Topics in Statistical Mechanics, Combinatorics, and Representation Theory

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1 Basics of statistical mechanics

Throughout this course we will consider only spin models in the equilibrium statistical mechanics.

1. Gibbs measures, Boltzmann weights for models on graphs with local interaction. Examples.
2. The thermodynamical limit and Gibbs measures for local models on infinite graphs. Correlation functions, slopes, decay of correlations.
3. Phase transitions, scaling, universality.
4. Large deviations

2 Dimers and perfect matchings on surface graphs

2.1 Perfect matchings on surface graphs and related combinatorial problems

1. Perfect matchings and tilings.
2. Perfect matchings on bipartite graphs and height functions.
3. The double of a surface graph. Spanning trees on a surface graph are equivalent to perfect matchings on its double.

2.2 Pfaffian solution and discrete spin structures

1. Edge weighted dimers (perfect matchings).
2. Kasteleyn orientations of a surface graph and spin structures.
3. Local correlations for perfect matchings on a surface graphs are sums of Pfaffians.

2.3 Discrete Dirac operator

1. Kenyon's construction of the Dirac operator for planar graphs and the equivalence of it to the Kasteleyn matrix.

2. Surface graphs.

2.4 Perfect matchings on the hexagonal lattice and Gelfand-Zetlin patterns

1. Gelfand-Zetlin tableaux and weight multiplicities for gl_n.

2. Tilings of a triangular lattice by rhombi is equivalent to perfect matchings on the hexagonal lattice, to plane partitions, and to Gelfand-Zetlin patterns.

3. Vertex operators for gl_∞. The partition function for perfect matchings in terms of vertex operators. Generating functions for the number of plane partitions, MacMahon formulae.

2.5 Interacting dimers and the 6-vertex model

1. Perfect matchings (dimers) on a planar graph interacting around faces.

2. The 6-vertex model. Different ways to write 6-vertex model and different ways to present it as interacting dimers. The 5-vertex model and the crystal limit.

2.6 "Local integrability"

Local transformations of weights which do not change the partition functions. Yang-Baxter relation. The 6-vertex case. The corresponding quantum algebra.

3 The thermodynamical limit for dimer models

3.1 Phase diagrams for dimer models

Dimer models on periodically weighted planar graphs. The partition functions and correlations in the large volume limit.

3.2 The limit shape phenomenon

Limit shapes for periodically weighted dimer models and amoebas.

3.3 Dimer models and large random Hermitian matrices

Basic facts about large random Hermitian matrices. The limit shape phenomenon is related to the semicircle distribution.

3.4 Fluctuations

Gaussian fluctuations, Airy process, Pearcy process, the universality conjecture. Similar processes in the theory random matrices.

4 The thermodynamical limit in the 6-vertex model

4.1 The phase diagram

Frozen, critical and anti-ferroelectric phases. The phase diagram in the (H, V) plane where H and V are horizontal and vertical magnetic field respectively.

4.2 New features comparing to dimer models

Corner singularities. The free energy near singularities.

4.3 Correlation functions for the anti-ferroelectric phase and vertex operators for $U_q(\widehat{sl}_2)$.

The definition of $U_q(\widehat{sl}_2)$ and the relation between the 6-vertex model and evaluation representations of this algebra. The "spin" model for irreducible level 1 integrable representations. Correlation functions as matrix elements of vertex operators.

5 Criticality and conformal field theory

5.1 Critical phenomena and conformal invariance

The idea of the universality of critical phenomena. Conformal invariance at criticality. Conformal invariance in two dimensions.

5.2 Conformal invariance of local operators

The Virasoro algebra and its centarl extension. Its action of local operators.

5.3 Primary operators

Primary operators and irreducible representations of the Virasoro algebra. Operator product expansion and vertex operator algebras.

5.4 Rational conformal field theories

5.5 Minimal conformal field theories

Gaussian conformal field theories.

5.6 Off-critical behavior and perturbations of conformal field theories