partial \xi_n}{\partial t} &= d(n)\Delta_x \xi_n + \mathcal{L}_n^c \boldsymbol{\xi} + \mathcal{L}_n^f \boldsymbol{\xi} + \gamma_n, \\ \xi_n(x,0) &= \bar{\xi}_n(x), \end{aligned} \tag{2.11}$$

where $\boldsymbol{\xi} = (\xi_n : n \in \mathbb{N})$, and

$$\mathcal{L}_n^c = \mathcal{L}_n^{+,c} - \mathcal{L}_n^{-,c}, \quad \mathcal{L}_n^f = \mathcal{L}_n^{+,f} - \mathcal{L}_n^{-,f}, \tag{2.12}$$

with

$$\mathcal{L}_n^{+,c}\boldsymbol{\xi} = \sum_{m=1}^{n-1} \hat{\alpha}(m, n-m) f_m \xi_{n-m}, \quad \mathcal{L}_n^{-,c}\boldsymbol{\xi} = 2\sum_{m=1}^{\infty} \hat{\alpha}(m,n)(f_m \xi_n + f_n \xi_m), \tag{2.13}$$

$$\mathcal{L}_n^{+,f}\boldsymbol{\xi} = \sum_{m=1}^{\infty} \hat{\beta}(m,n)\xi_{n+m}, \qquad \mathcal{L}_n^{-,f}\boldsymbol{\xi} = \frac{1}{2}\sum_{m=1}^{n-1} \hat{\beta}(m, n-m)\xi_n, \tag{2.14}$$

and γ_n is a space-time white noise with variance given by

$$\begin{aligned} \left\langle \left(\sum_n \iint J_n \gamma_n dx dt \right)^2 \right\rangle &= 2 \iint \sum_n d(n) f_n |\nabla J_n|^2 dx dt \\ &+ \frac{1}{2} \iint \sum_{m,n} \hat{\alpha}(m,n) f_n f_m (J_{n+m} - J_n - J_m)^2 dx dt \qquad (2.15) \\ &+ \frac{1}{2} \iint \sum_{m,n} \hat{\beta}(m,n) f_{n+m} (J_{n+m} - J_n - J_m)^2 dx dt \qquad (2.16) \end{aligned}$$

for any smooth test function $J = (J_n : n \in \mathbb{N})$ of compact support in $\mathbb{R}^d \times (0,\infty)$.

In fact γ belongs to a suitable negative Sobolev space and the integral of $J_n\gamma_n$ must be understood as the value of the distribution γ_n at the smooth test function J_n. See the next section or the beginning of Section 8 for the precise definition of ξ and γ and the meaning of the equation (2.11).

The main result of this paper asserts that Conjecture 2.1 is valid if the initial distribution of the cluster is chosen according to a reversible equilibrium state and $d = 2$.

3 Equilibrium Fluctuations

We start with constructing reversible invariant measures for the process $\omega(t)$. For this we take a collection of positive numbers $\boldsymbol{\lambda} = (\lambda_n : n \in \mathbb{N})$ such that $\sum_n \lambda_n < \infty$, and

$$\alpha(m,n)\lambda_n\lambda_m = \beta(m,n)\lambda_{n+m} \tag{3.1}$$

for every $m, n \in \mathbb{N}$. Note that for such a collection, the functions $f_n(x,t) \equiv \lambda_n$ do solve the Smoluchowski's equation because by (2.7) and (3.1),

$$\hat{\alpha}(m,n)\lambda_n\lambda_m = \hat{\beta}(m,n)\lambda_{n+m} \tag{3.2}$$

and this in turn implies

$$Q_n^{+,c}(\boldsymbol{\lambda}) = Q_n^{-,f}(\boldsymbol{\lambda}), \quad Q_n^{-,c}(\boldsymbol{\lambda}) = Q_n^{+,f}(\boldsymbol{\lambda}). \tag{3.3}$$

Given such $\boldsymbol{\lambda}$, we construct a reversible invariant measure $\mu_{\boldsymbol{\lambda}}$ for our process $\omega(t)$: Let $\mathbf{x}^n$ to be a Poisson point process with intensity $K_\varepsilon\lambda_n$. Assume that $(\mathbf{x}^n, n \in \mathbb{N})$ are independent and define $\omega = (\mathbf{x}, \mathbf{m})$ by $\mathbf{x} = \bigcup_{n=1}^\infty \mathbf{x}^n$ and $\mathbf{m}(a) = n$ for $a \in \mathbf{x}^n$. In words, particles of size n form a Poisson point process of intensity of $K_\varepsilon\lambda_n$ and these processes are independent for different choices of n. We note that if Λ is a bounded subset of $\mathbb{R}^d$, then

$$\int M_\Lambda d\mu_{\boldsymbol{\lambda}} = |\Lambda| K_\varepsilon \sum_n \lambda_n$$

where

$$M_\Lambda^n(\omega) = M_\Lambda^n(\mathbf{x}, \mathbf{m}) = \#\{a \in \mathbf{x} : a \in \Lambda, \mathbf{m}(a) = n\}, \quad M_\Lambda = \sum_{n=1}^\infty M_\Lambda^n.$$

Hence, if we assume that $\sum_n \lambda_n < \infty$, then there are finitely many clusters in a bounded domain almost surely with respect to $\mu_{\boldsymbol{\lambda}}$.

We now assert that $\mu_{\boldsymbol{\lambda}}$ is indeed reversible. To explain this, let us take two bounded local C^2 functions $F, G : \Omega \to \mathbb{R}$. By a local function F we mean that there exists a positive constant c_0 such that F depends only on particles (x_i, m_i) such that $|x_i|, m_i \le c_0$. We then have

$$\int G\ \mathcal{A}F\ d\mu_{\boldsymbol{\lambda}} = \int F\ \mathcal{A}G\ d\mu_{\boldsymbol{\lambda}}. \tag{3.4}$$

Indeed,

$$\int G\ \mathcal{A}_0 F\ d\mu_{\boldsymbol{\lambda}} = -\int \sum_i d(m_i) \nabla_{x_i} F \cdot \nabla_{x_i} G\ d\mu_{\boldsymbol{\lambda}}, \tag{3.5}$$

$$\int G\ \mathcal{A}_c^+ F\ d\mu_{\boldsymbol{\lambda}} = \int F\ \mathcal{A}_f^- G\ d\mu_{\boldsymbol{\lambda}}, \quad \int G\ \mathcal{A}_c^- F\ d\mu_{\boldsymbol{\lambda}} = \int F\ \mathcal{A}_f^+ G\ d\mu_{\boldsymbol{\lambda}}. \tag{3.6}$$

Note that (3.6) is the microscopic analog of (3.3), and together with (3.5) imply (3.4). The proof of (3.5) follows from an integration by parts. As for (3.6), observe that for any bounded set Λ,

$$\mu^{\Lambda}_{\boldsymbol{\lambda}}(d\omega_\Lambda) = \sum_{L_1,L_2,\ldots} \prod_{n=1}^{\infty} \left[\prod_{i=1}^{L_n} \mathbb{1}(m_{ni} = n,\ x_{ni} \in \Lambda) dx_{ni} \right] \frac{(\lambda_n K_\varepsilon)^{L_n}}{L_n!} e^{-\lambda_n K_\varepsilon |\Lambda|} \tag{3.7}$$

where ω_Λ is the configuration in the set Λ and $\mu^{\Lambda}_{\boldsymbol{\lambda}}$ is the law of ω_Λ under $\mu_{\boldsymbol{\lambda}}$. Here we have labeled particles of size n by $n1, n2, \ldots, nL_n$ and $L_n = M^n_\Lambda$ is the number of such particles. Using the representation (3.7), one can readily verify (3.6).

Let us write $\mathbb{P}^\omega_\varepsilon$ and $\mathbb{E}^\omega_\varepsilon$ for the probability and the expectation with respect to the process $\omega(\cdot)$ subject to the initial condition $\omega(0) = \omega$. When $\omega(0)$ is distributed according to an invariant measure $\mu_{\boldsymbol{\lambda}}$, we write $\mathbb{P}^{eq}_\varepsilon$ and $\mathbb{E}^{eq}_\varepsilon$ instead. Given $\omega(\cdot)$, we define

$$\xi^\varepsilon_n(t, J) = \sqrt{K_\varepsilon} \left(\frac{1}{K_\varepsilon} \sum_i J(x_i(t)) \mathbb{1}(m_i(t) = n) - \lambda_n \int J(x) dx \right) \tag{3.8}$$

for every smooth $J : \mathbb{R}^d \to \mathbb{R}$ of compact support. Let $\mathcal{D}$ denote the space of smooth functions of compact support and let $\mathcal{D}'$ denote the space of distributions (the dual of $\mathcal{D}$). We regard ξ^ε_n as an element of the Skorohod space $\mathbb{D} = D([0, T], (\mathcal{D}')^{\mathbb{N}})$. The transformation $\omega(\cdot) \mapsto \xi^\varepsilon$ induces a probability measure $\mathcal{P}^\varepsilon$ on $\mathbb{D}$. We regard $\xi^\varepsilon_n(t, J)$ as the value of the distribution $\xi^\varepsilon_n(t)$ at J

To state our assumptions, take a nondecreasing function $a \geq 1$, such that $\alpha'(m, n) = \alpha(m, n)/(d(m) + d(n)) \leq a(n) + a(m)$, and set $\beta'(n) = \sum_{m=1}^{n-1} \beta(n - m, m)$.

Hypothesis 3.1. The function $d(\cdot)$ is bounded. Moreover for some $\theta > 1/2$,

$$\lim_{\varepsilon \to 0} \tau(\varepsilon) := \lim_{\varepsilon \to 0} K_\varepsilon^{1/2} \sum_{2\varepsilon r(n) > \delta(\varepsilon)} a(n) \lambda_n = 0, \tag{3.9}$$

where $\delta(\varepsilon) = |\log \varepsilon|^{-\theta}$, and

$$\sum_n [a(n)(r(n) + \beta'(n) \log n) + a(n)^2 (a(n) + \log n)] \lambda_n < \infty. \tag{3.10}$$

Remark 3.1. Note that by detailed balance, we have that $\beta(n, m) = \alpha(m, n)\lambda_n \lambda_m / \lambda_{m+n}$. Hence, if α and $\boldsymbol{\lambda}$ are known, then β is determined. As an example, consider the case with λ_n decaying like e^{-cn}, as $n \to \infty$. In this case, we can readily see that if $a(n)$ is growing at most like a polynomial as n gets large, then both (3.9) and (3.10) are satisfied.

Theorem 3.1 *Assume Hypothesis 3.1 and that the dimension $d = 2$. Then the finite dimensional marginals of the sequence $\mathcal{P}^\varepsilon$ converges to the finite dimensional marginals of $\mathcal{P}$, where $\mathcal{P}$ is the distribution of a stationary Ornstein–Uhlenbeck Gaussian process with covariance*

$$\iint \sum_{n=1}^{\infty} \xi_n(t, J_n)\xi_n(0, H_n)\mathcal{P}(d\xi) = \int \sum_{n=1}^{\infty} (T_t J_n)(x) H_n(x)\lambda_n dx. \tag{3.11}$$

Here $J_n, H_n \in \mathcal{D}$ for $n \in \mathbb{N}$ and T_t is the semigroup generated by the linear Smoluchowski's operator

$$\begin{aligned}(\Gamma \mathbf{J})_n = {} & d(n)\Delta_x J_n + \sum_{m=1}^{\infty} \hat{\beta}(m, n) J_{n+m} - \frac{1}{2}\sum_{m=1}^{n-1} \hat{\beta}(m, n-m) J_n \\ & + \sum_{m=1}^{n-1} \hat{\alpha}(m, n-m) J_{n-m}\lambda_m - \sum_{m=1}^{\infty} \hat{\alpha}(m, n)(J_n\lambda_m + J_m\lambda_n).\end{aligned} \tag{3.12}$$

Remark 3.2. Note that the macroscopic coagulation and fragmentation rates $\hat{\alpha}$ and $\hat{\beta}$ are strictly smaller than their microscopic counterparts α and β. We refer the reader to Section 4 for a heuristic explanation and how a fundamental auxiliary function u^ε would allow us to switch from the microscopic rates α and β to macroscopic rates $\hat{\alpha}$ and $\hat{\beta}$. Note also that even though the "strengths" of the noises associated with the coagulation and fragmentation are given by α and β, the corresponding macroscopic "strengths" are given by $\hat{\alpha}$ and $\hat{\beta}$ as the expressions (2.15) and (2.16) indicate. In fact this reduction in the strength happens in a very curious way:

- The auxiliary function u^ε corrects the original noises coming from the coagulation and fragmention by reducing their strengths to $\tilde{\alpha} = \alpha\eta^2$ and $\tilde{\beta} = \beta\eta^2$. (See formulas (8.33) and (8.37) and the definitions of A_{c0} and A_{f0} which are given right after (8.26) and (8.34).)
- The Brownian part of the dynamics uses the corrector u^ε and produces some noise which enhances the reduced strengths $\tilde{\alpha}$ and $\tilde{\beta}$ to their final values $\hat{\alpha}$ and $\hat{\beta}$. (See formula (8.24), expression $A_{02121111}$ which is defined right before (8.23), and the final step of the proof of (8.4).)

Remark 3.3. In fact what we can prove is somewhat stronger than what has appeared in the statement of Theorem 3.1. We will show that the process $\xi^\varepsilon = \xi' - \xi''$ with both ξ' and ξ'' stationary processes in time, where the law of ξ' under $\mathbb{P}_\varepsilon^{eq}$ converges to $\mathcal{P}$, and

$$\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon^{eq} |\xi''(t, J)| = 0$$

for every t, $n \in \mathbb{N}$, and test function J. We refer the reader to Section 9 for the details.

An alternative description of the law $\mathcal{P}$ is the martingale formulation of Holley and Stroock [HS] that will be defined in Section 8. It is this formulation which we use for the proof of Theorem 3.1.

Remark 3.4. The intuition behind (3.11) is the standard *dissipation-fluctuation principle*. This principle is used to predict the form of the diffusion coefficient once the drift and the invariant measure for the fluctuation fields are known. In fact (3.11) is equivalent to saying that the process ξ is a solution to the stochastic differential equation

$$d\xi = \Gamma\xi dt + \mathcal{B}d\mathbf{W}_t \tag{3.13}$$

where $d\mathbf{W}_t = (dW_{1,t}, \dots, dW_{n,t}, \dots)$ with $(dW_n : n \in \mathbb{N})$ independent space-time white noises and the operator $\mathcal{B}$ is determined by

$$\begin{aligned}\int \sum_{n=1}^{\infty} (\mathcal{B}\boldsymbol{\zeta})_n (\mathcal{B}\mathbf{H})_n dx = 2 \int \sum_{n=1}^{\infty} \lambda_n \nabla_x J_n \cdot \nabla_x H_n dx \\ + \frac{1}{2} \int \sum_{m,n=1}^{\infty} \hat{\alpha}(m,n)\lambda_n\lambda_m (J_{n+m} - J_n - J_m)(H_{n+m} - H_n - H_m) dx \\ + \frac{1}{2} \int \sum_{m,n=1}^{\infty} \hat{\beta}(m,n)\lambda_{n+m} (J_{n+m} - J_n - J_m)(H_{n+m} - H_n - H_m) dx.\end{aligned}$$

Indeed if we start with the ansatz that ξ satisfies an Ornstein-Uhlenbeck equation of the form (3.13), then we have an obvious guess for the linear drift $\Gamma\xi$, namely the linearization of the right-hand side of the macroscopic equation (2.6). We also have a candidate for its invariant measure, namely the measure $\mathcal{P}^0$ given by (3.11) at $t = 0$;

$$\iint \sum_{n=1}^{\infty} \xi_n^0(J_n)\xi_n^0(H_n)\mathcal{P}^0(d\xi^0) = \int \sum_{n=1}^{\infty} J_n(x)H_n(x)\lambda_n dx.$$

We then select the diffusion operator $\mathcal{B}$ to be compatible with what we have for the drift and the invariant measure of the process ξ.

Remark 3.5. As our final remark, we comment that it is not obvious that our Markov process $\omega(\cdot)$ exists because we are dealing with infinitely many interacting diffusions. However, since we are only interested in the process $\omega(\cdot)$ at equilibrium, its existence can be shown by rather standard arguments which we now sketch.

– (i) Observe that if initial macroscopic densities $(f_n^0 : n \in \mathbb{N})$ satisfy $\sum_n \int f_n^0 dx < \infty$, then we can construct our process by starting from N independent particles

$(x_1, m_1), \ldots, (x_N, m_N)$ satisfying (2.5), where N and ε are related by the equation $N = K_\varepsilon \sum_n \int f_n^0 dx$. In other words, if the total density is finite macroscopically, then initially we are dealing with finitely many particles almost surely and the existence of the process $\omega(\cdot)$ is obvious. However, since at equilibrium $f_n^0 \equiv \lambda_n$ is not integrable, we need to consider a Poisson point process with infinitely many particles.

– (ii) We now argue that we can construct our process if we make two assumptions:

$$\sum_n \int_{|x|\le r} n f_n^0(x) dx < \infty, \tag{3.14}$$

$$\alpha(m,n) = \beta(m,n) = 0 \quad \text{if} \quad m+n > \ell, \tag{3.15}$$

for every $r > 0$ and some $\ell > 0$. In other words, we assume that locally the total mass is finite macroscopically but now we assume that no interaction occurs if particles are large. To construct $\omega(\cdot)$ in this case, we first replace f^0 with $f^0 \mathbb{1}(|x| \le k)$. Our process exists for such an initial macroscopic density by (i). The corresponding process is denoted by $\omega_k(\cdot)$. We now want to send k to infinity and show that the sequence $(\omega_k : k \in \mathbb{N})$ is tight and that any of its limit point ω is a solution to the martingale problem associated with the generator $\mathcal{A}$. That is, $F(\omega(t)) - \int_0^t \mathcal{A}F(\omega(s)ds$, is a martingale for every C^2 local function F. This can be readily achieved by establishing a control on the total number of particles in a ball $\{x : |x| \le r\}$. Here is a way of establishing such a control uniformy in k: Pick a positive smooth function J which equals to $\exp(-|x|)$ for large x, and set $H(x) = -\int_{|y|\le 1} \log|y| J(x-y)dy$. We can readily show that that $H > 0$ and that $\Delta H \le c_0 H$ for a constant c_0. Then use the martingale $M(t) = F(\omega_k(t)) - \int_0^t \mathcal{A}F(\omega_k(s)ds$ for $F(\omega) = \sum_i H(x_i)m_i$ to show

$$\sup_k \mathbb{E} \sup_{t\in[0,T]} F(\omega_k(t))^2 < \infty, \tag{3.16}$$

for every T. This can be used to establish the tightness of ω_k and the existence of our process provided that (3.14) and (3.15) are true.

– (iii) It remains to relax the restriction (3.15). We now would like to take advantage of the fact that we only need to consider $f_n^0 \equiv \lambda_n$. More precisely, by (ii), we know that $\mathbb{P}^{eq}$ exists if we assume that (3.15) is true. Let us write ω^ℓ for our process when α and β are replaced with $\alpha_\ell(m,n) = \alpha(m,n)\mathbb{1}(m+n \le \ell)$, $\beta_\ell(m,n) = \beta(m,n)\mathbb{1}(m+n) \le \ell)$. Again, we need to show the tightness of ω^ℓ and pass to the limit in the martingale formulation of our process. For this, we need to show something like (3.16) for the sequence ω^ℓ. This can be readily achieved by bounding various terms that appear in the martigale $M(\cdot)$, using the fact that the process ω^ℓ is stationary in time.

4 A Sketch of the Proof

We aim to show that the expression

$$X_\varepsilon(\omega(t)) = K_\varepsilon^{-1/2} \sum_i J(x_i(t), m_i(t)), \tag{4.1}$$

with $J(x,n) = J_n(x)$ satisfying $\int J_n dx = 0$, is close to $\sum_n \xi_n(t, J_n)$, with the distributions $(\xi_n : n \in \mathbb{N})$ solving (3.13) in the weak sense. To derive (3.13), we use Markov property of the process $\omega(t)$ to write

$$\begin{aligned} X_\varepsilon(\omega(t)) &= X_\varepsilon(\omega(0)) + \sum_n \int_0^t \mathcal{A}_0 X_\varepsilon(\omega(s))ds + \int_0^t \mathcal{A}_c X_\varepsilon(\omega(s))ds \\ &\quad + \int_0^t \mathcal{A}_f X_\varepsilon(\omega(s))ds + M_\varepsilon(t) \\ &=: Y_\varepsilon^1 + Y_\varepsilon^2(t) + Y_\varepsilon^3(t) + Y_\varepsilon^4(t) + M_\varepsilon(t), \end{aligned} \tag{4.2}$$

with M_ε a martingale for which

$$N_\varepsilon(t) = M_\varepsilon(t)^2 - \int_0^t (\mathcal{A}X_\varepsilon^2 - 2X_\varepsilon \mathcal{A}X_\varepsilon)(\omega(s))ds, \tag{4.3}$$

is a martingale.

The identity (4.2) should be compared to what we have as the weak form of (3.13), namely

$$\begin{aligned} \sum_n \xi_n(J_n, t) &= \sum_n \xi_n(J_n, 0) + \int_0^t \sum_n d(n)\xi_n(\Delta J_n, s)ds \\ &\quad + \int_0^t \sum_{m,n} \hat{\alpha}(m,n)\lambda_m \xi_n(J_{n+m} - J_m - J_n, s)ds \\ &\quad + \int_0^t \sum_{m,n} \hat{\beta}(m,n)\xi_{n+m}(J_{n+m} - J_m - J_n, s)ds + M(t) \\ &=: Y^1 + Y^2(t) + Y^3(t) + Y^4(t) + M(t), \end{aligned} \tag{4.4}$$

where the process $M(t)$ is a martingale for which

$$N(t) = M(t)^2 - t\int 2\sum_n d(n)\lambda_n|\nabla J_n(x)|^2dx$$
$$- \frac{t}{2}\int\sum_{m,n}\hat{\alpha}(m,n)\lambda_m\lambda_n(J_{n+m}-J_n-J_m)^2(x)dx$$
$$- \frac{t}{2}\int\sum_{m,n}\hat{\beta}(m,n)\lambda_{n+m}(J_{n+m}-J_n-J_m)^2(x)dx,$$

is a martingale.

To establish Theorem 3.1, we may try to show

$$Y_\varepsilon^j \to Y^j, \quad M_\varepsilon \to M,$$

for $i = 1, \ldots, 4$. It turns out that this is not what is going on! Firstly, it is rather straightforward to show that $Y_\varepsilon^1 \to Y^1$ by the classical central limit theorem with Y^1 a Gaussian random variable with variance $\sum_n \lambda_n \int J_n^2 dx$. Also, virtually by definition, we have that if $\boldsymbol{\xi}^\varepsilon$ converges to $\boldsymbol{\xi}$, then $Y_\varepsilon^2 \to \sum_n d(n)\int_0^t \xi_n(\Delta J_n, s)ds$. This stems from the fact that Y_ε^2 corresponds to the "non-interacting" part of the evolution, namely the Laplacian operator Δ. However we need to split the "interacting" part of the microscopic evolution into 3 distinct parts of completely different natures. Indeed, we have a decomposition

$$Y_\varepsilon^3 = Y_\varepsilon^{3,1} + Y_\varepsilon^{3,2} + Y_\varepsilon^{3,3}, \tag{4.5}$$

where $Y_\varepsilon^{3,1} \to Y^3$ as $\varepsilon \to 0$, the term $Y_\varepsilon^{3,2}$ contributes to the fragmentation term so that $Y_\varepsilon^{3,2} + Y_\varepsilon^4 \to Y^4$, and $Y_\varepsilon^{3,3}$ contributes to the martingale part. That is, $Y_\varepsilon^{3,3} + M_\varepsilon \to M$. It is as if a part of the microscopic "drift" becomes some type of "white noise" as $\varepsilon \to 0$. Perhaps this is the most surprising aspect of the present work and is in complete contrast with some earlier works on the equilibrium and non-equilibrium fluctuations on models with diffusive scaling [CY], [C] and a stochastic model with kinetic scaling [R1]. This ramification of the diffusion coefficient by the "drift" is reminiscent of a similar phenomenon for the tagged particles in the exclusion processes (see Kipnis-Vardhan [KV]). In our setting however, the ramification of the noise happens in a rather curious way as we explained in Remark 3.2.

To explain the decomposition (4.5), and sketch our method of proof further, we need to recall how the Smoluchowski's equation has been derived from our microscopic model in the articles [HR1, HR2], [R2] and [HRY]. For this derivation, we need to understand how the microscopic coagulation (respectively fragmentation) rate $\alpha(m,n)$

(respectively $\beta(m,n)$) leads to the macroscopic coagulation rate $\hat{\alpha}(m,n)$ (respectively $\hat{\beta}(m,n)$).

For the derivation of (2.6), we start from the expression

$$\hat{X}_\varepsilon(\omega(t)) = K_\varepsilon^{-1/2} X_\varepsilon(\omega(t)) = K_\varepsilon^{-1} \sum_i J(x_i(t), m_i(t)),$$

and study the corresponding (4.2) which we obtain by multiplying both sides of (4.2) by $K_\varepsilon^{-1/2}$. Since $K_\varepsilon^{-1/2} M_\varepsilon \to 0$, we only need to concentrate on $K_\varepsilon^{-1/2} Y_\varepsilon^3$ and $K_\varepsilon^{-1/2} Y_\varepsilon^4$. The term $K_\varepsilon^{-1/2} Y_\varepsilon^4$ is in some sense linear and all challenges come from $K_\varepsilon^{-1/2} Y_\varepsilon^3$. It turns out that there is a splitting $K_\varepsilon^{-1/2} Y_\varepsilon^3 = Z_\varepsilon^1 + Z_\varepsilon^2$ with Z_ε^1 converging to $\int_0^t \int \sum_n J_n(x) Q_n^c(\mathbf{f})(x,t) dx$ and $Z_\varepsilon^2 + K_\varepsilon^{-1/2} Y_\varepsilon^4$ converging to $\int_0^t \int \sum_n J_n(x) Q_n^f(\mathbf{f})(x,t) dx$. This splitting is not hard to justify; when a fragmentation occurs, a pair of particles are produced which are within a distance of order $O(\varepsilon)$ and prone to coagulate. Of course such a coagulation undoes the fragmentation that has just been occurred. Indeed Z_ε^2 is negative which results in a macroscopic fragmentation $\hat{\beta}$ strictly less than β.

To describe the decomposition (4.5), let us observe

$$\begin{aligned} Y_\varepsilon^3 &= \frac{1}{2} K_\varepsilon^{-1/2} \sum_{i,j} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \tilde{J}(x_i, m_i, x_j, m_j) \\ &= \frac{1}{2} K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) V^\varepsilon(x_i - x_j; m_i, m_j) \tilde{J}(x_i, m_i, x_j, m_j), \end{aligned} \tag{4.6}$$

where $V^\varepsilon = K_\varepsilon V_\varepsilon$ and $\tilde{J}(x_i, m_i, x_j, m_j)$ is given by

$$\frac{m_i}{m_i + m_j} J(x_i, m_i + m_j) + \frac{m_j}{m_i + m_j} J(x_j, m_i + m_j) - J(x_i, m_i) - J(x_j, m_j). \tag{4.7}$$

Our goal would be a decomposition of the form

$$\int_0^t Y_\varepsilon^3(s) ds = \int_0^t B_\varepsilon^z(\omega(s)) ds + \int_0^t C_\varepsilon(\omega(s)) ds + D_\varepsilon(t) + Error, \tag{4.8}$$

where

$$B_\varepsilon^z(\omega) = \frac{1}{2} K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) W^\varepsilon(x_i - x_j + z; m, n) \tilde{J}(x_i, m_i, x_j, m_j), \tag{4.9}$$

for a suitable function W^ε which will be defined shortly, and *Error* represents a term that will go to zero as $\varepsilon \to 0$ and $|z| \to 0$. The form of W^ε would allow us to replace

α with its macroscopic counterpart $\hat{\alpha}$. The term C_ε is given by

$$\int K_\varepsilon^{-3/2} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) V^\varepsilon(x_i - y; m, m_i - m) \\ u^\varepsilon(x_i - y; m, m_i - m) \tilde{J}(x_i, m, y, m_i - m) dy,$$

and the term $D_\varepsilon(t)$ is a martingale. It is the decomposition (4.8) that leads to the decomposition (4.5).

To achieve the decomposition (4.8), fix z and start from the expression

$$G_\varepsilon(\omega) = K_\varepsilon^{-3/2} \sum_{i,j} \hat{u}^\varepsilon(x_i - x_j; m_i, m_j) \tilde{J}(x_i, m_i, x_j, m_j), \tag{4.10}$$

where $\hat{u}^\varepsilon(a; m, n) = u^\varepsilon(a+z; m, n) - u^\varepsilon(a; m, n)$, with $u^\varepsilon(a; m, n)$ satisfying the equation

$$(d(m) + d(n))\Delta u^\varepsilon(x; m, n) = \alpha(m, n) \left[V_\varepsilon(x; m, n) u^\varepsilon(x; m, n) + V^\varepsilon(x; m, n) \right]. \tag{4.11}$$

(The functions V^ε and V_ε were defined by (2.4) and right before (2.2) rescepectively.) We then apply the martingale decomposition as in (4.2) to assert

$$G_\varepsilon(\omega(t)) = G_\varepsilon(\omega(0)) + \int_0^t \mathcal{A} G_\varepsilon(\omega(s)) + E_\varepsilon(t), \tag{4.12}$$

with $E_\varepsilon(t)$ a martingale. This involves various terms as we apply the operators $\mathcal{A}_0$, $\mathcal{A}_c$ and $\mathcal{A}_f$ on G_ε. As it turns out, the first term $G_\varepsilon(\omega(0))$ and many other terms on the right-hand side of (4.12) are small if $|z|$ is sufficiently small. However, the choice of u^ε results in a component in $(\mathcal{A}_0 + \mathcal{A}_c)G_\varepsilon$, which is exactly our $2(B_\varepsilon^z - Y_\varepsilon^3)$ in (4.9), and a component in $\mathcal{A}_f G_\varepsilon$ which is exactly C_ε. The function W^ε in (4.9) is given by

$$W^\varepsilon(a; m, n) = V^\varepsilon(a; m, n)(1 + K_\varepsilon^{-1} u^\varepsilon(a; m, n)). \tag{4.13}$$

Of course we need to show that all other components in $(\mathcal{A}_0 + \mathcal{A}_c)G_\varepsilon$, and $\mathcal{A}_f G_\varepsilon$ are small if ε and $|z|$ are small. This can be achieved by rather straightforward reasoning if we require

$$K_\varepsilon^{1/2} |z| |\log |z|| \to 0, \qquad K_\varepsilon^{-1/2} |\log |z|| \to 0. \tag{4.14}$$

(In higher dimension, the second condition is replaced with $K_\varepsilon^{-1/2} |z|^{2-d} \to 0$, which is inconsistent with the first condition if $d \geq 3$.) These two conditions are satisfied if $|z| = |\log \varepsilon|^{-\theta}$, for some $\theta > 1/2$. At this stage, we simply use the smallness of $\hat{u}^\varepsilon$ for z satisfying $\varepsilon << |z| << 1$. In other words, we do not take advantage of the fact

that J is of 0 average and do not apply any central limit-type arguments. (For higher dimensions, this line of reasoning is not applicable and we really need to establish a central limit-type theorem to show that the error term in (4.8) is small.) Of course we may try to square the error term and take advantage of the fact that J is of 0 average and that particles are independent at equilibrium. This turns out to be rather technical and challenging and will be dealt with in a future publication.

So far we have learned that the expression Y_ε^3 can be replaced with the right hand-side of (4.8). Once this is achieved, we take a smooth function ζ of compact support, set $\zeta^\delta(a) = \delta^{-d}\zeta\left(\frac{a}{\delta}\right)$, and define

$$f_n^\delta(x,t) = K_\varepsilon^{-1}\sum_i \zeta^\delta(x_i(t)-x)\mathbb{1}(m_i(t)=n).$$

We think of this as an approximation of the density of particles of cluster size n. We choose $\delta = \delta(\varepsilon) = |\log\varepsilon|^{-\theta}$ with $\theta > 1/2$. The outcome $\tilde{f}_n^\varepsilon(x,t) = f_n^{\delta(\varepsilon)}(x,t)$ converges weakly to λ_n as $\varepsilon \to 0$. So far we have not carried out any CLT. We know that if $|z_2 - z_1| \le \delta(\varepsilon)$, then

$$\int_0^t Y_\varepsilon^3(s)ds = \int_0^t B_\varepsilon^{z_2-z_1}(\omega(s))ds + \int_0^t C_\varepsilon(\omega(s))ds + D_\varepsilon(t) + \mathrm{Error}^1(\varepsilon) \tag{4.15}$$

with $\lim_{\varepsilon\to 0}\mathrm{Error}^1(\varepsilon) = 0$ and $D_\varepsilon(t)$ a martingale. We multiply both sides of (4.15) by $\zeta^\delta(z_1)\zeta^\delta(z_2)$ and integrate with respect to z_1 and z_2. After a change of variables $z_1 \mapsto z_1 - x_i$, $z_2 \mapsto z_2 - x_j$, we obtain

$$\frac{1}{2}\int_0^t\iint K_\varepsilon^{-3/2}\sum_{i,j}\alpha(m_i,m_j)W^\varepsilon(z_1-z_2;m_i,m_j)\tilde{J}(x_i,m_i,x_j,m_j)$$
$$\zeta^{\delta(\varepsilon)}(x_i-z_1)\zeta^{\delta(\varepsilon)}(x_j-z_2)dz_1dz_2ds,$$

for the first term of the right-hand side of (4.15). Since J is smooth and ζ is of compact support, we may replace $\tilde{J}(x_i,m_i,x_j,m_j)$ with $\tilde{J}(z_1,m_i,z_2,m_j)$ for an error of order $O(\delta(\varepsilon))$. We then carry out the summation over i and j to obtain

$$\frac{1}{2}\sum_{m,n}\alpha(m,n)\int_0^t\iint K_\varepsilon^{1/2}f_n^{\delta(\varepsilon)}(z_1,s)f_m^{\delta(\varepsilon)}(z_2,s)W^\varepsilon(z_1-z_2;m,n)\tilde{J}(z_1,m,z_2,n)dz_1dz_2ds.$$

Since J is of zero average, the integrand

$$\Lambda := K_\varepsilon^{1/2}(f_n^{\delta(\varepsilon)}(z_1,s)f_m^{\delta(\varepsilon)}(z_2,s) - \lambda_n\lambda_m)W^\varepsilon(z_1-z_2;m,n)\tilde{J}(z_1,m,z_2,n),$$

can be written as

$$\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3, \tag{4.16}$$

where

$$\Lambda_1 = K_\varepsilon^{1/2}(f_n^{\delta(\varepsilon)}(z_1,s)-\lambda_n)\lambda_m W^\varepsilon(z_1-z_2;m,n)\tilde{J}(z_1,m,z_2,n),$$
$$\Lambda_2 = K_\varepsilon^{1/2}(f_m^{\delta(\varepsilon)}(z_1,s)-\lambda_m)\lambda_n W^\varepsilon(z_1-z_2;m,n)\tilde{J}(z_1,m,z_2,n),$$
$$\Lambda_3 = K_\varepsilon^{1/2}[(f_n^{\delta(\varepsilon)}(z_1,s)-\lambda_n)(f_m^{\delta(\varepsilon)}(z_2,s)-\lambda_m)]W^\varepsilon(z_1-z_2;m,n)\tilde{J}(z_1,m,z_2,n).$$

To achieve our goals, we wish to show that Λ_3 is small in average. Formally, if Λ_1 is a bounded quantity, then Λ_3 is smaller than Λ_1 because of the additional small term $f_m^{\delta(\varepsilon)}-\lambda_m$. This turns out to be wrong; the term $f_m^\delta-\lambda_m$ is small only in a weak sense and its product with $f_n^\delta-\lambda_n$ is no longer small. This is not surprising at all because $\delta=\delta(\varepsilon)$ is not sufficiently large enough for a central limit theorem to take place. Indeed the support of ζ^δ is a set of volume $O(\delta^d)$ and as a result, the particle density f_n^δ involves $O(K_\varepsilon\delta^d)$ many particles in average. For a CLT taking place, we need a density which deals with a large number of particles. In other words, we expect Λ_3 to be small only when $K_\varepsilon\delta^d\to\infty$ as $\varepsilon\to\infty$. This would not be the case if $\delta=|\log\varepsilon|^{-\theta}$ for a $\theta>1/2$.

In order to figure out a successful way of going beyond $|\log\varepsilon|^{-\theta}$ and reach a density f^δ with δ satisfying $K_\varepsilon\delta^d\to\infty$, we need to review what has been achieved so far and what to learn from it.

Basically our goal is a central limit theorem (CLT) for the particle density (4.1) and for this we need to perform some type of CLT for the time average of (4.6). Note that Y_ε^3 is in some sense singular because the function V^ε is a delta-type expression. That is, in a region of volume $O(\varepsilon^d)$, V^ε is of order $O(\varepsilon^{-d})$. In fact if we calculate $\mathbb{E}_\varepsilon^{eq}Y_\varepsilon^3(\omega)^2$, we get an expression that blows up as $\varepsilon\to0$. All this ultimately stems from the fact that the coagulation occurs when particles are microscopically close. We wish to replace V^ε with a smoother kernel and this is exactly what purpose (4.8) serves. We try to replace x_i-x_j, the argument of V^ε, with x_i-x_j+z. That is, we try to figure out the coagulation rate when particles x_i and x_j are not microscopically close but only macroscopically close, i.e., $x_i-x_j=z+O(\varepsilon)$ with $|z|\to0$ after sending $\varepsilon\to0$. (For example $|z|$ could be as "large" as $|\log\varepsilon|^{-\theta}$.) However there is a price to pay for such a replacement; we need to replace V^ε with W^ε and modify the fragmentation term (we are referring to the term C_ε), and even the noise is modified (the term D_ε). To carry this out, we encountered various additional terms which are presumably small. We have a relatively easy ride, if $|z|<<|\log\varepsilon|^{-1/2}$. Even though we have not reached out ultimate goal $|z|>>|\log\varepsilon|^{-1/2}$, we have already achieved three important tasks:

- (i) The correctors C_ε and D_ε would modify the fragmentation and martingle terms as required in the proof of the main result Theorem 3.1.
- (ii) The term W^ε would allow us to replace α with $\hat{\alpha}$ because $\lim\int W^\varepsilon=\eta$ as $\varepsilon\to0$.

– (iii) We have been able to replace the singular term $V^\varepsilon(a;m,n)$ with a less singular term $\bar{W}^\varepsilon(a;m,n) = \int W^\varepsilon(a+z;m,n)\zeta^{\delta(\varepsilon)}(z)dz$, where $\delta(\varepsilon) = |\log\varepsilon|^{-\theta}$ for some $\theta > 1/2$.

We are now in a position to explain the central role of the equation (4.11). Because of the time average in $\int_0^t Y_\varepsilon^3(s)ds$, we are dealing with an expression which is almost as smooth as $\mathcal{A}^{-1}Y_\varepsilon^3$. Of course $\mathcal{A}^{-1}$ is too complicated to use. The message behind the equation (4.11) and its use is that we only need to consider 2-particles dynamics. Namely, the fact that $x_i - x_j$ is a diffusion with generator $(d(m_i) + d(m_j))\Delta$, and that once a coagulation occurs with rate $\alpha(m_i, m_j)$ between the i-th and j-th particles, (x_i, x_j) as a pair no longer exists and hence the dynamics of $x_i - x_j$ has an infinitesimal generator of a killed Brownian motion:

$$\Gamma^\varepsilon = (d(m) + d(n))\Delta - \alpha(m,n)V_\varepsilon(\cdot;m,n),$$

with $m = m_i$ and $n = m_j$. Now the function $u^\varepsilon = \Gamma_\varepsilon^{-1}V^\varepsilon$ is smoother than V^ε and this allows us to perturb its argument by a small vector z and obtain (4.8). By (iii), we are now dealing $\bar{W}^\varepsilon$ in place of V^ε. We note that $\bar{W}^\varepsilon = O(\delta(\varepsilon)^{-d})$, and has a support of diameter $O(\delta(\varepsilon))$. To replace $\bar{W}^\varepsilon$ with $\tilde{W}^\varepsilon(a;m,n) = \int W^\varepsilon(a+z;m,n)\zeta^{\delta'(\varepsilon)}(z)dz$, for some $\delta'(\varepsilon) >> |\log\varepsilon|^{-1/2}$, we almost repeat the formula (4.12) where V^ε is replaced with $\bar{W}^\varepsilon$, and u^ε is replaced with v^ε which now solves

$$(d(m) + d(n))\Delta v^\varepsilon(x;m,n) = \alpha(m,n)\bar{W}^\varepsilon(x;m,n). \tag{4.17}$$

This time we can show that various terms that appear in $\mathcal{A}G_\varepsilon$ are small provided that $|z| \le \delta'(\varepsilon)$ for δ' that is now can be chosen as large as $|\log\log\varepsilon|^{-\theta'}$ for any $\theta' > 1/2$. For this step of the proof we show that all the error terms have small second moments, in other words, a CLT is taking place and the errors have small variances. (See Section 7).

As a consequence of the main result of Section 7, we have the decomposition (4.15) where $\delta(\varepsilon)$ is replaced with $\delta'(\varepsilon)$. We can now rigorously show that Λ_3 is small by ignoring the time integration and showing that the integrand has a small second moment with respect to the equilibrium measure. As for Λ_1, we first carry out dz_2 integration and use the fact that $\lim \int W^\varepsilon(a;m,n)da = \eta(m,n)$, as $\varepsilon \to 0$. (This was proved as Theorem 3.2 in [HR2].) After some straightforward manipulations,

$$\int_0^t \iint \Lambda_1 \, dz_1 dz_2 ds = \left[\int_0^t \xi_n(t,J)ds\right]\eta(m,n)\lambda_m + \text{Error}^2(\varepsilon).$$

As for Λ_2, we first replace $J(z_1)$ with $J(z_2)$ for a small error because $|z_1 - z_2| = O(\varepsilon)$. We then integrate with respect to z_1 and repeat our reasoning for Λ_1 to obtain

$$\int_0^t \iint \Lambda_2 \, dz_1 dz_2 ds = \left[\int_0^t \xi_m(s,J)ds\right]\eta(m,n)\lambda_n + \text{Error}^3(\varepsilon).$$

In summary

$$(4.18)\quad \int_0^t Y_\varepsilon^3(s)ds = \frac{1}{2}\sum_{m,n}\hat{\alpha}(m,n)\int_0^t(\xi_m(s,J)\lambda_n+\xi_n(s,J)\lambda_m)ds+D_\varepsilon(t)+Error(\varepsilon)$$

for an error Error(ε) that goes to 0 on $\varepsilon\to 0$.

5 Regularity of the Coagulation Term, Part I

As we mentioned in Section 4, the main ingredient for the proof of Theorem 3.1 is the statement (4.18). In this section this statement is partially established and the full proof of of (4.18) will be achieved in Section 7.

We now prepare for the main result of this section, which will appear as Theorem 5.1 at the end of the section. The proof of Theorem 5.1 will be given in Section 6. Note that the function J in (4.1) is of compact support and satisfies $\int J(x,n)dx=0$, for every n. In fact we only need to consider $J(x,n)=\bar{J}(x)\mathbb{1}(n=\bar{m})$ with $\int\bar{J}(x)dx=0$. Evidently for such a function J, we have $\int J(x,n)dx=0$ for every n. Note that $\tilde{J}$ of (4.7) is not of compact support. However, for some positive l, we have that $\tilde{J}(x,m,y,n)=0$, if either $m,n\geq l$ or $|x_i|,|x_j|\geq l$. Because of the V_ε term in the definition of Y_ε^3, we may replace $\tilde{J}$ with

$$\hat{J}(x_i,m_i,x_j,m_j)=\tilde{J}(x_i,m_i,x_j,m_j)K(x_i-x_j),$$

for a smooth symmetric function $K(a)$ of compact support which is 1 whenever $|a|\leq 1$. The advantage of $\hat{J}$ to $\tilde{J}$ is that the former is of compact support in the spatial variables. Note however, the term V_ε only implies that $|x_i-x_j|\leq c_0\varepsilon r(m_i,m_j)$ for a constant c_0. Hence such a replacement is valid only if $c_0\varepsilon r(m_i,m_j)\leq 1$. This causes an error that can be readily handled with the aid of our hypothesis (3.9). (See the first step of the proof of Theorem 8.1 in Section 8.)

Recall that $u^\varepsilon(x;m,n)$ solves

$$(d(m)+d(n))\Delta u^\varepsilon(x;m,n)=\alpha(m,n)[V_\varepsilon(x;m,n)u^\varepsilon(x;m,n)+V^\varepsilon(x;m,n)]$$

where $V^\varepsilon(x;m,n)=\varepsilon^{-2}V(x/\varepsilon;m,n)$, and $V_\varepsilon(x;m,n)=K_\varepsilon^{-1}\varepsilon^{-2}V(x/\varepsilon;m,n)$. Given such a function u^ε, we define

$$(5.1)\qquad G(\omega;z)=G(\omega)=K_\varepsilon^{-3/2}\sum_{i,j}\hat{u}^\varepsilon(x_i-x_j;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j),$$

where $\hat{u}^\varepsilon(a;m,n)=u^\varepsilon(a+z;m,n)-u^\varepsilon(a;m,n)$. We have

$$G(\omega(t))=G(\omega(0))+\int_0^t\mathcal{A}G(\omega(s))ds+M_t$$

where M_t is a martingale. We write

$$\mathcal{A}G = \mathcal{A}_0 G + \mathcal{A}_c G + \mathcal{A}_f G =: H_1 + H_2 + H_3. \tag{5.2}$$

We now study various terms which appeared on the right-hand side. We write $\hat{J}_x$ and $\hat{J}_y$ for the derivatives of $\hat{J}$ with respect to its first and second spatial arguments. We then write

$$H_1 = H_{11} + H_{12} + H_{13},$$

with

$$\begin{aligned}
H_{11}(\omega) &= K_\varepsilon^{-3/2} \sum_{i,j} \hat{u}^\varepsilon(x_i - x_j; m_i, m_j)[(d(m_i)\Delta_{x_i} + d(m_j)\Delta_{x_j})\hat{J}(x_i, m_i, x_j, m_j)] \\
H_{12}(\omega) &= K_\varepsilon^{-3/2} \sum_{i,j} d(m_i)\hat{u}_x^\varepsilon(x_i - x_j; m_i, m_j) \cdot \hat{J}_x(x_i, m_i, x_j, m_j) \\
&\quad - K_\varepsilon^{-3/2} \sum_{i,j} d(m_j)\hat{u}_x^\varepsilon(x_i - x_j; m_i, m_j) \cdot \hat{J}_y(x_i, m_i, x_j, m_j) \\
&=: H_{121}(\omega) - H_{122}(\omega), \\
H_{13}(\omega) &= K_\varepsilon^{-3/2} \sum_{i,j} (d(m_i) + d(m_j))\Delta\hat{u}^\varepsilon(x_i - x_j; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j) \\
&=: H_{13}^z(\omega) - H_{131}^0(\omega) - H_{132}^0(\omega)
\end{aligned}$$

where

$$H_{13}^z(\omega) = K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) W^\varepsilon(x_i - x_j + z; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j)$$

with $W^\varepsilon(a; m, n) = u^\varepsilon(a; m, n)V_\varepsilon(a; m, n) + V^\varepsilon(a; m, n)$, and

$$\begin{aligned}
H_{131}^0(\omega) &= K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) u^\varepsilon(x_i - x_j; m_i, m_j)V_\varepsilon(x_i - x_j; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j), \\
H_{132}^0(\omega) &= K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) V^\varepsilon(x_i - x_j; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j).
\end{aligned}$$

We also write

$$H_2 = H_{21} + H_{22}, \quad H_{21} = H_{21}^z - H_{21}^0,$$

with $H_{21}^z(\omega)$ given by

$$\begin{aligned}
&-\frac{1}{2} K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \\
&\qquad \Big[u^\varepsilon(x_i - x_j + z; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j) + u^\varepsilon(x_j - x_i + z; m_i, m_j)\hat{J}(x_j, m_j, x_i, m_i) \Big] \\
&= -K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j) u^\varepsilon(x_i - x_j + z; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j),
\end{aligned}$$

Moreover,

$$
\begin{aligned}
H_{22}(\omega) = \frac{1}{2}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)K_\varepsilon^{-3/2} \\
\sum_k \Big\{ \frac{m_i}{m_i+m_j}[\hat{u}^\varepsilon(x_i-x_k;m_i+m_j,m_k)\hat{J}(x_i,m_i+m_j,x_k,m_k) \\
+ \hat{u}^\varepsilon(x_k-x_i,m_k,m_i+m_j)\hat{J}(x_k,m_k,x_i,m_i+m_j)] \\
+ \frac{m_i}{m_i+m_j}[\hat{u}^\varepsilon(x_j-x_k;m_i+m_j,m_k)\hat{J}(x_j,m_i+m_j,x_k,m_k) \\
+ \hat{u}^\varepsilon(x_k-x_j;m_k,m_i+m_j)\hat{J}(x_k,m_k,x_j,m_i+m_j)] \\
- [\hat{u}^\varepsilon(x_i-x_k;m_i,m_n)\hat{J}(x_i,m_i,x_k,m_k) \\
+ \hat{u}^\varepsilon(x_k-x_i;m_k,m_i)\hat{J}(x_k,m_k,x_i,m_i)] \\
- [\hat{u}^\varepsilon(x_j-x_k;m_j,m_k)\hat{J}(x_j,m_j,x_k,m_k) \\
+ \hat{u}^\varepsilon(x_k-x_j;m_k,m_j)\hat{J}(x_k,m_k,x_j,m_j)]\Big\}.
\end{aligned}
$$

The expression H_{22} arises from the changes in the function G when a coagulation occurs due to the influence of the appearance and disappearance of particles on other particles that are not directly involved. The expression H_{21} represents those terms in G that are absent after a coagulation. Note that for our formula for H_{12}, we used the fact that K is symmetric and since V is symmetric, the function u^ε is also symmetric.

As for the fragmentation part of dynamics, we have

$$H_3 = H_{31} + H_{32} + H_{33},$$

where $H_{31} = H_{311} + H_{312}$, with

$$
\begin{aligned}
H_{311}(\omega) = \frac{1}{2}\int K_\varepsilon^{-3/2}\sum_{i,j}\sum_{m=1}^{m_i-1}\beta(m,m_i-m)V^\varepsilon(x_i-y;m,m_i-m) \\
\big[\hat{u}^\varepsilon(x_i-x_j;m,m_j)\hat{J}(x_i,m,x_j,m_j) - \hat{u}^\varepsilon(x_i-x_j;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j)\big]dy, \\
H_{312}(\omega) = \frac{1}{2}\int K_\varepsilon^{-3/2}\sum_{i,j}\sum_{m=1}^{m_j-1}\beta(m,m_j-m)V^\varepsilon(x_j-y;m,m_j-m) \\
\big[\hat{u}^\varepsilon(x_i-x_j;m_i,m)\hat{J}(x_i,m_i,x_j,m) - \hat{u}^\varepsilon(x_i-x_j;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j)\big]dy.
\end{aligned}
$$

We carry out dy integration and use symmetry to obtain that $H_{31} = 2H_{311}$, where

$$H_{311}(\omega) = \frac{1}{2}\int K_\varepsilon^{-3/2} \sum_{i,j} \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \\ \left[\hat{u}^\varepsilon(x_i - x_j; m, m_j)\hat{J}(x_i, m, x_j, m_j) - \hat{u}^\varepsilon(x_i - x_j; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j)\right].$$

Also, $H_{32} = H_{321} + H_{322}$, with

$$H_{321}(\omega) = \frac{1}{2}\int K_\varepsilon^{-3/2} \sum_{i,j} \sum_{m=1}^{m_i-1} \beta(m, m_i - m) V^\varepsilon(x_i - y; m, m_i - m) \\ \hat{u}^\varepsilon(y - x_j; m, m_j)\hat{J}(y, m, x_j, m_j)dy,$$

$$H_{322}(\omega) = \frac{1}{2}\int K_\varepsilon^{-3/2} \sum_{i,j} \sum_{m=1}^{m_j-1} \beta(m, m_j - m) V^\varepsilon(x_j - y; m, m_j - m) \\ \hat{u}^\varepsilon(x_i - y; m_i, m)\hat{J}(x_i, m_i, y, m)dy,$$

and $H_{33} = H_{33}^z - H_{33}^0$, with

$$H_{33}^z(\omega) = \int K_\varepsilon^{-3/2} \sum_{i} \sum_{m=1}^{m_i-1} \beta(m, m_i - m) V^\varepsilon(x_i - y; m, m_i - m) \\ u^\varepsilon(x_i - y + z; m, m_i - m)\hat{J}(x_i, m, y, m_i - m)\, dy.$$

Note that $H_{131}^0 + H_{21}^0 = 0$. We may rewrite (5.2) as

$$\mathcal{A}G + [H_{132}^0 - H_{13}^z + H_{33}^0] = (H_{11} + H_{12}) + H_{21}^z + (H_{22} + H_{31} + H_{32}) + H_{33}^z. \tag{5.3}$$

We are now ready to state the main result of this section.

Theorem 5.1 *Let $\hat{J}$ be as above and assume that $\sqrt{\varepsilon} < |z| < 1$. Then*

$$\mathbb{E}_\varepsilon^{eq}\left|\int_0^t \mathcal{A}G(\omega(s))ds + \int_0^t [H_{132}^0(\omega(s)) - H_{13}^z(\omega(s)) + H_{33}^0(\omega(s))]ds\right| \\ \le C_0 t\left[K_\varepsilon^{1/2}|z||\log|z|| + K_\varepsilon^{-1/2}|\log|z||\right]. \tag{5.4}$$

We establish Theorem 5.1 by examining various terms that appeared on the right-hand side of (5.3). Indeed we show

$$\mathbb{E}_\varepsilon^{eq}|G(\omega(t))||G(\omega(0))| \le C_0' K_\varepsilon^{1/2}|z|, \tag{5.5}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{11}(\omega(s))| \le C_0' K_\varepsilon^{1/2}|z|, \tag{5.6}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{12}(\omega(s))| \le C_0' K_\varepsilon^{1/2}|z||\log|z||, \tag{5.7}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{22}(\omega(s))| \le C_0' K_\varepsilon^{1/2}|z|, \tag{5.8}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{31}(\omega(s))| \le C_0' K_\varepsilon^{1/2}|z|, \tag{5.9}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{32}(\omega(s))| \le C_0' K_\varepsilon^{1/2}|z|, \tag{5.10}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{21}^z(\omega(s))| \le C_0' K_\varepsilon^{-1/2}\,|\log|z||\,, \tag{5.11}$$

$$\mathbb{E}_\varepsilon^{eq}\,|H_{33}^z(\omega(s))| \le C_0' K_\varepsilon^{-1/2}\,|\log|z||\,. \tag{5.12}$$

Theorem 5.1 is an immediate consequence of (5.5-11). The bound (5.5) will be used for the proof of Theorem 3.1.

As we mentioned in Section 4, our method of proof can be used to establish a law of large number (LLN) for the expression $\int_0^t K_\varepsilon^{1/2} Y_\varepsilon^3(s)ds$ with Y_ε^3 as in (4.4). This can be achieved as in [HR2] by using the regularity of the coagulation term and this time z can be chosen to be any small vector. Moreover for $\tilde{J}$, we may choose any smooth function of compact support. Note that since we are at equilibrium, the proof of LLN is much easier than what we have in [HR2] because all the correlation bounds needed for the proof are trivially true. This would allow us to find the limit of $\int_0^t K_\varepsilon^{1/2} Y_\varepsilon^3(s)ds$ as $\varepsilon \to 0$. Since this limit is not random, the limit can be calculated by passing to the limit in $\mathbb{E}_\varepsilon^{eq}\int_0^t K_\varepsilon^{1/2} Y_\varepsilon^3(s)ds = t\mathbb{E}_\varepsilon^{eq} K_\varepsilon^{1/2} Y_\varepsilon^3(0)$. In summary,

Lemma 5.1 *Let $K(x,m,y,n)$ by any smooth function of compact support. Then*

$$\lim_{\varepsilon\to 0}\mathbb{E}_\varepsilon^{eq}\left|\int_0^t Z^\varepsilon(\omega(s))ds - t\bar{Z}\right| = 0, \tag{5.13}$$

where

$$Z^\varepsilon(\omega) = K_\varepsilon^{-2}\sum_{i,j}\alpha(m_i,m_j)V^\varepsilon(x_i-x_j;m_i,m_j)K(x_i,m_i,x_j,m_j),$$

$$\bar{Z} = \sum_{m,n}\lambda_m\lambda_n\alpha(m,n)\int K(x,m,x,n)dx.$$

Lemma 5.1 will be needed in Section 8. In Section 8 we also need another LLN which can be established with a similar argument. This time our $Z^\varepsilon(\omega)$ is given by

$$(5.14)\quad K_\varepsilon^{-2}\sum_{i,j} d(m_i)\left(K_\varepsilon^{-1}|\nabla u^\varepsilon(x_i-x_j;m_i,m_j)|^2\right)\tilde{J}(x_i,m_i,x_i,m_j)^2 \mathbb{1}(|x_i-x_j|\le 1).$$

As we will see in Lemma 6.1 of Section 6, the function

$$W^\varepsilon(a;m,n)=K_\varepsilon^{-1}|\nabla u^\varepsilon(a;m,n)|^2\mathbb{1}(|a|\le 1),$$

is almost as singular as $V^\varepsilon(a;m,n)$ because $W^\varepsilon(a;m,n)=O(\varepsilon^{-2}K_\varepsilon^{-1})$ when $|a|\le\varepsilon$. However $\int W^\varepsilon da$ stays bounded as $\varepsilon\to 0$. We will calculate $\gamma=\lim_{\varepsilon\to 0}\int W^\varepsilon da$ in Section 8 (see the final step of the proof of (8.4).) We have,

Lemma 5.2 *Let Z^ε be as in* (5.14)*. Then* (5.13) *is true for*

$$\bar{Z}=\sum_{m,n}\lambda_m\lambda_n d(m)\gamma(m,n)\int \tilde{J}(x,m,x,n)^2dx.$$

This lemma can be proved in a similar way. This time we start with a function $w^\varepsilon(x;m,n)$ that now solves

$$(d(m)+d(n))\Delta w^\varepsilon(x;m,n)=\alpha(m,n)V_\varepsilon(x;m,n)w^\varepsilon(x;m,n)+d(m)W^\varepsilon(x;m,n),$$

and define

$$(5.15)\qquad G(\omega)=K_\varepsilon^{-2}\sum_{i,j}\hat{w}^\varepsilon(x_i-x_j;m_i,m_j)\tilde{J}(x_i,m_i,x_i,m_j)^2,$$

where $\hat{w}^\varepsilon(a;m,n)=w^\varepsilon(a+z;m,n)-w^\varepsilon(a;m,n)$. Again, using the same method of proof as [HR2] we can show that the limit in (5.13) exists and then by taking the expectation of Z^ε, we identify the limit.

6 Proof of Theorem 5.1

In this section, we establish (5.5)– (5.12). As a preliminary step, we state a lemma about the regularity of the function u^ε. Recall that u^ε satisfies (4.11) or equivalently

$$(d(m)+d(n))\Delta_x u^\varepsilon(x;m,n)=\alpha(m,n)V^\varepsilon(x;m,n)\left[|\log\varepsilon|^{-1}u^\varepsilon(x;m,n)+1\right],$$

In fact $\log \varepsilon \leq u^\varepsilon$ and u^ε is given by

$$\frac{1}{2\pi}\alpha'(m,n)\int \log|x-y|V^\varepsilon(y;m,n)\left[|\log\varepsilon|^{-1}u^\varepsilon(y;m,n)+1\right]dy,$$

where $\alpha'(m,n)=\alpha(m,n)/(d(m)+d(n))$.

To ease the notation, we do not display the dependence of $\alpha'(m,n)$ and $r(m,n)$ on m and n.

Lemma 6.1 *There exist positive constants C_1 and C_2 such that for all x,*

$$|u^\varepsilon(x;m,n)| \leq C_1\alpha' \min\left\{1+\left|\log\frac{|x|}{r}\right|, |\log\varepsilon|\right\}, \tag{6.1}$$

$$|\nabla u^\varepsilon(x;m,n)| \leq C_1\alpha' \min\left\{|x|^{-1},(r\varepsilon)^{-1}\right\}, \tag{6.2}$$

and for $|x|\geq 2|z|+C_2r\varepsilon$,

$$|\nabla u^\varepsilon(x+z;m,n)-\nabla u^\varepsilon(x;m,n)| \leq C_1\alpha'|x|^{-2}|z|. \tag{6.3}$$

Also,

$$\int_{|a|\leq l}|\nabla u^\varepsilon(a;m,n)|da \leq C_1\alpha' l, \tag{6.4}$$

$$\int_{|a|\leq l}|\hat{u}^\varepsilon(a;m,n)|da \leq C_1\alpha'(l+|z|)|z|, \tag{6.5}$$

$$\int_{|a|\leq l}|\nabla\hat{u}^\varepsilon(a;m,n)|da \leq C_1\alpha'\left\{|z|\left[|\log(|z|+r\varepsilon)|+1+\log^+ l\right]+r\varepsilon\right\}, \tag{6.6}$$

$$\int_{|a|\leq l}u^\varepsilon(a;m,n)^2da \leq C_1\alpha'^2\left[r^2\varepsilon^2|\log\varepsilon|^2+l^2\left(\log^+\frac{l}{r}\right)^2+1\right], \tag{6.7}$$

$$\int_{|a|\leq l}|\nabla u^\varepsilon(a;m,n)|^2da \leq C_1\alpha'^2\left[1+\log^+\frac{l}{r\varepsilon}+r^2\right], \tag{6.8}$$

Proof. The proofs of (6.1), (6.2) and (6.3) are omitted and can be found in Section 2.2 of [HR2]. Note however that in [HR2] we are assuming that $\chi = 0$ and that we were dealing with $V^\varepsilon(x)=\varepsilon^{-2}V(x/\varepsilon)$ instead of $(\varepsilon r)^{-2}V(x/(r\varepsilon))$. Since we have $u^\varepsilon(x;m,n)=v^\varepsilon(x/r)$ for $r=r(m,n)$ and v^ε solving

$$(d(n)+d(m))\Delta v^\varepsilon(x)=\alpha(m,n)V^\varepsilon(x)\left[|\log\varepsilon|^{-1}v^\varepsilon(x)+1\right],$$

we can readily use the results of [HR2] to obtain (6.1), (6.2) and (6.3).

As for (6.4), we apply (6.2) to assert

$$\int_{|a|\le l} |\nabla u^\varepsilon(a;m,n)|da \le c_1\alpha' \int_{|a|\le l} \min\left\{|a|^{-1},(r\varepsilon)^{-1}\right\} da \le c_2\alpha' l.$$

As for (6.5), we simply write,

$$\begin{aligned}\int_{|a|\le l} |\hat{u}^\varepsilon(a;m,n)|da &= \int_{|a|\le l} \left| \int_0^1 \nabla u^\varepsilon(a+tz;m,n)\cdot z dt \right| da \\ &\le |z| \int_{|a|\le l+|z|} |\nabla u^\varepsilon(a;m,n)|da,\end{aligned}$$

and apply (6.4).

As for (6.6), we use (6.3) and (6.4) to write

$$\begin{aligned}\int |\nabla \hat{u}^\varepsilon(a;m,n)|da &\le \int_{|a|\le 2|z|+C_2r\varepsilon} |\nabla \hat{u}^\varepsilon(a;m,n)|da \\ &\quad + C_1 \int_{2|z|+C_2r\varepsilon\le|a|\le l} \alpha'|a|^{-2}|z|da \\ &\le c_2\alpha' \left[(|z|+r\varepsilon)+|z||\log(|z|+r\varepsilon)|+|z||\log l|\right].\end{aligned}$$

For the proof of (6.7), let us write $A(l;m,n)$ for the left-hand side of (6.7). We use (6.1) to assert that if $l \le \varepsilon r$, then

$$A(l;m,n) \le c_2 l^2 \alpha'^2 |\log\varepsilon|^2 \le c_2\alpha'^2 r^2\varepsilon^2 |\log\varepsilon|^2,$$

and if $\varepsilon r < l$, then $A(l;m,n)$ is bounded above by

$$\begin{aligned}&c_3\alpha'^2 \left[r^2\varepsilon^2|\log\varepsilon|^2 + \int \mathbb{1}(|a|\in(\varepsilon r,l)) \left|\log\frac{|a|}{r}\right| da \right] \\ &\le c_4\alpha'^2 \left[r^2\varepsilon^2|\log\varepsilon|^2 + l^2\left|\log\frac{l}{r}\right|^2 + 1 \right],\end{aligned}$$

completing the proof of (6.7). In the same fashion, we can readily establish (6.8).

□

Proof of (5.5)**,** (5.6) ***and*** (5.7)**.** We omit the proof of (5.6) because its proof is very similar to the proof of (5.5).

Evidently

$$\begin{aligned}\mathbb{E}_\varepsilon^{eq}|G(\omega(0))| &\le c_1 K_\varepsilon^{1/2}\int_{|a|\le 1}\sum_{m,n}\lambda_m\lambda_n|\hat{u}^\varepsilon(a;m,n)|da\\ &\le c_2 K_\varepsilon^{1/2}|z|\sum_{m,n}\alpha'(m,n)\lambda_m\lambda_n \le c_3 K_\varepsilon^{1/2}|z|,\end{aligned}$$

where we used (6.5) and (3.10) for the the second and third inequalities respectively. This proves (5.5).

We now turn to the proof of (5.7). We certainly have

$$\begin{aligned}\mathbb{E}_\varepsilon^{eq}|H_{12}(\omega(0))| \le& c_1 K_\varepsilon^{1/2}\int_{|a|\le 1}\sum_{m,n}\lambda_m\lambda_n|\nabla\hat{u}^\varepsilon(a;m,n)|da\\ &\le c_2 K_\varepsilon^{1/2}\sum_{m,n}\alpha'(m,n)\left(r(m,n)\varepsilon + |z|\log(|z|+r(m,n)\varepsilon)\right)\lambda_m\lambda_n\\ &\le c_3 K_\varepsilon^{1/2}|z||\log|z||,\end{aligned}$$

by (6.6) of Lemmas 6.2. We now use (3.10) to deduce (5.7). □

Proof of (5.8)**.** Evidently the expression $\mathbb{E}_\varepsilon^{eq}|H_{22}(\omega(0))|$ is bounded by

$$\begin{aligned}&c_1 K_\varepsilon^{1/2}\int\int_{|a|\le 1}\sum_{m,n,p}\alpha(m,n)V^\varepsilon(b;m,n)\left(|\hat{u}^\varepsilon(a;m,p)|+|\hat{u}^\varepsilon(a;m+n,p)|\right)\lambda_m\lambda_n\lambda_p dadb\\ &\le c_2 K_\varepsilon^{1/2}|z|\sum_{m,n,p}\alpha(m,n)(\alpha'(n,p)+\alpha'(m+n,p))\lambda_m\lambda_n\lambda_p \le c_3 K_\varepsilon^{1/2}|z|,\end{aligned}$$

where we used (6.5) and (3.10) for the the second and third inequalities respectively. This proves (5.8). □

Proof of (5.9) ***and*** (5.10)**.** We start with the proof (5.9). We have $H_{311} = H_{3111} - H_{3112}$, where

$$H_{3112}(\omega) = \frac{1}{2}\int K_\varepsilon^{-3/2}\sum_{i,j}\sum_{m=1}^{m_i-1}\beta'(m_i)\hat{u}^\varepsilon(x_i-x_j;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j),$$

with

$$\beta'(m_i) = \sum_{m=1}^{m_i-1}\beta(m,m_i-m).$$

Repeating the proof of (5.5) yields that $\mathbb{E}_\varepsilon^{eq}|H_{3112}(\omega(0))| \leq c_1 K_\varepsilon^{1/2}|z|$. The term H_{3111} is treated in the same fashion:

$$\mathbb{E}_\varepsilon^{eq}|H_{3111}(\omega(0))| \leq c_2 K_\varepsilon^{1/2}|z| \sum_{n,p} \sum_{m=1}^{n-1} \beta(m, n-m)(a(p)+a(m))\lambda_p \lambda_n \leq c_3 K_\varepsilon^{1/2}|z|.$$

This completes the proof of (5.9).

We now turn to the proof of (5.10). The terms H_{321} and H_{322} are similar and both can be treated as (5.9). We only treat the latter. We certainly have that $\mathbb{E}_\varepsilon^{eq}|H_{321}(\omega(0))|$ is bounded by

$$\begin{aligned}
&c_1 \int K_\varepsilon^{1/2} \sum_{n,p} \sum_{m=1}^{n-1} \beta(m, n-m) \int V^\varepsilon(a-y; m, n-m)|\hat{u}^\varepsilon(y-b; m, p)| \\
&\qquad\qquad \mathbb{1}(|y-b| \leq 1, |y|, |b| \leq l) da db dy \\
&= c_1 \int K_\varepsilon^{1/2} \sum_{n,p} \sum_{m=1}^{n-1} \beta(m, n-m) \int |\hat{u}^\varepsilon(y-b; m, p)| \mathbb{1}(|y-b| \leq 1, |y| \leq l) db dy \\
&\leq c_2 K_\varepsilon^{1/2}|z| \sum_{n,p} \sum_{m=1}^{n-1} \beta(m, n-m)(a(p)+a(m))\lambda_p \lambda_n \leq c_3 K_\varepsilon^{1/2}|z|,
\end{aligned}$$

completing proof of (5.10). □

Proof of (5.11) ***and*** (5.12)**.** We certainly have that the term $\mathbb{E}_\varepsilon^{eq}|H_{21}^z(\omega)|$ is bounded

above by

$$
\begin{aligned}
&c_1\mathbb{E}_\varepsilon^{eq}K_\varepsilon^{-3/2}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)|u^\varepsilon(x_i-x_j+z;m_i,m_j)|\mathbb{1}(|x_i|\le l)\\
&\quad\le c_2K_\varepsilon^{-1/2}\sum_{m,n}\alpha(m,n)\alpha'(m,n)\left|\log\frac{c_2|z|}{r(m,n)}\right|\mathbb{1}(\varepsilon r(m,n)\le|z|)\lambda_m\lambda_n\\
&\quad+c_2K_\varepsilon^{1/2}\sum_{m,n}\alpha(m,n)\alpha'(m,n)\lambda_m\lambda_n\mathbb{1}(\varepsilon r(m,n)>|z|)\\
&\quad\le c_3K_\varepsilon^{-1/2}|\log|z||+c_3K_\varepsilon^{-1/2}\sum_{m,n}\alpha(m,n)\alpha'(m,n)\log r(m,n)\ \lambda_m\lambda_n\\
&\qquad+c_3K_\varepsilon^{1/2}\sum_n a(n)^2\lambda_n\mathbb{1}(\varepsilon r(n)>|z|)\\
&\quad\le c_3K_\varepsilon^{-1/2}|\log|z||+c_4K_\varepsilon^{-1/2}\sum_n a(n)^2\log n\ \lambda_n+c_4K_\varepsilon^{1/2}\sum_n a(n)^2\mathbb{1}(\varepsilon r(n)>|z|)\lambda_n\\
&\quad\le c_3K_\varepsilon^{-1/2}|\log|z||+c_5K_\varepsilon^{1/2}\left(\log\frac{|z|}{\varepsilon}\right)^{-1}\sum_n a(n)^2\log n\ \lambda_n\\
&\quad\le c_3K_\varepsilon^{-1/2}|\log|z||+c_5K_\varepsilon^{-1},
\end{aligned}
$$

where we used Lemma 6.1 for the first inequality. This completes the proof of (5.11). Similarly the term $\mathbb{E}_\varepsilon^{eq}|H_{33}^z(\omega)|$ is bounded above by

$$
\begin{aligned}
&c_1\mathbb{E}_\varepsilon^{eq}\int K_\varepsilon^{-3/2}\sum_i\sum_{m=1}^{m_i-1}\beta(m,m_i-m)V^\varepsilon(x_i-y;m,m_i-m)\\
&\qquad|u^\varepsilon(x_i-y+z;m,m_i-m)|\mathbb{1}(|x_i|\le l)dy\\
&\quad\le c_2K_\varepsilon^{-1/2}\sum_n\sum_{m=1}^{n-1}\beta(m,n-m)\alpha'(m,n-m)\\
&\qquad\left|\log\frac{c_2|z|}{r(m,n-m)}\right|\mathbb{1}(\varepsilon r(m,n-m)\le|z|)\lambda_n\\
&\qquad+c_2K_\varepsilon^{1/2}\sum_n\sum_{m=1}^{n-1}\beta(m,n-m)\alpha'(m,n-m)\mathbb{1}(\varepsilon r(m,n-m)>|z|)\lambda_n\\
&\quad\le c_3K_\varepsilon^{-1/2}|\log|z||+c_3K_\varepsilon^{-1}.
\end{aligned}
$$

This completes the proof of (5.12). □

7 Regularity of the Coagulation Term, Part II

As we explained in Section 4, one of the main step of the proof of Theorem 3.1 is the replacement of the expression $V^\varepsilon(\cdot)$ in the collision term H^0_{132} with a more manageable expression $W^\varepsilon(\cdot+z)$ for small z. Ultimately we average out $W^\varepsilon(\cdot+z)$ over z and apply a CLT. For this to succeed, we need to make sure that we can afford a small z which is as big as $|\log\varepsilon|^{-a}$ for some $a<1/2$. In Section 5, we used the auxiliary function G in order to relate H_{132} to H^z_{13} provided that $|z|$ is of order $\delta(\varepsilon)=|\log\varepsilon|^{-\theta}$ for $\theta>1/2$. In this section, we would like to fill the gap by showing that in fact z can be chosen so that $|z|$ is as large as $\delta'(\varepsilon)=|\log\log\varepsilon|^{-\theta'}$ provided that $\theta'\in(0,1/2)$. To achieve this, we fix a $\theta\in(0,1/2)$ and set $\bar H_{13}(\omega)$ to be equal

$$\int H^z_{13}(\omega)\zeta^{\delta(\varepsilon)}(z)dz = K_\varepsilon^{-3/2}\sum_{i,j}\alpha(m_i,m_j)\bar W^\varepsilon(x_i-x_j;m_i,m_j)\hat J(x_i,m_i,x_j,m_j),$$

where $\bar W^\varepsilon(a;m,n)=\int W(a+z;m,n)\zeta^{\delta(\varepsilon)}(z)dz$ and W^ε was defined by (4.13). First observe that there exists a constant c_1 such that the function $\bar W^\varepsilon$ has a support that is contained in a ball of center 0 and radius $\delta(\varepsilon;m,n)=c_1\delta(\varepsilon)+r(m,n)\varepsilon$. For our purposes, it is more convenient to assume that $r(m,n)\varepsilon\le\delta(\varepsilon)$ so that for a constant c_2, the support $\bar W^\varepsilon$ is contained in a ball of center 0 and radius $c_2\delta(\varepsilon)$, with $c_2=c_1+1$, and that $|\bar W^\varepsilon|\le c_2\delta(\varepsilon)^{-2}$. Such a restriction causes a small error. Indeed, if we set

$$\bar H'_{13}(\omega) := K_\varepsilon^{-3/2}\sum_{i,j}\alpha(m_i,m_j)\bar W^\varepsilon(x_i-x_j;m_i,m_j)\bar J(x_i,m_i,x_j,m_j), \tag{7.1}$$

with

$$\bar J(x_i,m_i,x_j,m_j)=\tilde J(x_i,m_i,x_j,m_j)\mathbb{1}(r(m,n)\varepsilon\le\delta(\varepsilon)),$$

then

$$\begin{aligned}\mathbb{E}^{eq}_\varepsilon|\bar H'_{13}(\omega)-\bar H_{13}(\omega)| &\le c_1K_\varepsilon^{1/2}\sum_{m,n}\alpha(m,n)\lambda_m\lambda_n\mathbb{1}(r(m,n)\varepsilon>\delta(\varepsilon))\\ &\le c_2K_\varepsilon^{1/2}\sum_n a(n)\lambda_n\mathbb{1}(2r(n)\varepsilon>\delta(\varepsilon)),\end{aligned} \tag{7.2}$$

which goes to 0 by to our assumption (3.9).

Define v^ε by

$$v^\varepsilon(x;m,n)=\frac{1}{2\pi}\int\log|x-y|\bar W^\varepsilon(y;m,n)dy. \tag{7.3}$$

We then set

$$G'(\omega;z) = G'(\omega) = K_\varepsilon^{-3/2} \sum_{i,j} \hat{q}^\varepsilon(x_i - x_j; m_i, m_j) \bar{J}(x_i, m_i, x_j, m_j), \tag{7.4}$$

where $\hat{q}^\varepsilon(a;m,n) = v^\varepsilon(a+z;m,n)K(a+z) - v^\varepsilon(a;m,n)K(a)$. We have

$$G'(\omega(t)) = G'(\omega(0)) + \int_0^t \mathcal{A}G'(\omega(s))ds + M'_t$$

where M'_t is a martingale. Note that G' is very similar to G of Section 5; $\hat{u}$ is replaced with $\hat{q}$ and $\bar{J}$ is replaced with $\tilde{J}$. The latter difference has to do with the fact that now the function K appears in the definition of $\hat{q}$ and we no longer need to multiply $\tilde{J}$ with a cut-off function. We write

$$\mathcal{A}G' = \mathcal{A}_0 G' + \mathcal{A}_c G' + \mathcal{A}_f G' =: H'_1 + H'_2 + H'_3. \tag{7.5}$$

We now study various terms which appeared on the right-hand side. We write

$$H'_1 = H'_{11} + H'_{12} + H'_{13},$$

We do not repeat the definition of various H'-expressions which are all correspond to H-expressions of Section 5. However, since v^ε satisfies (7.3), we have a different decomposition for H'_{13}. The decomposition

$$\begin{aligned}
\Delta\hat{q}^\varepsilon(a;m,n) &= \Delta v^\varepsilon(a+z;m,n)K(a+z) - \Delta v^\varepsilon(a;m,n)K(a) \\
&\quad + \nabla v^\varepsilon(a+z;m,n)\cdot\nabla K(a+z) - \nabla v^\varepsilon(a;m,n)\cdot\nabla K(a) \\
&\quad + v^\varepsilon(a+z;m,n)\Delta K(a+z) - v^\varepsilon(a;m,n)\Delta K(a) \\
&=: q_1^\varepsilon(a;m,n) + q_2^\varepsilon(a;m,n) + q_3^\varepsilon(a;m,n),
\end{aligned}$$

results in a decomposition

$$H'_{13} = H'_{131} + H'_{132} + H'_{133},$$

where

$$H'_{13r} = K_\varepsilon^{-3/2} \sum_{i,j} \hat{q}_r^\varepsilon(x_i - x_j; m_i, m_j) \tilde{J}(x_i, m_i, x_j, m_j).$$

We may rewrite (7.5) as

$$\begin{aligned}
\int_0^t H'_{131}(\omega(s))ds &= G(\omega(t)) - G(\omega(0)) - M'_t \\
&\quad - \int_0^t (H'_{11} + H'_{12} + H'_{132} + H'_{133} + H'_2 + H'_3)(\omega(s))ds.
\end{aligned} \tag{7.6}$$

We are now ready to state the main result of this section.

Theorem 7.1 *Assume that* $\delta(\varepsilon) < |z| < 1$. *Then*

$$\mathbb{E}_\varepsilon^{eq} \left| \int_0^t H'_{131}(\omega(s))ds \right| \leq C_0(t+1) \left[|z| + K_\varepsilon^{-1/2} \right] |\log \delta(\varepsilon)|^{1/2}.$$

Remark 7.1. With the aid of this theorem, we can readily improve the z-average from $|z| = O(\delta(\varepsilon))$ to $|z| = O(\delta'(\varepsilon))$. Indeed H'_{131} is given by

$$K_\varepsilon^{-3/2} \sum_{i,j} \bar{W}^\varepsilon(x_i - x_j + z; m_i, m_j) K(x_i - x_j + z) \bar{J}(x_i, m_i, x_j, m_j) - \bar{H}'_{13},$$

and by (7.2), the term $\bar{H}'_{13}$ can be replaced with $\bar{H}_{13}$, for an error that goes to 0 as $\varepsilon \to 0$. From this and Theorem 7.1 we deduce that $\bar{H}_{13} = \int H^z_{13}(\omega)\zeta^{\delta(\varepsilon)}(z)dz$ can be replaced with

$$K_\varepsilon^{-3/2} \sum_{i,j} \bar{W}^\varepsilon(x_i - x_j + z; m_i, m_j) K(x_i - x_j + z) \tilde{J}(x_i, m_i, x_j, m_j),$$

so long as $|z| = \delta'(\varepsilon)$.

We establish Theorem 7.1 by examining various terms that appeared on the right-hand side of (7.6). Indeed we show

$$\mathbb{E}_\varepsilon^{eq} |G'(\omega(t))| \leq C'_0 \left(K_\varepsilon^{-1/2} + |z| \right), \tag{7.7}$$

$$\mathbb{E}_\varepsilon^{eq} |(H'_{11} + H'_{133})(\omega(s))| \leq C'_0 \left(K_\varepsilon^{-1/2} + |z| \right), \tag{7.8}$$

$$\mathbb{E}_\varepsilon^{eq} |(H'_{12} + H'_{132})(\omega(s))| \leq C'_0 K_\varepsilon^{1/2} |z| |\log |z||, \tag{7.9}$$

$$\mathbb{E}_\varepsilon^{eq} |H'_{22}(\omega(s))| \leq C'_0 |z| |\log \delta(\varepsilon)|^{1/2}, \tag{7.10}$$

$$\mathbb{E}_\varepsilon^{eq} |(H'_{31} + H'_{32})(\omega(s))| \leq C'_0 \left(K_\varepsilon^{-1/2} + |z| \right), \tag{7.11}$$

$$\mathbb{E}_\varepsilon^{eq} |(H'_{21} + H'_{33})(\omega(s))| \leq C'_0 K_\varepsilon^{-1/2} |\log \delta(\varepsilon)|, \tag{7.12}$$

$$\mathbb{E}_\varepsilon^{eq} [M'_t]^2 \leq C'_0 t \left(K_\varepsilon^{-1} \delta(\varepsilon) + |z|^2 (\log |z|)^2 \right). \tag{7.13}$$

To prepare for the proof of Theorem 7.1, we start with an elementary lemma.

Lemma 7.1 *Assume that* $\int G(x,y,m,n)dxdy = 0$*, for every* $m,n \in \mathbb{N}$*. Then* $\int Y^2 d\nu_{\boldsymbol{\lambda}} = Z_1 + Z_2 + Z_3$*, where*

$$Z_1 = N(N-1)(N-2)\int \sum_{n_1,n_2,n_3} G(y_1,y_2,n_1,n_2)G(y_1,y_3,n_1,n_3)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}dy_1dy_2dy_3,$$

$$Z_2 = N(N-1)(N-2)\int \sum_{n_1,n_2,n_3} G(y_1,y_2,n_1,n_2)G(y_3,y_2,n_3,n_1)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}dy_1dy_2dy_3,$$

$$Z_3 = N(N-1)\int \sum_{n_1,n_2} G(y_1,y_2,n_1,n_2)^2 dy_1dy_2\lambda_{n_1}\lambda_{n_2}.$$

The straightforward proof of Lemma 7.1 is omitted. See also Lemma 3.3 of [R1] where a similar lemma is proved. As our next lemma we state some bounds on the function v^ε. The proof of this lemma is omitted because it is identical to the proof of Lemma 6.1.

Lemma 7.2 *There exist positive constants* C_1 *and* C_2 *such that for all* x*,*

$$|v^\varepsilon(x;m,n)| \le C_1\alpha' \min\left\{1 + |\log|x||\,, |\log\delta(\varepsilon)|\right\}, \tag{7.14}$$

$$|\nabla v^\varepsilon(x;m,n)| \le C_1\alpha' \min\left\{|x|^{-1}, \delta(\varepsilon)^{-1}\right\}, \tag{7.15}$$

and for $|x| \ge 2|z| + C_2\delta(\varepsilon)$*,*

$$|\nabla v^\varepsilon(x+z;m,n) - \nabla v^\varepsilon(x;m,n)| \le C_1\gamma(\varepsilon;m,n)\alpha'|x|^{-2}|z|. \tag{7.16}$$

Also,

$$\int |\nabla q^\varepsilon(a;m,n)|da \le C_1\alpha', \tag{7.17}$$

$$\int |\hat{q}^\varepsilon(a;m,n)|da \le C_1\alpha'|z|, \tag{7.18}$$

$$\int |\nabla \hat{q}^\varepsilon(a;m,n)|dx \le C_1\alpha'\left\{|z|[|\log(|z| + \delta(\varepsilon))| + 1] + \delta(\varepsilon)\right\}, \tag{7.19}$$

$$\int q^\varepsilon(a;m,n)^2 da \le C_1\alpha'^2, \tag{7.20}$$

$$\int |\nabla q^\varepsilon(a;m,n)|^2 da \le C_1\alpha'|\log\delta(\varepsilon)|, \tag{7.21}$$

Proof of (7.7) ***and*** (7.8)**.** We only prove (7.7) because (7.8) can be proved by a verbatim argument. To apply Lemma 7.1, we need to check that for every n_1 and n_2,

$$\int \hat{q}^\varepsilon(y_1 - y_2; n_1, n_2)\tilde{J}(y_1,n_1,y_2,n_2)dy_1dy_2 = 0. \tag{7.22}$$

We certainly have

$$\int \hat{q}^{\varepsilon}(y_1 - y_2; n_1, n_2) J(y_1, n_1) dy_1 dy_2 = \int J(y, n_1) dy \int \hat{q}^{\varepsilon}(a; n_1, n_2) da = 0.$$

The same is true if we replace $J(y_1, n_1)$ with $J(y_1, n_1 + n_2)$. This completes the proof of (7.22). In view of Lemma 7.1,

$$\mathbb{E}_{\varepsilon}^{eq} G'^2(\omega(0)) = \int G'^2(\omega) \nu_{\boldsymbol{\lambda}}(d\omega) = R_1 + R_2 + R_3,$$

with

$$\begin{aligned} R_1 = K_{\varepsilon}(K_{\varepsilon} - 1)(K_{\varepsilon} - 2) K_{\varepsilon}^{-3} \int \sum_{n_1, n_2, n_3} \hat{q}^{\varepsilon}(y_1 - y_2; n_1, n_2) \hat{q}^{\varepsilon}(y_1 - y_3; n_1, n_3) \\ \bar{J}(y_1, n_1, y_2, n_2) \bar{J}(y_1, n_1, y_3, n_3) \lambda_{n_1} \lambda_{n_2} \lambda_{n_3} dy_1 dy_2 dy_3 \\ R_3 = K_{\varepsilon}(K_{\varepsilon} - 1) K_{\varepsilon}^{-3} \int \sum_{n_1, n_2} \hat{q}^{\varepsilon}(y_1 - y_2; n_1, n_2)^2 \bar{J}^2(y_1, n_1, y_2, n_2) \lambda_{n_1} \lambda_{n_2} dy_1 dy_2 \end{aligned}$$

and R_2 is given by an expression similar to R_1.

We start with bounding R_3. We certainly have

$$R_3 \leq c_1 K_{\varepsilon}^{-1} \int \sum_{n_1, n_2} \lambda_{n_1} \lambda_{n_2} \left[q^{\varepsilon}(a; n_1, n_2)^2 + q^{\varepsilon}(a + z; n_1, n_2)^2 \right] da.$$

By Lemmas 7.2,

$$R_3 \leq c_2 K_{\varepsilon}^{-1} \sum_{n_1, n_2} \alpha'(n_1, n_2)^2 \lambda_{n_1} \lambda_{n_2} \leq c_3 K_{\varepsilon}^{-1} \sum_{n} a(n)^2 \lambda_n \leq c_4 K_{\varepsilon}^{-1}. \tag{7.23}$$

We now turn to R_1. First observe that $R_1 \leq R_1'$, where

$$R_1' = c_1 \sum_{n_1, n_2, n_3} \lambda_{n_1} \lambda_{n_2} \lambda_{n_3} \int |\hat{q}^{\varepsilon}(a; n_1, n_2)| da \int |\hat{q}^{\varepsilon}(a; n_1, n_3)| da.$$

By Lemma 7.2 we deduce

$$R_{11} \leq c_1 |z|^2 \sum_{n} a(n)^2 \lambda_n \leq c_2 |z|^2. \tag{7.24}$$

From this and (7.23) we deduce (7.7). □

Proof of (7.9)**.** Since H'_{12} is very similar to H'_{132}, we only establish (7.9) for H'_{12}. In view of Lemma 7.1,

$$\mathbb{E}_\varepsilon^{eq} H'^2_{12}(\omega(0)) = R_1 + R_2 + R_3,$$

with

$$R_1 = K_\varepsilon(K_\varepsilon - 1)(K_\varepsilon - 2)K_\varepsilon^{-3} \int \sum_{n_1,n_2,n_3} \nabla \hat{q}^\varepsilon(y_1 - y_2; n_1, n_2) \cdot \nabla \hat{q}^\varepsilon(y_1 - y_3; n_1, n_3)$$
$$\bar{J}(y_1, n_1, y_2, n_2)\bar{J}(y_1, n_1, y_3, n_3)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3} dy_1 dy_2 dy_3$$
$$R_3 = K_\varepsilon(K_\varepsilon - 1)K_\varepsilon^{-3} \int \sum_{n_1,n_2} |\nabla \hat{q}^\varepsilon(y_1 - y_2; n_1, n_2)|^2 \bar{J}^2(y_1, n_1, y_2, n_2)\lambda_{n_1}\lambda_{n_2} dy_1 dy_2$$

and R_2 is given by an expression similar to R_1.

We start with bounding R_3. We certainly have

$$R_3 \le c_1 K_\varepsilon^{-1} \int \sum_{n_1,n_2} \lambda_{n_1}\lambda_{n_2} \left[|\nabla q^\varepsilon(a; n_1, n_2)|^2 + |\nabla q^\varepsilon(a + z; n_1, n_2)|^2 \right] da.$$

By Lemmas 7.2,

$$R_3 \le c_2 |\log \delta(\varepsilon)| K_\varepsilon^{-1} \sum_{n_1,n_2} \alpha'(n_1, n_2)^2 \lambda_{n_1}\lambda_{n_2}$$
$$\le c_3 K_\varepsilon^{-1} |\log \delta(\varepsilon)| \sum_n a(n)^2 \lambda_n \le c_4 K_\varepsilon^{-1} |\log \delta(\varepsilon)|. \tag{7.25}$$

We now turn to R_1. First observe that $R_1 \le R'_1$, where

$$R'_1 = c_1 \sum_{n_1,n_2,n_3} \lambda_{n_1}\lambda_{n_2}\lambda_{n_3} \int |\hat{q}^\varepsilon(a; n_1, n_2)| da \int |\hat{q}^\varepsilon(a; n_1, n_3)| da.$$

By Lemma 7.2 we deduce

$$R_1 \le c_1 |z|^2 \sum_{n_1,n_2,n_3} \alpha'(n_1, n_2)\alpha'(n_1, n_3)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3} \le c_2 |z|^2. \tag{7.26}$$

From this and (7.25) we deduce (7.9). □

Proof of (7.11)**.** As in the proof of (5.8) and (5.9), we have that $H'_{31} = 2H'_{311}$, $H'_{311} = H'_{3111} - H'_{3112}$, where

$$H'_{3112}(\omega) = \frac{1}{2} K_\varepsilon^{-3/2} \sum_{i,j} \beta'(m_i) \hat{q}^\varepsilon(x_i - x_j; m_i, m_j) \tilde{J}(x_i, m_i, x_j, m_j).$$

Repeating the proof of (7.9) yields

$$\mathbb{E}^{\varepsilon}_{eq}\left[H'_{3112}(\omega(s))\right]^2 \leq c_3(|z|^2 + K_{\varepsilon}^{-1}).$$

The term H'_{3111} is handled in just the same way we handle H'_{32} below.

We now turn to H'_{32}. The terms H'_{321} and H'_{322} are similar and both can be treated as (7.11). We only treat the latter. We apply Lemma 7.1 for $G(x_i, x_j, m_i, m_j)$ given by

$$\frac{1}{2}\int \sum_{m=1}^{m_j-1} \beta(m, m_j - m) V^{\varepsilon}(x_j - y; m, m_j - m)\hat{q}^{\varepsilon}(x_i - y; m_i, m)\bar{J}(x_i, m_i, y, m)dy.$$

As a result,

$$\mathbb{E}^{eq}_{\varepsilon}\left[H_{322}(\omega(0))\right]^2 = R_1 + R_2 + R_3,$$

with R_1, R_2, and R_3 corresponding to Z_1, Z_2, and Z_3 in Lemma 7.1.

We first treat R_3. For this term we need to bound $|G(x_i, x_j, m_i, m_j)|$. In this case we simply move the absolute value inside the summation and replace $|\hat{q}^{\varepsilon}(a; m_i, m)|$ with a constant multiple of $|q^{\varepsilon}(a; m_i, m)| + |q^{\varepsilon}(a + z; m_i, m)|$. We then apply Lemma 6.1 to assert

$$|G(x_i, x_j, m_i, m_j)| \leq S(x_i - x_j, m_i, m_j) + S(x_i - x_j + z, m_i, m_j),$$

with $S(a, p, n)$ given by

$$c_2 \int \sum_{0}^{n-1} \beta(m, n-m) V^{\varepsilon}(y; m, n-m)\alpha'(p, m) \min\left\{|\log \delta(\varepsilon)|, |\log|a + y||\right\} \mathbb{1}(|a| \leq 2)dy.$$

We have that there exists constants c_3 and c_4 such that

$$S(a, p, n) \leq \begin{cases} c_3 \sum_0^{n-1} \beta(m, n-m)\alpha'(p, m)(|\log|a|| + 1), & \text{if } |a| \geq c_4(r(n)\varepsilon + \delta(\varepsilon)), \\ c_3 \sum_0^{n-1} \beta(m, n-m)\alpha'(p, m)|\log \delta(\varepsilon)|, & \text{otherwise.} \end{cases}$$

From this we can readily deduce that $R_3 \leq c_5 K_{\varepsilon}^{-1}$, as in the proof of (7.11). (Note that $\sum_0^{n-1} \beta(m, n-m)\alpha'(p, m) \leq (a(n) + a(p))\beta'(n)$ because by our choice, the function a is non-decreasing.)

We now turn to R_1 and R_2. We certainly have that $|G(x_i, x_j, m_i, m_j)|$ is bounded above by

$$|z| \int_0^1 \int \sum_{m=1}^{m_j-1} \beta(m, m_j - m) V^{\varepsilon}(x_j - y; m, m_j - m)|\nabla q^{\varepsilon}(x_i - y + tz; m_i, m)\bar{J}(x_i, m_i, y, m)|dydt.$$

We then apply Lemma 7.2 to assert

$$|G(x_i, x_j, m_i, m_j)| \leq |z| \int_0^1 L(x_i - x_j + tz, m_i, m_j)dt,$$

with $L(a, p, n)$ given by

$$c_5 \int \sum_0^{n-1} \beta(m, n-m) V^\varepsilon(y; m, n-m)\alpha'(p, m) \\ \min\left\{\delta(\varepsilon)^{-1}, |a+y|^{-1}\right\} \mathbb{1}(|a| \leq 2)dy.$$

Again we can readily show

$$L(a, n) \leq \begin{cases} c_3 \sum_0^{n-1} \alpha'(m, n)\beta(m, n-m)|a|^{-1}, & \text{if } |a| \geq c_4(r(n)\varepsilon + \delta(\varepsilon)), \\ c_3 \sum_0^{n-1} \alpha'(m, n)\beta(m, n-m)\delta(\varepsilon)^{-1}, & \text{otherwise.} \end{cases}$$

Repeating the proof of (7.7) yields that $R_1 + R_2 \leq c_7|z|^2$, completing the proof of (7.11). □

Proof of (7.12)**.** We only establish (7.12) for H'_{21} because H'_{33} can be treated by an identical argument. Choose c_1 so that $V(a) = 0$ if $|a| > c_1$. We certainly have that the expression $\mathbb{E}^{eq}_\varepsilon |H^z_{21}(\omega)|$ is bounded above by

$$\begin{aligned} &\mathbb{E}^{eq}_\varepsilon K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j)|\hat{q}^\varepsilon(x_i - x_j + z; m_i, m_j)| \\ &\quad \leq c_1 |\log \delta(\varepsilon)| K_\varepsilon^{-1/2} \sum_{m,n} \alpha(m, n)\alpha'(m, n)\lambda_m \lambda_n \\ &\quad \leq c_2 |\log \delta(\varepsilon)| K_\varepsilon^{-1/2}, \end{aligned}$$

where we used Lemma 6.1 for the first inequality. This completes the proof of (7.12). □

Proof of (7.10)**.** We note that H'_{22} is a sum of eight terms H'_{22i}, $i = 1, \ldots, 8$, and we establish (7.10) by showing the analogous bound for each H'_{22i}. Since all the eight terms can be treated in the same way, we only treat the sixth term which is given by

$$H'_{226}(\omega) = \frac{1}{2} K_\varepsilon^{-3/2} \sum_{i,j,k} \alpha(m_i, m_j) V_\varepsilon(x_i - x_j; m_i, m_j) q^\varepsilon(x_k - x_i; m_k, m_i) \bar{J}(x_k, m_k, x_i, m_i).$$

We note that $\bar{J}$ is a sum of 4 terms which yields a decomposition

$$H'_{226} = H'_{2261} + H'_{2262} - H'_{2263} - H'_{2264}. \tag{7.27}$$

Again all the 4 terms can be treated in the same way, so we only treat H'_{2264} which is given by

$$\frac{1}{2}K_\varepsilon^{-3/2}\sum_{i,j,k}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)\tilde{q}^\varepsilon(x_k-x_i;m_k,m_i)J(x_i,m_i),$$

where $\tilde{q}(a;m_k,m_i)=\hat{q}(a;m_k,m_i)\mathbb{1}(r(m_i,m_k)\varepsilon\le\delta(\varepsilon))$. We use the elementary inequality $|a|\le\delta+\delta^{-1}a^2$ to assert

$$|H'_{2264}|\le H'_{22641}+H'_{22642}, \tag{7.28}$$

where H'_{22641} and H'_{22642} are respectively given by

$$\frac{\delta}{2}K_\varepsilon^{-1}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)|J(x_i,m_i)|$$

$$\frac{\delta^{-1}}{2}K_\varepsilon^{-1}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)|J(x_i,m_i)|\left[K_\varepsilon^{-1/2}\sum_k\tilde{q}^\varepsilon(x_k-x_i;m_k,m_i)\right]^2.$$

Evidently, $\mathbb{E}^\varepsilon_{eq}H'_{22641}\le c_1\delta$, for some constant c_1. Moreover, by squaring the expression in the brackets, we learn that $H'_{22642}=H'_{226421}+H'_{226422}$, where

$$\begin{aligned}H'_{226421}=&\delta^{-1}K_\varepsilon^{-2}\sum_{i,j,k}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)|J(x_i,m_i)|\ \tilde{q}^\varepsilon(x_k-x_i;m_k,m_i)^2,\\ H'_{226422}=&\delta^{-1}K_\varepsilon^{-2}\sum_{i,j,\ k\neq l}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)|J(x_i,m_i)|\\ &\tilde{q}^\varepsilon(x_k-x_i;m_k,m_i)\tilde{q}^\varepsilon(x_l-x_i;m_l,m_i).\end{aligned}$$

Becasue of our choice of $\tilde{q}$, we have that $\int\tilde{q}(a;m,n)da=0$. As a consequence,

$$\mathbb{E}^\varepsilon_{eq}H'_{226422}=0. \tag{7.29}$$

We certainly have that $\mathbb{E}^\varepsilon_{eq}H'_{226421}$ is bounded above by

$$\begin{aligned}&c_2\delta^{-1}K_\varepsilon\sum_{n_1,n_2,n_3}\alpha(n_1,n_2)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\int V_\varepsilon(a;n_1,n_2)da\int\hat{q}^\varepsilon(b;n_3,n_1)^2db\\ &=c_2\delta^{-1}\sum_{n_1,n_2,n_3}\alpha(n_1,n_2)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\int\hat{q}^\varepsilon(b;n_3,n_1)^2db.\end{aligned}$$

On the other hand, by Lemma 7.2,

$$\int\hat{q}^\varepsilon(b;n_3,n_1)^2db\le|z|^2\int\nabla q^\varepsilon(b;n_3,n_1)|^2db\le c_3\alpha'(n_3,n_1)^2|\log\delta(\varepsilon)||z|^2.$$

As a result, $\mathbb{E}^{\varepsilon}_{eq} H'_{226421}$ is bounded above by

$$c_4\delta^{-1}|z|^2|\log\delta(\varepsilon)|\sum_{n_1,n_2,n_3}\alpha(n_1,n_2)\alpha'(n_3,n_1)^2\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\leq c_5\delta^{-1}|z|^2|\log\delta(\varepsilon)|.$$

In summary, from this (7.28), and (7.29) we deduce

$$\mathbb{E}^{\varepsilon}_{eq}|H'_{2264}|\leq c_1\delta+c_5\delta^{-1}|z|^2|\log\delta(\varepsilon)|.$$

By choosing $\delta=|z||\log\delta(\varepsilon)|^{1/2}$ we deduce (7.10). □

Proof of (7.13)**.** As it is well-known,

$$\mathbb{E}^{eq}_{\varepsilon}[M'_t]^2=\mathbb{E}^{eq}_{\varepsilon}\int_0^t(\mathcal{A}G'-2G'\mathcal{A}G')(\omega(s))ds=t(Z_1+Z_2+Z_3),$$

where

$$\begin{aligned}Z_1&=2\mathbb{E}^{eq}_{\varepsilon}(\mathcal{A}_0G'-2G'\mathcal{A}_0G')(\omega),\\ Z_2&=\mathbb{E}^{eq}_{\varepsilon}(\mathcal{A}_cG'-2G'\mathcal{A}_cG')(\omega),\\ Z_3&=\mathbb{E}^{eq}_{\varepsilon}(\mathcal{A}_fG'-2G'\mathcal{A}_fG')(\omega).\end{aligned}$$

We start with bounding Z_1:

$$Z_1=K_{\varepsilon}^{-3}\mathbb{E}^{eq}_{\varepsilon}\sum_i d(m_i)|\nabla_{x_i}G'(\omega)|^2\leq Z_{11}+Z_{12}+Z_{13}+Z_{14},$$

where

$$\begin{aligned}Z_{11}=&4K_{\varepsilon}^{-3}\mathbb{E}^{eq}_{\varepsilon}\sum_i d(m_i)\left|\sum_j\nabla\hat{q}^{\varepsilon}(x_i-x_j;m_i,m_j)\bar{J}(x_i,m_i,x_j,m_j)\right|^2\\ Z_{12}=&4K_{\varepsilon}^{-3}\mathbb{E}^{eq}_{\varepsilon}\sum_i d(m_i)\left|\sum_j\hat{q}^{\varepsilon}(x_i-x_j,m_i,m_j)\nabla_{x_i}\bar{J}(x_i,m_i,x_j,m_j)\right|^2.\end{aligned}$$

The term Z_{13} and Z_{14} are given by similar expression; x_i and x_j are swapped inside the absolute values. We only bound Z_{11} because Z_{11} involves $\nabla\hat{q}^{\varepsilon}$ which is more singular than $\hat{q}^{\varepsilon}$. The remaining Z_{1r} can be bounded in a similar way. Squaring yields

$$\begin{aligned}Z_{11}\leq&\,4K_{\varepsilon}^{-3}\mathbb{E}^{eq}_{\varepsilon}\sum_i d(m_i)\sum_{j\neq k}\nabla\hat{q}^{\varepsilon}(x_i-x_j;m_i,m_j)\cdot\nabla\hat{q}^{\varepsilon}(x_i-x_k;m_i,m_k)\\ &\bar{J}(x_i,m_i,x_j,m_j)\bar{J}(x_i,m_i,x_k,m_k)\\ &+4K_{\varepsilon}^{-3}\mathbb{E}^{eq}_{\varepsilon}\sum_i d(m_i)\sum_j|\nabla\hat{q}^{\varepsilon}(x_i-x_j;m_i,m_j)|^2\bar{J}(x_i,m_i,x_j,m_j)^2\\ =:&\,Z_{111}+Z_{112}.\end{aligned}$$

Use Lemma 7.2 to deduce

$$Z_{112} \leq c_1 K_\varepsilon^{-1} |\log \delta(\varepsilon)| \sum_{n_1,n_2} d(n_1)\alpha'(n_1,n_2)^2 \lambda_{n_1}\lambda_{n_2} \leq c_2 K_\varepsilon^{-1} |\log \delta(\varepsilon)|.$$

We now turn to Z_{111}. By Lemma 7.2,

$$\begin{aligned} Z_{111} &\leq c_1 \sum_{n_1,n_2,n_3} d(n_1) \left[||z|| \log |z|| + \delta(\varepsilon) \right]^2 \alpha'(n_1,n_2)\alpha'(n_1,n_3)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3} \\ &\leq c_2 \left[||z|| \log |z|| + \delta(\varepsilon) \right]^2 \leq 4c_2 \left[||z|| \log |z|| \right]^2. \end{aligned}$$

In summary

$$Z_1 \leq c_3 K_\varepsilon^{-1} |\log \delta(\varepsilon)| + c_3 [||z|| \log |z||]^2. \tag{7.30}$$

We now look at Z_2. We have

$$Z_2 = \frac{1}{2} K_\varepsilon^{-3} \mathbb{E}_\varepsilon^{eq} \sum_{i,j} \alpha(m_i,m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \left\{ \sum_k \left[\Gamma_{i,j}(0) + \sum_{p=1}^{8} \Gamma_{i,j,k}(p) \right] \right\}^2$$

where $\sum_{p=1}^{8} \Gamma_{i,j,k}(p)$ represents the eight terms that appeared in the definition of $\bar{H}'_{22}$ and $\Gamma_{i,j}(0) = -\hat{q}^\varepsilon(x_i - x_j; m_i, m_j)\bar{J}(x_i, m_i, x_j, m_j)$. An application of the inequality

$$\left(\sum_{p=0}^{8} a_p \right)^2 \leq 9 \sum_{p=0}^{8} a_p^2,$$

yields that Z_2 is bounded by

$$\frac{9}{2} K_\varepsilon^{-3} \mathbb{E}_\varepsilon^{eq} \sum_{i,j} \alpha(m_i,m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \left[\Gamma_{i,j}(0)^2 + \sum_{p=1}^{8} \left(\sum_k \Gamma_{i,j,k}(p) \right)^2 \right] =: \sum_{p=0}^{8} Z_{2p}$$

with for example,

$$\begin{aligned} Z_{28} \leq \frac{9}{2} \mathbb{E}_\varepsilon^{eq} K_\varepsilon^{-3} &\sum_{i,j} \alpha(m_i,m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \\ &\left[\sum_k \hat{q}^\varepsilon(x_k - x_j; m_k, m_j) \bar{J}(x_k, m_k, x_j, m_j) \right]^2. \end{aligned}$$

We only treat Z_{20} and Z_{28} as the other terms Z_{2r} for $r = 1, \dots, 7$ can be treated as Z_{28}. We have $Z_{28} = Z_{281} + Z_{282}$ where

$$Z_{281} = \frac{9}{2}\mathbb{E}_\varepsilon^{eq}\ K_\varepsilon^{-3} \sum_{i,j} V_\varepsilon(x_i - x_j; m_i, m_j)\alpha(m_i, m_j)$$
$$\sum_{k\neq l} \hat{q}^\varepsilon(x_k - x_j; m_k, m_j)\hat{q}^\varepsilon(x_l - x_j; m_l, m_j)\bar{J}(x_k, m_k, x_j, m_j)\bar{J}(x_l, m_l, x_j, m_j),$$

$$Z_{282} = \frac{9}{2}\mathbb{E}_\varepsilon^{eq}\ K_\varepsilon^{-3} \sum_{i,j} V_\varepsilon(x_i - x_j; m_i, m_j)\alpha(m_i, m_j)$$
$$\sum_{k} \hat{q}^\varepsilon(x_k - x_j; m_k, m_j)^2\bar{J}(x_k, m_k, x_j, m_j)^2.$$

We start with the former

$$Z_{281} \leq c_1 \sum_{n_1,n_2,n_3,n_4} \alpha(n_1, n_2)\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\lambda_{n_4} \int |\hat{q}^\varepsilon(a; n_3, n_2)|da \int |\hat{q}^\varepsilon(a; n_4, n_2)|da \leq c_2|z|^2.$$

As for Z_{282} we have

$$Z_{282} \leq c_1 K_\varepsilon^{-1} \sum_{n_1,n_2,n_3} \alpha(n_1, n_2) \int \hat{q}^\varepsilon(a; n_3, n_2)^2 da \lambda_{n_1}\lambda_{n_2}\lambda_{n_3} \leq c_2 K_\varepsilon^{-1}.$$

Finally

$$Z_{280} \leq c_1 K_\varepsilon^{-2} \sum_{n_1,n_2} \alpha(n_1, n_2)\lambda_{n_1}\lambda_{n_2} \int \hat{q}^\varepsilon(a; n_1, n_2)^2 da \leq c_2 K_\varepsilon^{-2}.$$

In summary,

$$Z_2 \leq c_1 \left(K_\varepsilon^{-1} + |z|^2\right). \tag{7.31}$$

We now turn to Z_3. We have

$$Z_3 = \frac{1}{2}\mathbb{E}_\varepsilon^{eq} K_\varepsilon^{-3} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m)$$
$$\left[\Gamma_i(y, m) + \sum_{p=1}^{4} \Gamma_i(p; y, m)\right]^2 dy$$

where for example

$$\Gamma_i(y, m) = \hat{q}^\varepsilon(x_i - y; m, m_i - m)\bar{J}(x_i, m, m_i - m, y),$$
$$\Gamma_i(3; y, m) = \sum_j \bar{q}^\varepsilon(y - x_j; m, m_j)\bar{J}(y, m, x_j, m_j).$$

Again $Z_3 \le 5\sum_0^4 Z_{3r}$ with for example

$$Z_{33} = \frac{5}{2}\mathbb{E}_\varepsilon^{eq} K_\varepsilon^{-3} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m)\Gamma_i(3; y, m)^2 dy.$$

We can now repeat the line of argument we had for Z_2 by squaring out Γ_i and use Lemma 7.2 to get

$$Z_3 \le c_1 \left(K_\varepsilon^{-1} + |z|^2\right). \tag{7.32}$$

From (7.30), (7.31) and (7.32) we deduce (7.13). □

8 Kinetic Limit

In this section we establish the main claim of Theorem 3.1. We now state the *martingale formulation* of the Ornstein–Uhlenbeck diffusion which uniquely determines the solution of the equation (3.13).

Definition 8.1. We say ξ is a solution of (3.13) if for any smooth function J of compact support with $\int J = 0$, the following processes are martingales:

$$M_J(t) = M(t) = \xi(t, J) - \xi(0, J) - \int_0^t \Gamma\xi(s, J)ds,$$
$$N_J(t) = N(t) = M(t)^2 - tA(J).$$

Here $J = (J_n : n \in \mathbb{N})$ with $J_n : \mathbb{R}^d \to \mathbb{R}$ and $\int J_n(x)dx = 0$, $\Gamma = \Gamma_0 + \Gamma_c + \Gamma_f$, and

$$\xi(t, J) = \sum_n \xi_n(t, J_n),$$
$$\Gamma_0\xi(t, J) = \sum_n d(n)\xi_n(t, \Delta_x J_n)$$
$$\Gamma_c\xi(t, J) = \sum_{m,n} \hat{\alpha}(m, n)\lambda_n\xi_n(t, J_{n+m} - J_n - J_m)$$
$$\Gamma_f(\xi, J) = \sum_{m,n} \hat{\beta}(m, n)\xi_n(t, J_n + J_m - J_{n+m}),$$
$$A(J) = 2\int \sum_n d(n)\lambda_n|\nabla_x J_n|^2 dx + \frac{1}{2}\int \sum_{m,n} \hat{\alpha}(m, n)\lambda_n\lambda_m(J_{n+m} - J_n - J_m)^2 dx$$
$$+ \frac{1}{2}\int \sum_{m,n} \hat{\beta}(m, n)\lambda_{n+m}(J_{n+m} - J_n - J_m)^2 dx.$$

We note that the last two terms in the definition of $A(J)$ are equal by the detailed balance assumption.

Ideally, we would like to show that the family $\mathcal{P}^\varepsilon$ is tight as $\varepsilon \to 0$ and that any limit point solves (3.13). Unfortunately we have not been able to establish the tightness and the difficulty comes from two error terms which go to 0 as $\varepsilon \to 0$ for each t. More precisely, let us define $\xi'(t,J) = \xi(t,J) + \xi''(t,J)$, where

$$\xi(t,J) = K_\varepsilon^{1/2} \sum_i J(x_i(t), m_i(t)),$$

and $\xi''(t,J) = \frac{1}{2}\bar{G}_\varepsilon(\omega(t))$, with

$$\bar{G}_\varepsilon(\omega) = \int G(\omega; z)\zeta^{\delta(\varepsilon)}(z)dz + \iint G'(\omega; z_2 - z_1)\zeta^{\delta'(\varepsilon)}(z_2)\zeta^{\delta'(\varepsilon)}(z_1)dz_1 dz_2. \tag{8.1}$$

Here G and G' are as in (5.1) and (7.3), and $\zeta^\delta(a) = \delta^{-2}\zeta(a/\delta)$ with ζ a smooth non-negative symmetric function of compact support satisfying $\int \zeta(a)da = 1$. We take a countable dense subset $\mathcal{D}_0$ of smooth functions of compact support and write $\mathcal{H} = L^1([0,T];\mathbb{R})^{\mathcal{D}_0}$. The transformation $\omega(\cdot) \mapsto (\xi'(\cdot, J) : J \in \mathcal{D}_0)$ induces a probability measure $\hat{\mathcal{P}}^\varepsilon$ on $\mathcal{H}$. Let us write $\mathcal{P}$ for the distribution of a process ξ which solves (3.13) and is subject to the following initial condition: $\xi(0,J)$ is a Gaussian random variable with variance

$$\int \xi(0,J)^2 \mathcal{P}(d\xi) = \sum_n \lambda_n \int J^2(x,n)dx.$$

Note that $\xi(\cdot, J)$ is stationary under $\mathcal{P}$. Note also that $\mathcal{P}$ can be regarded as a probability measure on $\mathcal{H}$. It turns out that the tightness of the sequence $\hat{\mathcal{P}}^\varepsilon$ can be shown by standard arguments.

Theorem 8.1 *The sequence $\hat{\mathcal{P}}^\varepsilon$ converges to $\mathcal{P}$ as $\varepsilon \to 0$. Moreover,*

$$\lim_\varepsilon \mathbb{E}_\varepsilon^{eq} \xi''(t,J) = 0, \tag{8.2}$$

for every t.

We note that (8.2) is an immediate consequence of (5.5) and (7.7). The proof of he convergence of $\hat{\mathcal{P}}^\varepsilon$ is naturally divided into two steps. The first step is devoted to the proof of the tightness of the family $\hat{\mathcal{P}}^\varepsilon$. This step will be carried out in Section 9. For the second step, we show that any limit point solves (3.13). This is a rather straight forward consequence of Theorem 8.2 below. This theorem is also the main ingredient for the proof of Theorem 3.1. We note that by a celebrated result of Holley and Stroock [HS], (3.13) has a unique solution in the sense of Definition 8.1.

Theorem 8.2 *There exist martingales M_ε and N_ε, and processes* $\mathrm{Err}^{1,\varepsilon}$ *and* $\mathrm{Err}^{2,\varepsilon}$ *such that*

$$\xi'(t,J) - \xi'(0,J) - \int_0^t \Gamma\xi'(s,J)ds = M_\varepsilon(t) + \mathrm{Err}^{1,\varepsilon}(t), \tag{8.3}$$

$$M_\varepsilon(t)^2 - tA(J) = N_\varepsilon(t) + \mathrm{Err}^{2,\varepsilon}, \tag{8.4}$$

where $A(J)$ was defined in Definition 8.1, and

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq} \left|\mathrm{Err}^{1,\varepsilon}(t)\right| = \lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq} \left|\mathrm{Err}^{2,\varepsilon}(t)\right| = 0.$$

The proof of Theorem 8.2 is naturally divided into two parts.

Proof of (8.3)**. Step 1:** Let us write $\bar{X}_\varepsilon(\omega) = X_\varepsilon(\omega) + \frac{1}{2}\bar{G}_\varepsilon(\omega)$ where $X_\varepsilon(\omega)$ and $\bar{G}^\varepsilon(\omega)$ were defined by (4.1) and (8.1) respectively. As it is a well-known fact for Markov processes, the following process is a martingale:

$$\bar{M}_\varepsilon(t) := \bar{X}_\varepsilon(\omega(t)) - \bar{X}_\varepsilon(\omega(0)) - \int_0^t \mathcal{A}\bar{X}_\varepsilon(\omega(s))ds.$$

Note that by definition, $\bar{X}_\varepsilon(\omega(t)) = \xi'(t,J)$.

Let us study the term $\mathcal{A}\bar{X}_\varepsilon$. We certainly have

$$\mathcal{A}\bar{X}_\varepsilon = \mathcal{A}_0 X_\varepsilon + \mathcal{A}_c X_\varepsilon + \mathcal{A}_f X_\varepsilon + \frac{1}{2}\mathcal{A}\bar{G}_\varepsilon.$$

Note that the term $\mathcal{A}X_\varepsilon$ involves $\tilde{J}$ whereas $\mathcal{A}\bar{G}_\varepsilon$ involves $\hat{J}$. We replace $\tilde{J}$ of $\mathcal{A}X_\varepsilon$ with $\hat{J}$. This causes an error Err_0 which is small because $\mathbb{E}_\varepsilon^{eq}|\,\mathrm{Err}_0\,|$ is bounded above by

$$c_1 K_\varepsilon^{1/2} \sum_{n,m} \alpha(m,n)\mathbb{1}(c_0\varepsilon r(m,n) \geq 1)\lambda_m\lambda_n \leq c_2 K_\varepsilon^{1/2} \sum_n a(n)\mathbb{1}(2c_0\varepsilon r(n) \geq 1)\lambda_n.$$

As a result, we may use (3.9) to deduce

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq}|\,\mathrm{Err}_0\,| = 0.$$

As a consequence of Theorems 5.1 and 7.1 (see Remark 7.1), we have

$$\int_0^t \left(\mathcal{A}_c X_\varepsilon + \frac{1}{2}\mathcal{A}\bar{G}_\varepsilon\right)(\omega(s))ds = \int_0^t Q_\varepsilon(\omega(s))ds - \int_0^t H_{33}^0(\omega(s))ds + \int_0^t \mathrm{Err}_1\ ds,$$

where $Q_\varepsilon(\omega)$ equals

$$\frac{1}{2}\int K_\varepsilon^{-3/2}\sum_{i,j}\alpha(m_i,m_j)\bar{W}^\varepsilon(x_i-x_j+z_2-z_1;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j)\zeta^\delta(z_1)\zeta^\delta(z_2)dz_1dz_2,$$

and

$$\mathbb{E}_\varepsilon^{eq}|\,\mathrm{Err}_1\,| \le c_1\left[K_\varepsilon^{1/2}\delta(\varepsilon)+K_\varepsilon^{-1/2}\right]|\log\delta(\varepsilon)| + c_1\left[\delta'(\varepsilon)+K_\varepsilon^{-1/2}\right]|\log\delta(\varepsilon)|^{1/2}, \tag{8.5}$$

which goes to 0 in small ε limit.

Step 2: Recall that the summation is over distinct i and j by our overall convention. However, one can readily check that if we allow $i=j$ in the summation, then the discrepancy is of order $O(K_\varepsilon^{-1/2})$. Also, if we replace $\hat{J}$ with $\tilde{J}$, the error is of order $O(\tau(\varepsilon))$. The sum of these two errors is denoted by Err_2, and we have

$$\mathbb{E}_\varepsilon^{eq}|\,\mathrm{Err}_2\,| \le c_1(K_\varepsilon^{-1/2}+\tau(\varepsilon)), \tag{8.6}$$

which goes to 0 in small ε limit. Because of the form of $\tilde{J}$, we may write

$$Q_\varepsilon(\omega) = Q_\varepsilon^1 + Q_\varepsilon^2 - Q_\varepsilon^3 - Q_\varepsilon^4 + Err_2,$$

where for example $Q_\varepsilon^4(\omega)$, given by

$$\frac{1}{2}\sum_{i,j}\int K_\varepsilon^{-3/2}\alpha(m_i,m_j)\bar{W}^\varepsilon(x_i-x_j+z_2-z_1;m_i,m_j)J(x_j,m_j)\zeta^{\delta'(\varepsilon)}(z_1)\zeta^{\delta'(\varepsilon)}(z_2)dz_1dz_2,$$

with the summation over all i and j. We make a change of variables $x_i - z_1 = a_1$, $x_j - z_2 = a_2$ to write that $Q_\varepsilon^4(\omega)$ equals to

$$\begin{aligned}&\frac{1}{2}\sum_{i,j}K_\varepsilon^{-3/2}\int\alpha(m_i,m_j)\bar{W}^\varepsilon(a_1-a_2;m_i,m_j)J(x_j,m_j)\zeta^{\delta'(\varepsilon)}(x_i-a_1)\zeta^{\delta'(\varepsilon)}(x_j-a_2)da_1da_2\\ &= K_\varepsilon^{1/2}\sum_{m,n}\int\alpha(m,n)\bar{W}^\varepsilon(a_1-a_2;m,n)f^\varepsilon(a_1,m;\omega)f^\varepsilon(a_2,n;\omega;J)da_1da_2\end{aligned}$$

where

$$\begin{aligned}f^\varepsilon(a,m;\omega) &= K_\varepsilon^{-1}\sum_i\zeta^{\delta'(\varepsilon)}(x_i-a)\mathbb{1}(m_i=m),\\ f^\varepsilon(a,m;\omega;J) &= K_\varepsilon^{-1}\sum_i\zeta^{\delta'(\varepsilon)}(x_i-a)J(x_i,m)\mathbb{1}(m_i=m).\end{aligned}$$

We then have that $Q^4_\varepsilon(\omega) = Q^{41}_\varepsilon(\omega) + Q^{42}_\varepsilon(\omega) + \mathrm{Err}_3$, where

$$Q^{41}_\varepsilon(\omega) = \frac{1}{2}\int K_\varepsilon^{1/2} \sum_{m,n} \alpha(m,n)\bar{W}^\varepsilon(a_1 - a_2, m, n)\lambda_m f^\varepsilon(a_2, n; \omega; J)da_1 da_2,$$

$$Q^{42}_\varepsilon(\omega) = \frac{1}{2}\int K_\varepsilon^{1/2} \sum_{m,n} \alpha(m,n)\bar{W}^\varepsilon(a_1 - a_2; m, n)\lambda_n f^\varepsilon(a_1, m; \omega)\bar{J}^\varepsilon(a_2, n)da_1 da_2,$$

where $\bar{J}^\varepsilon(a,n) = \int \zeta^{\delta'(\varepsilon)}(x-a)J(x,n)dx$ and Err_3 is given by

$$\begin{aligned}
&\frac{1}{2}\int K_\varepsilon^{1/2} \sum_{m,n} \alpha(m,n)\bar{W}^\varepsilon(a_1 - a_2; m, n) \\
&\qquad (f^\varepsilon(a_1, m; \omega) - \lambda_m)(f^\varepsilon(a_2, n; \omega; J) - \lambda_n \bar{J}^\varepsilon(a_2, m))da_1 da_2 \\
&= \frac{1}{2}\int K_\varepsilon^{-3/2} \sum_{i,j} \alpha(m_i, m_j)\bar{W}^\varepsilon(a_1 - a_2; m_i, m_j) \\
&\qquad (\zeta^{\delta'(\varepsilon)}(x_i - a_1) - \lambda_{m_i})(\zeta^{\delta'(\varepsilon)}(x_j - a_2)J(x_j, m_j) - \lambda_{m_j}\bar{J}^\varepsilon(a_2, m_j))da_1 da_2.
\end{aligned}$$

Here we have used the assumption $\int J = 0$. We wish to show that Err_3 is small. We first observe that we can write $Err_3 = Err_{31} + Err_{32}$, where Err_{31} is what we obtain by restricting the summation to indices $i \neq j$, and Err_{32} corresponds to the case $i = j$. It is not hard to show that Err_{32} is of order $O(K_\varepsilon^{-1/2})$. On the other hand,

$$\mathbb{E}^{eq}_\varepsilon \mathrm{Err}^2_{31} \leq c_1 K_\varepsilon^{-1}\delta'(\varepsilon)^{-2}. \tag{8.7}$$

To see this, observe that Err^2_{31} equals to

$$\begin{aligned}
&\frac{1}{2}\int da_1 da_2 db_1 db_2 \; K_\varepsilon^{-3} \sum_{i,j,p,q} \alpha(m_i, m_j)\alpha(m_p, m_q) \\
&\bar{W}^\varepsilon(a_1 - a_2; m_i, m_j)\bar{W}^\varepsilon(b_1 - b_2; m_p, m_q)(\zeta^{\delta'(\varepsilon)}(x_i - a_1) - \lambda_{m_i})(\zeta^{\delta'(\varepsilon)}(x_p - b_1) - \lambda_{m_p}) \\
&(\zeta^{\delta'(\varepsilon)}(x_j - a_2)J(x_j, m_j) - \lambda_{m_j}\bar{J}^\varepsilon(a_2, m_j))(\zeta^{\delta'(\varepsilon)}(x_q - b_2)J(x_q, m_q) - \lambda_{m_q}\bar{J}^\varepsilon(b_2, m_q)) \\
&=: E_1 + E_2 + E_3
\end{aligned}$$

where E_s represents the above summation with $(i,j,p,q) \in I(s)$ with $I(1)$ corresponding to the cases $i \neq p, q$ or $p \neq i, j$ or $j \neq p, q$ or $q \neq i, j$, $I(2)$ corresponds to the case $i = p$ and $q = j$, and $I(3)$ corresponding to the case $i = q$ and $p = j$. (Recall that the summation in our expression for Err^2_{31} is over $i \neq j$ and $q \neq p$.) We can readily check

$$\mathbb{E}^{eq}_\varepsilon E_1 = 0. \tag{8.8}$$

On the other hand $\mathbb{E}_\varepsilon^{eq} E_2$ equals

$$\begin{aligned}
&\frac{1}{2}K_\varepsilon^{-1}\sum_{m,n}\int \bar{W}^\varepsilon(a_1-a_2;m,n)\bar{W}^\varepsilon(b_1-b_2;m,n)\alpha(m,n)^2(\lambda_m\gamma^{\delta'(\varepsilon)}(a_1-b_1)-\lambda_m^2)\\
&\qquad(\lambda_n\zeta^{\delta'(\varepsilon)}(y-a_2)\zeta^{\delta'(\varepsilon)}(y-b_2)J^2(y,n)-\lambda_n^2\bar{J}^\varepsilon(a_2,n)\bar{J}^\varepsilon(b_2,n))dyda_1da_2db_1db_2\\
&=\frac{1}{2}(E_{21}+E_{22}+E_{23}+E_{24}),
\end{aligned}$$

where $\gamma^{\delta'(\varepsilon)}(a)=\delta'(\varepsilon)^{-2}\gamma(a/\delta'(\varepsilon))$, for $\gamma(a)=\int\zeta(a+b)\zeta(b)db$, and E_{2r} for $r=1,\dots,4$, are given by

$$\begin{aligned}
E_{21}&=K_\varepsilon^{-1}\sum_{m,n}\lambda_m\lambda_n\int\bar{W}^\varepsilon(a_1-a_2;m,n)\bar{W}^\varepsilon(b_1-b_2;m,n)\alpha(m,n)^2\gamma^{\delta'(\varepsilon)}(a_1-b_1)\\
&\qquad\zeta^{\delta'(\varepsilon)}(y-a_2)\zeta^{\delta'(\varepsilon)}(y-b_2)J^2(y,n)dyda_1da_2db_1db_2,\\
E_{22}&=K_\varepsilon^{-1}\sum_{m,n}\lambda_m^2\lambda_n\int\bar{W}^\varepsilon(a_1-a_2;m,n)\bar{W}^\varepsilon(b_1-b_2;m,n)\alpha(m,n)^2\\
&\qquad\zeta^{\delta'(\varepsilon)}(y-a_2)\zeta^{\delta'(\varepsilon)}(y-b_2)J^2(y,n)dyda_1da_2db_1db_2,\\
E_{23}&=K_\varepsilon^{-1}\sum_{m,n}\lambda_m\lambda_n^2\int\bar{W}^\varepsilon(a_1-a_2;m,n)\bar{W}^\varepsilon(b_1-b_2;m,n)\alpha(m,n)^2\gamma^{\delta'(\varepsilon)}(a_1-b_1)\\
&\qquad\bar{J}^\varepsilon(a_2,n)\bar{J}^\varepsilon(b_2,n)dyda_1da_2db_1db_2,\\
E_{24}&=K_\varepsilon^{-1}\sum_{m,n}\lambda_m^2\lambda_n^2\int\bar{W}^\varepsilon(a_1-a_2;m,n)\bar{W}^\varepsilon(b_1-b_2;m,n)\alpha(m,n)^2\\
&\qquad\bar{J}^\varepsilon(a_2,n)\bar{J}^\varepsilon(b_2,n)dyda_1da_2db_1db_2.
\end{aligned}$$

We can readily see

$$|E_{22}|+|E_{23}|+|E_{24}|\le c_1K_\varepsilon^{-1},$$

for a constant c_1. As for E_{21} we have,

$$\begin{aligned}
E_{21}&\le c_2K_\varepsilon^{-1}\delta'(\varepsilon)^{-2}\sum_{m,n}\lambda_m\lambda_n\int\bar{W}^\varepsilon(a_1-a_2;m,n)\bar{W}^\varepsilon(b_1-b_2;m,n)\alpha(m,n)^2\gamma^{\delta'(\varepsilon)}(a_1-b_1)\\
&\qquad\zeta^{\delta'(\varepsilon)}(y-a_2)J^2(y,n)dyda_1da_2db_1db_2\\
&\le c_3K_\varepsilon^{-1}\delta'(\varepsilon)^{-2}.
\end{aligned}$$

Hence $\mathbb{E}_\varepsilon^{eq}E_2\le c_4K_\varepsilon^{-1}\delta'(\varepsilon)^{-2}$. Similarly $\mathbb{E}_\varepsilon^{eq}E_3\le c_2K_\varepsilon^{-1/2}\delta'(\varepsilon)^{-2}$ for a constant c_5. This and (8.8) yield (8.7).

Step 3: Note that $\hat{\alpha} = \alpha \lim_{\varepsilon\to 0} \int W^\varepsilon$ (this was proved in [HR1] as Theorem 3.2), and $\int \bar{W}^\varepsilon = \int W^\varepsilon$. Hence

$$Q_\varepsilon^{41}(\omega) = \frac{1}{2}\int K_\varepsilon^{1/2} \sum_{m,n} \hat{\alpha}(m,n)\lambda_m f^\varepsilon(a_2, n; \omega, J) da_2 + Err_4 \tag{8.9}$$
$$= \frac{1}{2} K_\varepsilon^{-1/2} \sum_j \sum_{m,n} \hat{\alpha}(m,n)\lambda_m J(x_j, n) \mathbb{1}(m_j = n) + Err_{41},$$

where Err_4 is the error we get by replacing $\int \bar{W}^\varepsilon = \int W^\varepsilon$ with its limit as $\varepsilon \to 0$. Since

$$\mathbb{E}_\varepsilon^{eq}\left[K_\varepsilon^{-1/2} \sum_j \sum_{m,n} \hat{\alpha}(m,n)\lambda_m J(x_j, n)\mathbb{1}(m_j = n)\right] \le c_1,$$

for a constant c_1 independent of ε, we deduce

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq} |Err_{41}|^2 = 0. \tag{8.10}$$

Moreover, since $\bar{J}^\varepsilon(a,n) = \lambda_n J(a,n) + O(\delta'(\varepsilon))$, we have that $Q_\varepsilon^{42}(\omega)$ equals to

$$\frac{1}{2}\int K_\varepsilon^{1/2} \sum_{m,n} \alpha(m,n)\bar{W}^\varepsilon(a_1 - a_2; m, n) J(a_2, m) f^\delta(a_1, n; \omega) da_1 da_2 + O(\delta)$$
$$= \frac{1}{2}\int K_\varepsilon^{1/2} \sum_{m,n} \hat{\alpha}(m,n)\lambda_n J(a_1, n) f^\delta(a_1, m; \omega) da_1 + \mathrm{Err}_{42} \tag{8.11}$$
$$= \frac{1}{2}\int K_\varepsilon^{-1/2} \sum_i \sum_{m,n} \hat{\alpha}(m,n)\lambda_n J(x_i, n)\mathbb{1}(m_i = m) + \mathrm{Err}_{42}$$

with

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq} |Err_{42}|^2 = 0. \tag{8.12}$$

The terms Q_ε^j for $j = 1, 2, 3$ can be treated likewise. From (8.6), (8.7), (8.9), (8.10), (8.11), (8.12) and (8.2) we deduce

$$\int_0^t Q_\varepsilon^\delta(\omega(s)) ds = \int_0^t \Gamma_c \xi(s, J) ds + \int_0^t \mathrm{Err}_5 \; ds, \tag{8.13}$$

with

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq} |Err_5| = 0. \tag{8.14}$$

Step 4: We now study the term H^0_{33}. Recall

$$H^0_{33}(\omega) = \frac{1}{2}K_\varepsilon^{-3/2}\sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \\ \int V^\varepsilon(x_i - y; m, m_i - m)u^\varepsilon(x_i - y; m, m_i - m)\hat{J}(x_i, m, y, m_i - m)dy.$$

As we discussed in Step 1, we have replaced $\hat{J}$ with $\tilde{J}$ for an error which vanishes in small ε limit. Moreover

$$\tilde{J}(x_i, m, y, m_i - m) = \tilde{J}(x_i, m, x_i, m_i - m) + O(r(m, m_i - m)\varepsilon),$$

whenever $V^\varepsilon(x_i - y; m, m_i - m) \neq 0$. Hence $-H^0_{33}(\omega) + \mathcal{A}_f X_\varepsilon$ equals

$$\begin{aligned}
&\frac{-1}{2}K_\varepsilon^{-1/2}\sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \int W^\varepsilon(x_i - y; m, m_i - m)\tilde{J}(x_i, m, x_i, m_i - m)dy + Err_6 \\
&\quad = \frac{-1}{2}K_\varepsilon^{-1/2}\sum_i \sum_{m=1}^{m_i-1} \hat{\beta}(m, m_i - m)\tilde{J}(x_i, m, x_i, m_i - m) + \mathrm{Err}_7 \\
&\quad = \Gamma_f(\xi, J) + \mathrm{Err}_7,
\end{aligned}$$

with

$$\lim_{\varepsilon\to 0} \mathbb{E}^{eq}_\varepsilon |Err_7| = 0. \tag{8.15}$$

This is proved as in the previous step. For example, we have used

$$\mathbb{E}^{eq}_\varepsilon \Big[K_\varepsilon^{-1/2}\sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \int W^\varepsilon(x_i - y; m, m_i - m) \\ \nabla \hat{J}(x_i, m, x_i + \theta(y - x_i), m_i - m) \cdot (y - x_i)d\theta\, dy \Big]^2 = O(\varepsilon^2),$$

where ∇ denotes the derivative with respect to the second spatial variable.

Final Step: From (8.4-6) and (8.13-15) we deduce that the martingale $\bar{M}_\varepsilon(t)$ satisfies

$$\begin{aligned}
\bar{M}_\varepsilon(t) &= \xi'(t, J) - \xi'(0, J) - \int_0^t \Gamma\xi(s, J)ds + \int_0^t \mathrm{Err}_8(\omega(s))ds \\
&= \xi'(t, J) - \xi'(0, J) - \int_0^t \Gamma\xi'(s, J)ds + \int_0^t \mathrm{Err}_9(\omega(s))ds \\
&\lim_{\varepsilon\to 0} \mathbb{E}^{eq}_\varepsilon |\,\mathrm{Err}_8(\omega(0))| = \lim_{\varepsilon\to 0} \mathbb{E}^{eq}_\varepsilon |\,\mathrm{Err}_9(\omega(0))| = 0. \qquad (8.16)
\end{aligned}$$

□

Proof of (8.4)**. Step 1:** Define $\hat{G}_\varepsilon(\omega) = \int G(\omega; z)\zeta^{\delta(\varepsilon)}(z)dz$ and let us write $\hat{X}_\varepsilon(\omega) = X_\varepsilon(\omega) + \frac{1}{2}\hat{G}_\varepsilon(\omega)$. Set,

$$\hat{M}_\varepsilon(t) := \hat{X}_\varepsilon(\omega(t)) - \hat{X}_\varepsilon(\omega(0)) - \int_0^t \mathcal{A}\hat{X}_\varepsilon(\omega(s))ds.$$

Note that $\hat{M}_\varepsilon(t) = \bar{M}_\varepsilon(t) + M'_t$, where M'_t was defined in Section 7. By (7.13),

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq}\left[\bar{M}_\varepsilon(t) - \hat{M}_\varepsilon(t)\right]^2 = 0. \tag{8.17}$$

As it is a well-known fact for Markov processes, the following process is also a martingale:

$$\hat{N}_\varepsilon(t) = \hat{M}_\varepsilon(t)^2 - \int_0^t (\mathcal{A}(\hat{X}_\varepsilon)^2 - 2\hat{X}_\varepsilon\mathcal{A}\hat{X}_\varepsilon)(\omega(s))ds.$$

We certainly have

$$A := \mathcal{A}(\hat{X}_\varepsilon)^2 - 2\hat{X}_\varepsilon\mathcal{A}\hat{X}_\varepsilon = A_0 + A_c + A_f, \tag{8.18}$$

with

$$A_0 = 2K_\varepsilon^{-1}\sum_i d(m_i)\left|\nabla_x J(x_i, m_i) + B_i(\omega)\right|^2,$$

where

$$\begin{aligned} B_i(\omega) = &\frac{1}{2}K_\varepsilon^{-1}\nabla_{x_i}\sum_j \bar{u}^\varepsilon(x_i - x_j; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j), \\ &+ \frac{1}{2}K_\varepsilon^{-1}\nabla_{x_i}\sum_j \bar{u}^\varepsilon(x_j - x_i; m_j, m_i)\hat{J}(x_j, m_j, x_i, m_i), \end{aligned}$$

with $\bar{u}^\varepsilon = \tilde{u}^\varepsilon - u^\varepsilon$, where $\tilde{u}^\varepsilon(a; m, n) = \int u^\varepsilon(a + z; m, n)\zeta^{\delta(\varepsilon)}(z)dz$. The exact form of A_c and A_f will be given in Steps 2 and 4 respectively. We have $B_i = B_{i1} + B_{i2}$, where $B_{i1}(\omega)$ and $B_{i2}(\omega)$ are given by

$$\frac{1}{2}K_\varepsilon^{-1}\sum_j \left(\bar{u}_x^\varepsilon(x_i - x_j; m_i, m_j)\hat{J}(x_j, m_j, x_i, m_i) - \bar{u}_x^\varepsilon(x_j - x_i; m_j, m_i)\hat{J}(x_i, m_i, x_j, m_j)\right),$$

$$\frac{1}{2}K_\varepsilon^{-1}\sum_j \left(\bar{u}^\varepsilon(x_i - x_j; m_i, m_j)\hat{J}_x(x_i, m_i, x_j, m_j) - \bar{u}^\varepsilon(x_j - x_i; m_j, m_i)\hat{J}_y(x_j, m_j, x_i, m_i)\right),$$

respectively. We have

$$A_0 = 2A_{00} + 2A_{01} + 2A_{02},$$

with

$$A_{00} = K_\varepsilon^{-1} \sum_i d(m_i) |\nabla_x J(x_i, m_i)|^2,$$
$$A_{01} = 2K_\varepsilon^{-1} \sum_i d(m_i)(B_{i1}(\omega) + B_{i2}(\omega)) \cdot \nabla_x J(x_i, m_i)$$
$$A_{02} = K_\varepsilon^{-1} \sum_i d(m_i) \left(B_{i1}(\omega) + B_{i2}(\omega)\right)^2.$$

We first show that the term A_{01} is small. Note that $|A_{01}| \leq A_{011} + A_{012}$, for

$$A_{01r} = c_1 K_\varepsilon^{-1} \sum_i d(m_i) |B_{ir}(\omega)|,$$

with $c_1 = 2\|\nabla J\|_\infty$. By Lemma 6.1,

$$\mathbb{E}^\varepsilon_{eq} A_{011} \leq c_2 \sum_{m,n} \lambda_m \lambda_n \int |\nabla \bar{u}^\varepsilon(a; m, n)| da \leq c_3 \delta(\varepsilon) \log \delta(\varepsilon),$$
$$\mathbb{E}^\varepsilon_{eq} A_{012} \leq c_2 \sum_{m,n} \lambda_m \lambda_n \int_{|a|\leq 1} |\bar{u}^\varepsilon(a; m, n)| da \leq c_3 \delta(\varepsilon),$$

as in the proof of (5.5) and (5.7). As a result,

$$|A_{01}| \leq 2c_3 \delta(\varepsilon) \log \delta(\varepsilon). \tag{8.19}$$

We now turn to A_{02}. We may write $A_{02} = A_{021} + A_{022} + A_{023}$, with

$$A_{021} = K_\varepsilon^{-1} \sum_i d(m_i) B_{i1}^2,$$
$$A_{022} = K_\varepsilon^{-1} \sum_i d(m_i) B_{i2}^2,$$
$$A_{023} = 2K_\varepsilon^{-1} \sum_i d(m_i) B_{i1} B_{i2}.$$

We now use Lemma 6.2 to show that A_{022} and A_{023} are small. Indeed after squaring,

$$\mathbb{E}^\varepsilon_{eq} A_{022} \leq c_4 \mathbb{E}^\varepsilon_{eq} K_\varepsilon^{-3} \sum_{i,j,k} d(m_i) |\bar{u}^\varepsilon(x_i - x_j; m_i, m_j)| \; |\bar{u}^\varepsilon(x_i - x_k; m_i, m_k)|$$
$$= A_{0221} + A_{0222},$$

where A_{0221} and A_{0222} correspond to the cases $k \neq j$ and $k = j$ respectively. By Lemma 6.2 and (3.10),

$$\mathbb{E}^{\varepsilon}_{eq} A_{0221} \leq c_5 \sum_{m,n,p} \lambda_m \lambda_n \lambda_p \int_{|a|\leq 1} |\bar{u}^{\varepsilon}(a;m,n)|da \int_{|a|\leq 1} |\bar{u}^{\varepsilon}(a;m,p)|da \leq c_6 \delta(\varepsilon)^2,$$

$$\mathbb{E}^{\varepsilon}_{eq} A_{0222} \leq c_5 K_{\varepsilon}^{-1} \sum_{m,n} \lambda_m \lambda_n \int_{|a|\leq 1} |\bar{u}^{\varepsilon}(a;m,n)|^2 da \leq c_6 K_{\varepsilon}^{-1}.$$

In the same fashion,

$$\begin{aligned}\mathbb{E}^{\varepsilon}_{eq} A_{023} &\leq c_7 \mathbb{E}^{\varepsilon}_{eq} K_{\varepsilon}^{-3} \sum_{i,j,k} d(m_i) |\bar{u}^{\varepsilon}(x_i - x_j; m_i, m_j)| \; |\nabla \bar{u}^{\varepsilon}(x_i - x_k; m_i, m_k)| \\ &= A_{0231} + A_{0232},\end{aligned}$$

where A_{0231} and A_{0232} correspond to the cases $k \neq j$ and $k = j$ respectively. By Lemma 6.2 and (3.10),

$$\mathbb{E}^{\varepsilon}_{eq} A_{0231} \leq c_8 \sum_{m,n,p} \lambda_m \lambda_n \lambda_p \int_{|a|\leq 1} |\bar{u}^{\varepsilon}(a;m,n)|da \int_{|a|\leq 1} |\nabla \bar{u}^{\varepsilon}(a;m,p)|da \leq c_9 \delta(\varepsilon)^2 |\log \delta(\varepsilon)|,$$

$$\begin{aligned}\mathbb{E}^{\varepsilon}_{eq} A_{0232} &\leq c_{10} K_{\varepsilon}^{-1} \sum_{m,n} \lambda_m \lambda_n \int_{|a|\leq 1} |\bar{u}^{\varepsilon}(a;m,n)||\nabla \bar{u}^{\varepsilon}(a;m,n)|da \\ &\leq c_{10} K_{\varepsilon}^{-1} \sum_{m,n} \lambda_m \lambda_n \left(\int_{|a|\leq 1} |\bar{u}^{\varepsilon}(a;m,n)|^2 da \right)^{1/2} \left(\int_{|a|\leq 1} |\nabla \bar{u}^{\varepsilon}(a;m,n)|^2 da \right)^{1/2} \\ &\leq c_{11} K_{\varepsilon}^{-1} \delta(\varepsilon) K_{\varepsilon}^{1/2} K_{\varepsilon}^{1/2} = c_{11} \delta(\varepsilon).\end{aligned}$$

As a result,

$$|A_{022} + A_{023}| \leq c_{12} \left(\delta(\varepsilon) + K_{\varepsilon}^{-1} \right). \tag{8.20}$$

We now concentrate on A_{021}. First observe that since V and ζ are symmetric, we learn that $\bar{u}^{\varepsilon}$ is symmetric. From this and symmetry of $\tilde{J}$ and K we learn

$$B_{i1} = K_{\varepsilon}^{-1} \sum_{j} \bar{u}^{\varepsilon}_x(x_i - x_j; m_i, m_j) \hat{J}(x_i, m_i, x_j, m_j).$$

After squaring, we obtain $A_{021} = A_{0211} + A_{0212}$ where

$$\begin{aligned}A_{0211} = 2K_{\varepsilon}^{-3} \sum_{i,j,k} & d(m_i) \bar{u}^{\varepsilon}_x(x_i - x_j; m_i, m_j) \cdot \bar{u}^{\varepsilon}_x(x_i - x_k; m_i, m_k) \\ & \hat{J}(x_i, m_i, x_j, m_j) \hat{J}(x_i, m_i, x_k, m_k), \\ A_{0212} = K_{\varepsilon}^{-3} \sum_{i,j} & d(m_i) |\bar{u}^{\varepsilon}_x(x_i - x_j; m_i, m_j)|^2 \hat{J}(x_i, m_i, x_j, m_j)^2,\end{aligned}$$

where $j \neq k$ in A_{0211}. Using Lemma 6.1 we deduce

$$\mathbb{E}^{\varepsilon}_{eq}|A_{0211}| \leq c_1 \sum_{m,n,p} \lambda_m\lambda_n\lambda_p \int_{|a|\leq 1} |\nabla \bar{u}^{\varepsilon}(a;m,n)|da \int_{|a|\leq 1} |\nabla \bar{u}^{\varepsilon}(a;m,p)|da \leq c_2\delta(\varepsilon)^2(\log\delta(\varepsilon))^2.$$

As for A_{0212}, we first write, $A_{0212} = A_{02121} + A_{02122} + A_{02123}$, where

$$\begin{aligned}
A_{02121} &= K_{\varepsilon}^{-3} \sum_{i,j} d(m_i)|u^{\varepsilon}_x(x_i - x_j; m_i, m_j)|^2 \hat{J}(x_i, m_i, x_j, m_j)^2,\\
A_{02122} &= K_{\varepsilon}^{-3} \sum_{i,j} d(m_i)|\tilde{u}^{\varepsilon}_x(x_i - x_j; m_i, m_j)|^2 \hat{J}(x_i, m_i, x_j, m_j)^2,\\
A_{02123} &= -2K_{\varepsilon}^{-3} \sum_{i,j} d(m_i)\left(\tilde{u}^{\varepsilon}_x \cdot u^{\varepsilon}_x\right)(x_i - x_j; m_i, m_j)\hat{J}(x_i, m_i, x_j, m_j)^2, .
\end{aligned}$$

We now argue that A_{02122} and A_{02123} are small. To see this, first observe that by Lemma 6.2,

$$\begin{aligned}
|\nabla \tilde{u}^{\varepsilon}(a;m,n)| &= \left| \iint \nabla u^{\varepsilon}(a+z;m,n)\zeta^{\delta(\varepsilon)}(z)dz \right|\\
&\leq c_3\delta(\varepsilon)^{-2} \int_{|z|\leq c_3\delta(\varepsilon)} |\nabla u^{\varepsilon}(a+z;m,n)|dz\\
&\leq c_4\delta(\varepsilon)^{-2}\alpha'(m,n) \int_{|z|\leq c_3\delta(\varepsilon)} |a+z|^{-1}dz\\
&\leq c_5\alpha'(m,n)\min\left\{|a|^{-1}, \delta(\varepsilon)^{-1}\right\}.
\end{aligned}$$

As a result,

$$\int_{|a|\leq 1} |\nabla \tilde{u}^{\varepsilon}(a;m,n)|^2 da \leq c_6\alpha'(m,n)^2|\log\delta(\varepsilon)|,$$

$$\int_{|a|\leq 1} |\nabla \tilde{u}^{\varepsilon}(a;m,n)\cdot\nabla u^{\varepsilon}(a;m,n)|da \leq c_6\alpha'(m,n)^2|\log\delta(\varepsilon)|^{1/2}|\log\varepsilon|^{1/2}.$$

From this we learn

$$|A_{02122}| \leq c_7\alpha'(m,n)^2\frac{|\log\delta(\varepsilon)|}{|\log\varepsilon|}, \quad |A_{02123}| \leq c_7\alpha'(m,n)^2\frac{|\log\delta(\varepsilon)|^{1/2}}{|\log\varepsilon|^{1/2}}. \tag{8.21}$$

From (8.18)-(8.21) we deduce that $A_0 = 2A_{00} + 2A_{02121} + Err_1$, with

$$\mathbb{E}^{eq}_{\varepsilon}|\operatorname{Err}_1| \leq c\left(\delta(\varepsilon)|\log\delta(\varepsilon)| + |\log\varepsilon|^{-1/2}|\log\delta(\varepsilon)|^{1/2}\right). \tag{8.22}$$

Furthermore, if we pick a small $\delta > 0$ and write $A_{02121} = A_{021211} + A_{021212}$, with

$$A_{021211} = K_\varepsilon^{-3} \sum_{i,j} d(m_i) |u_x^\varepsilon(x_i - x_j; m_i, m_j)|^2 \hat{J}(x_i, m_i, x_j, m_j)^2 \mathbb{1}(|x_i - x_j| \le \delta),$$
$$A_{021212} = K_\varepsilon^{-3} \sum_{i,j} d(m_i) |u_x^\varepsilon(x_i - x_j; m_i, m_j)|^2 \hat{J}(x_i, m_i, x_j, m_j)^2 \mathbb{1}(|x_i - x_j| > \delta),$$

then we have that $\mathbb{E}_\varepsilon^{eq} |A_{021212}| \le K_\varepsilon^{-1} \delta^{-2}$, and $A_{021211} = A_{0212111} + A_{0212112}$, where in the first term we replace x_j argument of $\hat{J}$ with x_i, and the second term is the error caused by such a replacement. More precisely,

$$A_{0212111} = K_\varepsilon^{-3} \sum_{i,j} d(m_i) |u_x^\varepsilon(x_i - x_j; m_i, m_j)|^2 \tilde{J}(x_i, m_i, x_i, m_j)^2 \mathbb{1}(|x_i - x_j| \le \delta),$$

$$\mathbb{E}_\varepsilon^{eq} |A_{0212112}| \le K_\varepsilon^{-1} \delta c_1 \sum_{m,n} \lambda_m \lambda_n \int_{|a| \le \delta} |\nabla u^\varepsilon(a; m, n)|^2 da \le c_2 \delta,$$

where we used the smoothness of J for the first inequality and (6.8) for the second inequality. Now that we have replaced x_j with x_i in $\hat{J}$, we can drop the condition $|x_i - x_j| \le \delta$. Indeed $A_{0212111} = A_{02121111} + A_{02121112}$, with

$$A_{02121111} = K_\varepsilon^{-3} \sum_{i,j} d(m_i) |\nabla u^\varepsilon(x_i - x_j; m_i, m_j)|^2 \tilde{J}(x_i, m_i, x_i, m_j)^2 \mathbb{1}(|x_i - x_j| \le 1),$$

$\mathbb{E}_\varepsilon^{eq} |A_{02121112}| \le c_3 K^{-1} \delta^{-2}$.

We now choose $\delta = |\log \varepsilon|^{-1/3}$. In summary, $A_{02121} = A_{02121111} + Err_2$, where

$$\mathbb{E}_\varepsilon^{eq} |Err_2| \le c_4 |\log \varepsilon|^{-1/3}. \tag{8.23}$$

On the other hand, by Lemma 5.1,

$$\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon^{eq} \left| \int_0^t 2A_{00}(\omega(s)) ds - tA_0(J) \right| = 0, \tag{8.24}$$
$$\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon^{eq} \left| \int_0^t 2A_{02121111}(\omega(s)) ds - tA_0'(J) \right| = 0,$$

where

$$A_0(J) = 2 \sum_n d(n) \lambda_n \int |\nabla J_n|^2 \, dx \tag{8.25}$$
$$A_0'(J) = \sum_{m,n} \lambda_m \lambda_n d(m) \gamma(m,n) \int |J_{m+n} - J_n - J_m|^2 \, dx,$$

with

$$\gamma(m,n) = \lim_{\varepsilon\to 0} |\log\varepsilon|^{-1} \int_{|a|\le 1} |\nabla u^\varepsilon(a;m,n)|^2 da.$$

Step 2: We now study A_c. Recall that $\bar{u}^\varepsilon = \tilde{u}^\varepsilon - u^\varepsilon$. We have

$$A_c(\omega) = \frac{1}{2} K_\varepsilon^{-1} \sum_{i,j} \alpha(m_i,m_j) V_\varepsilon(x_i - x_j; m_i, m_j) \left\{ S(i,j,\omega) + \sum_{r=0}^{8} R_r(i,j,\omega) \right\}^2$$

where

$$\begin{aligned}
S(i,j,\omega) &= \tilde{J}(x_i,m_i,x_j,m_j) + \hat{J}(x_i,m_i,x_j,m_j) K_\varepsilon^{-1} u^\varepsilon(x_i - x_j; m_i, m_j) \\
R_0(i,j,\omega) &= -K_\varepsilon^{-1} \hat{J}(x_i,m_i,x_j,m_j) \tilde{u}^\varepsilon(x_i - x_j; m_i, m_j) \\
R_1(i,j,\omega) &= K_\varepsilon^{-1} \sum_k \frac{m_i}{m_i+m_j} \bar{u}^\varepsilon(x_i - x_k; m_i+m_j, m_k) \hat{J}(x_i, m_i+m_j, x_k, m_k), \\
R_2(i,j,\omega) &= K_\varepsilon^{-1} \sum_k \frac{m_i}{m_i+m_j} \bar{u}^\varepsilon(x_k - x_i; m_k, m_i+m_j) \hat{J}(x_k, m_k, x_i, m_i+m_j), \\
R_3(i,j,\omega) &= K_\varepsilon^{-1} \sum_k \frac{m_j}{m_i+m_j} \bar{u}^\varepsilon(x_j - x_k; m_i+m_j, m_k) \hat{J}(x_j, m_i+m_j, x_k, m_k), \\
R_4(i,j,\omega) &= K_\varepsilon^{-1} \sum_k \frac{m_j}{m_i+m_j} \bar{u}^\varepsilon(x_k - x_j; m_k, m_i+m_j) \hat{J}(x_k, m_k, x_j, m_i+m_j), \\
R_5(i,j,\omega) &= -K_\varepsilon^{-1} \sum_k \bar{u}^\varepsilon(x_i - x_k; m_i, m_k) \hat{J}(x_i, m_i, x_k, m_k), \\
R_6(i,j,\omega) &= -K_\varepsilon^{-1} \sum_k \bar{u}^\varepsilon(x_k - x_i; m_k, m_i) \hat{J}(x_k, m_k, x_i, m_i), \\
R_7(i,j,\omega) &= -K_\varepsilon^{-1} \sum_k \bar{u}^\varepsilon(x_j - x_k; m_j, m_k) \hat{J}(x_j, m_j, x_k, m_k), \\
R_8(i,j,\omega) &= -K_\varepsilon^{-1} \sum_k \bar{u}^\varepsilon(x_k - x_j; m_k, m_j) \hat{J}(x_k, m_k, x_j, m_j),
\end{aligned}$$

where the summation is over k with $k \neq i,j$. Let us write $T(i,j,\omega) = \sum_{r=0}^{8} R_r(i,j,\omega)$. We then write

$$A_c(\omega) = A_{c0}(\omega) + A_{c1}(\omega) + A_{c2}(\omega) \tag{8.26}$$

with

$$A_{c0}(\omega) = \frac{1}{2}K_\varepsilon^{-1}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)S(i,j,\omega)^2,$$
$$A_{c1}(\omega) = K_\varepsilon^{-1}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)S(i,j,\omega)T(i,j,\omega),$$
$$A_{c2}(\omega) = \frac{1}{2}K_\varepsilon^{-1}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)T(i,j,\omega)^2.$$

Our goal is showing that both A_{c1} and A_{c2} are small. We start with the latter. We use the inequality $\left(\sum_0^8 R_r\right)^2 \le 9\sum_0^8 R_r^2$, to assert that $A_{c2} \le 9\sum_0^8 A_{c2r}$. We only bound A_{c20} and A_{c25} as the remaining A_{c2r} are similar to A_{c25}.

To bound A_{c25}, we simply square out the summation in k to obtain

$$A_{c25} = \frac{1}{2}K_\varepsilon^{-3}\sum_{i,j,k,l}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)\bar{u}^\varepsilon(x_i-x_k;m_i,m_k)\bar{u}^\varepsilon(x_i-x_l;m_i,m_l)$$
$$\hat{J}(x_i,m_i,x_k,m_k)\hat{J}(x_i,m_i,x_l,m_l) =: A_{c251}+A_{c252},$$

where A_{c251} and A_{c252} represent the cases $k\neq l$ and $k=l$ respectively. We then have

$$\mathbb{E}_\varepsilon^{eq}A_{c251} \le c_1\sum_{m,n,p,q}\alpha(m,n)\lambda_m\lambda_n\lambda_p\lambda_q\int_{|a|\le 1}|\bar{u}^\varepsilon(a;m,p)|da\int_{|a|\le 1}|\bar{u}^\varepsilon(a;m,q)|da$$
$$\le c_2\delta(\varepsilon)^2$$

by (6.5). In the same fashion, we may use (6.7) to show that $\mathbb{E}_\varepsilon^{eq}A_{c252} \le c_2K_\varepsilon^{-1}$. Treating other A_{c2r}, $r=1,\dots,8$ in the same way yields

$$\mathbb{E}_\varepsilon^{eq}\sum_{r=1}^8 A_{c2r} \le c_3\left(\delta(\varepsilon)^2+K_\varepsilon^{-1}\right). \tag{8.27}$$

As for A_{c20} we have that $\mathbb{E}_\varepsilon^{eq}A_{c20}$ is bounded by

$$\frac{1}{2}\mathbb{E}_\varepsilon^{eq}K_\varepsilon^{-3}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)\tilde{u}^\varepsilon(x_i-x_j;m_i,m_j)^2\hat{J}(x_i,m_i,x_j,m_j)^2$$
$$\le c_4K_\varepsilon^{-1}\sum_{m,n}\alpha(m,n)\lambda_m\lambda_n\int V_\varepsilon(a;m,n)\tilde{u}^\varepsilon(a;m,n)^2da.$$

If we restrict the summation to those m and n such that $r(m,n)\varepsilon \ge \delta(\varepsilon)$, then we simply use $\tilde{u}^\varepsilon \le c_1K^\varepsilon$ to show that the sum is bounded by a constant multiple of $\tau(\varepsilon)$,

which is small by our assumption (3.9). On the other hand, when $V_\varepsilon(a;m,n)\neq 0$ and $r(m,n)\varepsilon\leq\delta(\varepsilon)$,

$$
\begin{aligned}
|\tilde{u}^\varepsilon(a;m,n)| &= \left|\iint u^\varepsilon(a+z;m,n)\zeta^{\delta(\varepsilon)}(z)dz\right| \\
&\leq c_5\delta(\varepsilon)^{-2}\int_{|z|\leq c_5\delta(\varepsilon)}|u^\varepsilon(a+z;m,n)|dz \\
&\leq c_6\delta(\varepsilon)^{-2}\alpha'(m,n)\int_{|z|\leq c_5\delta(\varepsilon)+c_6\varepsilon r(m,n)}|\log|z||dz \\
&\leq c_7\alpha'(m,n)|\log\delta(\varepsilon)|.
\end{aligned}
\tag{8.28}
$$

Hence

$$
\mathbb{E}^{eq}_\varepsilon A_{c20}\leq c_8|\log\varepsilon|^{-2}|\log\delta(\varepsilon)|^2+c_8\tau(\varepsilon).
\tag{8.29}
$$

From this and (8.27) we deduce

$$
\mathbb{E}^{eq}_\varepsilon A_{c2}\leq c_9\left(\delta(\varepsilon)^2+|\log\varepsilon|^{-1}+|\log\varepsilon|^{-2}|\log\delta(\varepsilon)|^2+\tau(\varepsilon)\right).
\tag{8.30}
$$

Step 3: We now turn to A_{c1}. By Lemma 6.2, the expression $K_\varepsilon^{-1}u^\varepsilon(a;m,n)$ is uniformly bounded whenever $V_\varepsilon(a;m,n)\neq 0$. Hence

$$
|A_{c1}|\leq A'_{c1}=c_1K_\varepsilon^{-2}\sum_{i,j,k}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)|T(i,j,\omega)|.
$$

Again using the decomposition of T, we write $A'_{c1}=\sum_{r=0}^8 A_{c1r}$, with for example

$$
A_{c15}(\omega)=c_1K_\varepsilon^{-2}\sum_{i,j,k}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)\,|\bar{u}^\varepsilon(x_i-x_k;m_i,m_k)|\left|\hat{J}(x_i,m_i,x_k,m_k)\right|.
$$

By (6.5),

$$
\mathbb{E}^{eq}_\varepsilon A_{c15}\leq c_2\sum_{m,n,p}\alpha(m,n)\lambda_m\lambda_n\lambda_p\int_{|a|\leq 1}|\bar{u}^\varepsilon(a;m,p)|da\leq c_2\delta(\varepsilon).
\tag{8.31}
$$

Similarly

$$
A_{c10}(\omega)=c_1K_\varepsilon^{-2}\sum_{i,j}\alpha(m_i,m_j)V_\varepsilon(x_i-x_j;m_i,m_j)\,|\tilde{u}^\varepsilon(x_i-x_j;m_i,m_j)|\left|\hat{J}(x_i,x_j,m_i,m_j)\right|,
$$

which yields

$$
\mathbb{E}^{eq}_\varepsilon A_{c10}(\omega)\leq c_3\sum_{m,n}\alpha(m,n)\lambda_m\lambda_n\int_{|a|\leq 1}V_\varepsilon(a;m,n)|\tilde{u}^\varepsilon(a;m,n)|da.
$$

Again using (8.28) we obtain

$$\mathbb{E}_\varepsilon^{eq} A_{c10}(\omega) \le c_4 |\log\varepsilon|^{-1} |\log\delta(\varepsilon)| + c_4\tau(\varepsilon),$$

in the same way we obtained (8.28). From this and (8.30) we deduce

$$\mathbb{E}_\varepsilon^{eq} A_{c1} \le c_5 \left(\delta(\varepsilon) + |\log\varepsilon|^{-1} |\log\delta(\varepsilon)| + \tau(\varepsilon)\right).$$

From this, (8.26) and (8.30) we deduce

$$\int_0^t A_c(\omega(s))ds = \int_0^t A_{c0}(\omega(s))ds + \mathrm{Err}_2,$$

with

$$\mathbb{E}_\varepsilon^{eq} |\,\mathrm{Err}_2\,| \le c\left(\delta(\varepsilon) + |\log\varepsilon|^{-1} + |\log\varepsilon|^{-1} |\log\delta + \tau(\varepsilon)|\right). \tag{8.32}$$

On the other hand, by Law of Large Numbers,

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^{eq} \left| \int_0^t A_{c0}(\omega(s))ds - tA_c(J) \right| = 0 \tag{8.33}$$

where

$$A_c(J) = \frac{1}{2} \int \sum_{m,n} \tilde{\alpha}(m,n)\lambda_n\lambda_m (J_{n+m} - J_n - J_m)^2 dx$$

and

$$\tilde{\alpha}(m,n) = \alpha(m,n)\eta(m,n)^2 = \hat{\alpha}(m,n)\eta(m,n),$$

where $\eta = \hat{\alpha}/\alpha$. Here we have used $\lim_\varepsilon K_\varepsilon^{-1} u^\varepsilon = \eta - 1$ uniformly in the support of V_ε. (See Theorem 3.2 of [HR].)

Step 4: We now concentrate on A_f. We have

$$A_f(\omega) = \frac{1}{2} K_\varepsilon^{-1} \sum_i \sum_{m=1}^{m_i - 1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m) \left[S(i; y, m; \omega) + \sum_{r=0}^{4} R_r(i; y, m; \omega) \right]^2 dy$$

where

$$
\begin{aligned}
S(i;y,m;\omega) &= J(x_i,m) + J(y,m_i-m) - J(x_i,m_i) \\
&\qquad - K_\varepsilon^{-1} u^\varepsilon(x_i-y;m,m_i-m)\hat{J}(x_i,m,y,m_i-m), \\
R_0(i;y,m;\omega) &= K_\varepsilon^{-1}\tilde{u}^\varepsilon(x_i-y;m,m_i-m)\hat{J}(x_i,m,y,m_i-m), \\
R_1(i;y,m;\omega) &= K_\varepsilon^{-1}\sum_j [\bar{u}^\varepsilon(x_i-x_j;m,m_j)\hat{J}(x_i,m,x_j,m_j) \\
&\qquad - \bar{u}^\varepsilon(x_i-x_j;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j)], \\
R_2(i;y,m;\omega) &= K_\varepsilon^{-1}\sum_j [\bar{u}^\varepsilon(x_i-x_j;m_i,m)\hat{J}(x_i,m_i,x_j,m) \\
&\qquad - \bar{u}^\varepsilon(x_i-x_j;m_i,m_j)\hat{J}(x_i,m_i,x_j,m_j)], \\
R_3(i;y,m;\omega) &= K_\varepsilon^{-1}\sum_j \bar{u}^\varepsilon(y-x_j;m,m_j)\hat{J}(y,m,x_j,m_j), \\
R_4(i;y,m;\omega) &= K_\varepsilon^{-1}\sum_j \bar{u}^\varepsilon(x_j-y;m_j,m)\hat{J}(x_j,m_j,y,m).
\end{aligned}
$$

Let us write $T = \sum_0^4 R_r$, and

$$
A_f = A_{f0} + A_{f1} + A_{f2}, \tag{8.34}
$$

with

$$
\begin{aligned}
A_{f0} &= \frac{1}{2}K_\varepsilon^{-1}\sum_i \sum_{m=1}^{m_i-1} \beta(m,m_i-m)\int V^\varepsilon(x_i-y;m_i-m,m)S(i;y,m;\omega)^2 dy, \\
A_{f1} &= K_\varepsilon^{-1}\sum_i \sum_{m=1}^{m_i-1} \beta(m,m_i-m)\int V^\varepsilon(x_i-y;m_i-m,m)(ST)(i;y,m;\omega) dy, \\
A_{f2} &= \frac{1}{2}K_\varepsilon^{-1}\sum_i \sum_{m=1}^{m_i-1} \beta(m,m_i-m)\int V^\varepsilon(x_i-y;m_i-m,m)T(i;y,m;\omega)^2 dy.
\end{aligned}
$$

We have that $A_{f2} \le \frac{5}{2}\sum_0^4 A_{f2r}$ with

$$
A_{f2r}(\omega) = K_\varepsilon^{-1}\sum_i \sum_{m=1}^{m_i-1} \beta(m,m_i-m)\int V^\varepsilon(x_i-y;m_i-m,m)R_r(i;y,m;\omega)^2 dy.
$$

We then use (8.27) to learn that $A_{f20}(\omega)$ is bounded above by

$$c_1 K_\varepsilon^{-3} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m) \tilde{u}^\varepsilon(x_i - y; m, m_i - m)^2 dy$$
$$\leq c_2 K_\varepsilon^{-3} |\log \delta(\varepsilon)|^2 \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \alpha'(m, m_i - m)^2.$$

We then use (3.10) to deduce

$$\mathbb{E}_\varepsilon^{eq} A_{f20}(\omega) \leq c_3 |\log \varepsilon|^{-2} |\log \delta(\varepsilon)|^2. \tag{8.35}$$

On the other hand,

$$A_{f21}(\omega) = K_\varepsilon^{-1} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) R_1(i; y, m; \omega)^2,$$

is bounded above by $A_{f211} + A_{f212}$, with

$$A_{f211} = 2K_\varepsilon^{-3} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \left[\sum_j |\bar{u}^\varepsilon(x_i - x_j; m, m_j) \hat{J}(x_i, m, x_j, m_j)| \right]^2$$
$$A_{f212} = 2K_\varepsilon^{-3} \sum_i \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \left[\sum_j |\bar{u}^\varepsilon(x_i - x_j; m_i, m_j) \hat{J}(x_i, m_i, x_j, m_j)| \right]^2.$$

By squaring out the expression inside brackets, we can readily see

$$A_{f211} + A_{f212} \leq c_1 \left(\delta(\varepsilon)^2 + K_\varepsilon^{-1} \right).$$

by Lemma 6.1. The term A_{f22} is treated likewise. Hence,

$$\mathbb{E}_\varepsilon^{eq} \left(A_{f21}(\omega) + A_{f22}(\omega) \right) \leq c_2 \left(\delta(\varepsilon)^2 + K_\varepsilon^{-1} \right). \tag{8.36}$$

We now study A_{f23}:

$$A_{f23}(\omega) = K_\varepsilon^{-3} \sum_{i,j,k} \sum_{m=1}^{m_i-1} \beta(m, m_i - m) \int V^\varepsilon(x_i - y; m_i - m, m)$$
$$\bar{u}^\varepsilon(y - x_j; m, m_j) \hat{J}(y, m, x_j, m_j) \bar{u}^\varepsilon(y - x_k; m, m_k) \hat{J}(y, m, x_k, m_k) dy$$
$$= A_{f231} + A_{f232},$$

with A_{f231} and A_{f232} corresponding to the cases $k \neq j$ and $k = j$. With the aid of (6.5) and (6.8) , we can readily deduce

$$\mathbb{E}_\varepsilon^{eq} A_{f231} \leq c_3 \delta(\varepsilon)^2, \qquad \mathbb{E}_\varepsilon^{eq} A_{f232} \leq c_3 |\log \varepsilon|^{-1}.$$

The term A_{f24} is treated likewise. In summary

$$E_{eq}^\varepsilon A_{f2}(\omega) \leq c_4 \left(\delta(\varepsilon)^2 + |\log \varepsilon|^{-1}\right).$$

We can readily bound A_{f1} as in the previous step:

$$E_{eq}^\varepsilon A_{f1}(\omega) \leq c_5 \left(\delta(\varepsilon) + |\log \varepsilon|^{-1} |\log d(\varepsilon) + \tau(\varepsilon)|\right).$$

From all this we conclude

$$\int_0^t A_f(\omega(s))ds = \int_0^t A_{f0}(\omega(s))ds + \mathrm{Err}_3$$

with Err_3 satisfying (8.32). On the other hand

$$S(i; y, m; \omega) = -J(x_i, m, x_i, m_i - m)(1 + K_\varepsilon^{-1} u^\varepsilon(x_i - y; m, m_i - m)) + O(\varepsilon r(m, m_i - m))$$

whenever $V^\varepsilon(y - x_i; m, m_i - m) \neq 0$. From this and a law of large numbers, we can readily deduce

$$\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon^{eq} \left| \int_0^t A_{f0}(\omega(s))ds - tA_f(J) \right| = 0, \tag{8.37}$$

where

$$A_f(J) = \frac{1}{2} \int \sum_{m,n} \tilde{\beta}(m, n) \lambda_{n+m} (J_{n+m} - J_n - J_m)^2 dx$$

where

$$\tilde{\beta}(m, n) = \beta(m, n)\eta(m, n)^2 = \hat{\beta}(m, n)\eta(m, n).$$

Final Step: From (8.22), (8.23), (8.24), (8.32), (8.35) and (8.37) we learn that the process

$$\bar{M}_\varepsilon(t)^2 - tA'(J),$$

is a sum of a martingale and a small error, where

$$\begin{aligned} A'(J) = {} & 2 \int \sum_n \lambda_n d(n) |\nabla_x J_n|^2 dx + \int \sum_{n,m} \lambda_n \lambda_m d(n) \gamma(n, m) (J_{n+m} - J_n - J_m)^2 dx \\ & + \frac{1}{2} \int \sum_{m,n} \tilde{\alpha}(m, n) \lambda_n \lambda_m (J_{n+m} - J_n - J_m)^2 dx \\ & + \frac{1}{2} \int \sum_{m,n} \tilde{\beta}(m, n) \lambda_{n+m} (J_{n+m} - J_n - J_m)^2 dx. \end{aligned}$$

It remains to show that $A(J) = A'(J)$. Since

$$\lambda_m\lambda_n\tilde{\alpha}(m,n) = \tilde{\beta}(m,n)\lambda_{n+m}, \quad \lambda_m\lambda_n\hat{\alpha}(m,n) = \hat{\beta}(m,n)\lambda_{n+m},$$

it suffices to show

$$\begin{aligned}(8.38)\quad (d(m)+d(n))\gamma(m,n) &= \lim_{\varepsilon\to 0}(d(m)+d(n))K_\varepsilon^{-1}\int_{|a|\le 1}|\nabla u^\varepsilon(a;m,n)|^2da\\ &= \hat{\alpha}(m,n)-\tilde{\alpha}(m,n) = \alpha(m,n)(\eta(m,n)-\eta(m,n)^2).\end{aligned}$$

We have

$$\begin{aligned}&(d(m)+d(n))K_\varepsilon^{-1}\int_{|a|\le 1}|\nabla u^\varepsilon(a;m,n)|^2da\\ &\quad= -(d(m)+d(n))K_\varepsilon^{-1}\int_{|a|\le 1}u^\varepsilon(a;m,n)\Delta u^\varepsilon(a;m,n)da + O(K_\varepsilon^{-1})\\ &\quad= -\alpha(m,n)\int_{|a|\le 1}V^\varepsilon(a;m,n)K_\varepsilon^{-1}u^\varepsilon(a;m,n)\left(1+K_\varepsilon^{-1}u^\varepsilon(a;m,n)\right)da + O(K_\varepsilon^{-1}),\end{aligned}$$

where we integrated by parts and used (4.11). This and $\lim_\varepsilon K_\varepsilon^{-1}u^\varepsilon = \eta - 1$ (Theorem 3.2 of [HR1]) imply (8.38). □

9 Proofs of Theorems 8.1 and 3.1

In this section, we complete the proof of Theorems 8.1 and 3.1. We first show that the process $\xi'(t,J)$ is tight. More precisely,

Theorem 9.1 *For every smooth function J of compact support and positive T, there exists a constant $c(J,T)$ such that*

$$(9.1)\qquad \limsup_{\varepsilon\to 0}\mathbb{E}_\varepsilon^{eq}\int_0^T\sup_{0\le h\le\delta}|\xi'(t+h,J)-\xi'(t,J)|dt \le c(J,T)\delta^{1/2}.$$

Proof. Recall that by (8.16),

$$(9.2)\qquad \xi'(t,J)-\xi'(s,J) = \int_s^t\Gamma\xi(\theta,J)d\theta + \bar{M}_\varepsilon(t)-\bar{M}_\varepsilon(s) - \int_s^t \mathrm{Err}_8(\omega(\theta))\,d\theta,$$

where,

$$(9.3)\qquad \limsup_{\varepsilon\to 0}\mathbb{E}_\varepsilon^{eq}\sup_{0\le t\le T}\left|\int_0^t\mathrm{Err}_8(\omega(\theta))d\theta\right| \le T\limsup_{\varepsilon\to 0}\mathbb{E}_\varepsilon^{eq}|\mathrm{Err}_8(\omega(0))| = 0.$$

On the other hand, since

$$\sup_{\varepsilon} \mathbb{E}_{\varepsilon}^{eq} \Gamma\xi(\theta, J)^2 = \sup_{\varepsilon} \mathbb{E}_{\varepsilon}^{eq} \Gamma\xi(0, J)^2 < \infty,$$

we can readily deduce

$$\mathbb{E}_{\varepsilon}^{eq} \sup_{0\le s\le t\le s+\delta\le T} \left| \int_s^t \Gamma\xi(\theta, J) d\theta \right| \le (T+\delta)^{1/2} \left[\mathbb{E}_{\varepsilon}^{eq} \int_s^t \Gamma\xi(\theta, J)^2 d\theta \right]^{1/2} \le c_1 \delta^{1/2}. \tag{9.4}$$

See Section 5 and (5.7) of [R] for more details.

It remains to establish the tightness of the martingale $\bar{M}_\varepsilon$. For this, it suffices to show

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le T} \sup_{0<h<\delta} \mathbb{E}_{\varepsilon}^{eq} [\bar{M}_\varepsilon(t+h) - \bar{M}_\varepsilon(t)]^2 \le c_0(T)\delta. \tag{9.5}$$

This is an immediate consequence of Doob's inequality and (8.5). From (9.2), (9.3) and (9.5) we deduce (9.1). □

We are now ready to complete the proof of Theorems 3.1 and 8.1. Take a countable set $\{t_n : n \in \mathbb{N}\}$ that contains 0 and is dense in some fixed interval $[0, T]$. Write $\tilde{\mathcal{P}}^\varepsilon$ for the law of

$$((\xi'(\cdot, J) : J \in \mathcal{D}_0), (\xi'(t_n, J) : J \in \mathcal{D}_0, n \in \mathbb{N})) \in \mathcal{H} \times \mathbb{R}^{\mathcal{D}_0 \times \mathbb{N}},$$

with respect to the probability measure $\mathbb{P}_{\varepsilon}^{eq}$. Using Theorem 9.1, we can readily show that the sequence $\tilde{\mathcal{P}}^\varepsilon$ is tight. Let $\tilde{\mathcal{P}}$ by a limit point of the sequence $\tilde{\mathcal{P}}^\varepsilon$. Using Theorem 8.2, it is not hard to show that for every $J \in \mathcal{D}_0$, the sequences

$$\begin{aligned} &(M_J(t_n) := \xi'(t_n, J) - \xi'(0, J) - \int_0^{t_n} \Gamma\xi'(s, J) ds : n \in \mathbb{N}), \\ &(N_J(t_n) := M_J(t_n)^2 - t_n A(J) : n \in \mathbb{N}), \end{aligned}$$

are martingales with respect to the probability measure $\tilde{\mathcal{P}}$. We now extend M_J and N_J to the whole interval $[0, T]$. More precisely, for $t \notin \{t_n : n \in \mathbb{N}\}$, define

$$M_J(t) := \lim_{t_n \to t-} M_J(t_n), \qquad N_J(t) := \lim_{t_n \to t-} N_J(t_n),$$

which exist almost surely by Martingale Upcrossing Theorem. Here by $\lim_{t_n\to t-}$, we mean the limit with respect a subsequence of $\{t_n\}$ which increases to t from the left. As a result, $\bar{\xi}(t, J) := \lim_{t_n\to t-} \xi(t_n, J)$ also exists almost surely with respect to $\tilde{\mathcal{P}}$. The process $\bar{\xi}(t, J)$ is a solution to (3.13) in the sense of Definition 8.1. This completes the proof of Theorem 3.1 because the set $\{t_n : n \in \mathbb{N}\}$ can be chosen to include any given finite collection of points.

To complete the proof of Theorem 8.1, it suffices to show that $\xi'(t,J) = \bar{\xi}(t,J)$ almost everywhere. For this, let us assume that the set $\{t_n : n \in \mathbb{N}\}$ includes the points in the set $\{i/L : i \in \mathbb{N}\} \cap [0,T]$ for every poitive integer L. Write $T_L(t)$ for $[tL]/L$, where $[tL]$ denotes the integer part of tL. From (9.1) we can readily deduce

$$\int \int_0^T |\xi'(t,J) - \xi'(T_L(t),J)| dt d\tilde{\mathcal{P}} \leq c(J,T)L^{-1/2}.$$

Since $\lim_{L\to\infty} \xi'(T_L(t)) = \bar{\xi}(t,J)$ by definition, we deduce that $\xi'(t,J) = \bar{\xi}(t,J)$ for almost all t and almost surely. □

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