Lecture 9 The double construction.

Note Title 1/29/2009

Few more examples of Lie bialgebras:

Ex 2. $b \subset sl_n$, $H_i = e_{ii} - e_{i+1\,i+1}$,

$X_i = e_{i\,i+1}$, $X_\alpha = e_{ij} = X_{ij}$

$(\alpha = \varepsilon_i - \varepsilon_j)$

$\delta H_i = 0$, $\delta X_i = \frac{1}{2} H_i \wedge X_i$

b^* - dual vector space, dual basis: $H_i^\vee$, $X_{ij}^\vee$

h.w. problem: compute Lie brackets between $H_i^\vee$, $X_{ij}^\vee$.

Ex 3. $b \subset \mathfrak{g}$, $\mathfrak{g}$ - simple complex

H_i, X_i - generators $d_i = \frac{(\alpha_i, \alpha_i)}{2}$

the length of α_i

$(d_i = 1$ for $A, D, E)$

$\delta H_i = 0$, $\delta X_i = \frac{d_i}{2} H_i \wedge X_i$

Drinfeld double (bicross product)

If $\mathfrak{g}$ is a Lie algebra, it acts on $\mathfrak{g}^*$ via $ad^*_{\mathfrak{g}}$ action. The semidirect product:

$$\mathfrak{g} \ltimes \mathfrak{g}^* = \{ (x, \ell) \mid [(x,\ell), (y,m)] = ([x,y], ad^*_x m - ad^*_y \ell)$$

Example: $E_3 = SO(3) \ltimes \mathbb{R}^3$

Remark: more general: g_1, g_2 – Lie algebras, $\mathfrak{g}_1$ acts on $\mathfrak{g}_2$ by derivations $\Rightarrow g_1 \ltimes g_2$

Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra.

We have:
- Lie algebra $\mathfrak{g}$, acts on $\mathfrak{g}^*$ via $ad^*_{\mathfrak{g}}$
- Lie algebra $\mathfrak{g}^*$ acts on $\mathfrak{g}$ via $ad^*_{\mathfrak{g}^*}$

The bicross product $\mathfrak{g} \bowtie \mathfrak{g}^*$ (Drinfeld double):

Thm. $\exists!$ Lie algebra structure on $\mathfrak{g} \bowtie \mathfrak{g}^* = \mathfrak{g} \oplus \mathfrak{g}^*$, s.t.

- $\mathfrak{g}, \mathfrak{g}^* \subset$ Lie subalgebras.
- the canonical pairing

$$(\cdot,\cdot) : (\mathfrak{g} \oplus \mathfrak{g}^*)^{\otimes 2} \longrightarrow \mathbb{C}$$

$$((x,\ell),(y,m)) \mapsto \ell(y) + m(x)$$

is invariant.

Remark.
Such triple $(\mathfrak{g}, \mathfrak{g}^*, \mathfrak{g}\oplus\mathfrak{g}^*)$ is called Manin triple.

Remark. More general $\mathfrak{g} = \mathfrak{g}_1 \bowtie \mathfrak{g}_2$, $\mathfrak{g}_1$ acts on $\mathfrak{g}_2$, $\mathfrak{g}_2$ acts on $\mathfrak{g}_1$ and these actions agree.

Proof. First, let us compute the bracket

- $([\ell,x],m) + (x,[\ell,m]) = 0,$

$$[\ell,x](m) = -\delta(x)(\ell\wedge m),$$

- $([x,y],\ell) + (y,[x,\ell]) = 0$

$$([\ell,x],y) = [x,y](\ell)$$

From here, if $\{e_i\}$ is a basis in $\mathfrak{g}$, $\{e^i\}$ is a dual basis in $\mathfrak{g}^*$,

$$[e^i, e_j] = \sum_k c^i_{jk} e^k - \sum_k f^{ik}_j e_k ,$$

and by definition

$$[e_i, e_j] = \sum_k c^k_{ij} e_k, \quad [e^i, e^j] = \sum_k f^{ij}_k e^k$$

The proof of the Jacobi identity is an elementary algebra.

Note that the Jacobi identity for $\mathfrak{g} \bowtie \mathfrak{g}^*$ is equivalent to

$$(d_{\mathfrak{g}} + d_{\mathfrak{g}^*})^2 = 0$$

where $d_{\mathfrak{g}} + d_{\mathfrak{g}^*}$ is the diagonal differential in the bi-complex

$\Lambda^i \mathfrak{g} \otimes \Lambda^j \mathfrak{g}^*$ which was discussed earlier. $\blacksquare$

Thm. $\mathfrak{g} \bowtie \mathfrak{g}^*$ has a canonical Lie bialgebra structure such that the natural embeddings

$$\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \hookleftarrow (\mathfrak{g}^*)^{op}$$

are Lie bialgebra embeddings.

i.e. $\mathfrak{g} \bowtie \mathfrak{g}^* \simeq \mathfrak{g} \oplus (\mathfrak{g}^*)^{op}$ as a Lie coalgebra.

Proof. h.w.

$\boxtimes$

Thm. $\mathfrak{g} \bowtie \mathfrak{g}^*$ is a quasitriangular Lie bialgebra with

$$r = \sum_i e_i \otimes e^i \in \mathfrak{g} \otimes \mathfrak{g}^* \hookrightarrow (\mathfrak{g} \bowtie \mathfrak{g}^*)^{\otimes 2}$$

Proof.

$$\delta_r(e_i) = \sum_j [e_j, e_i] \otimes e^j + \sum_j e_j \otimes [e^j, e_i] =$$

$$=\sum_{j,k} c_{ji}^{k}\, e_k \otimes e^j + \sum_j e_j \otimes \left(c_{ik}^{j} e^k - f_i^{jk} e_k\right) =$$

$$= - f_i^{jk}\, e_j \otimes e_k\,,$$

$$\delta_r(e^i) = \sum_j [e_j, e^i] \otimes e^j - \sum_j e_j \otimes [e^i, e^j] =$$

$$= \sum_{j,k} \left(- c_{jk}^{i} e^k + f_j^{ik} e_k\right) \otimes e^j - \sum_{jk} e_j \otimes f_k^{ij} e^k$$

$$= - c_{jk}^{i}\, e^k \otimes e^j = c_{jk}^{i}\, e^j \otimes e^k\,,$$

$\blacksquare$

<u>Example</u>. • $\mathfrak{g} = \mathfrak{b}_+ = \mathbb{C}H \oplus \mathbb{C}X$,

$[H, X] = 2X$, $\delta H = 0$, $\delta X = \frac{1}{2} H \wedge X$,

• $\mathfrak{g}^* = \mathbb{C}H^\vee \oplus \mathbb{C}X^\vee$,

$[H^\vee, X^\vee] = X^\vee$, $\delta H^\vee = 0$, $\delta X^\vee = H^\vee \wedge X^\vee$

• $\mathfrak{g} \bowtie \mathfrak{g}^* = \{ \alpha H + \beta X + a H^\vee + b X^\vee \}$

$(x, x') = \alpha a' + \beta b' + \alpha' a + \beta' b$

• $([H^\vee, X], x) + (X, [H^\vee, x]) = 0$, $x \in \mathfrak{g}^*$

nonzero only
if $x = cX^\vee$

$([H^\vee, X], X^\vee) = -(X, X^\vee) = -1$,

• $([X, H^\vee], x) + (H^\vee, [X, x]) = 0 \qquad x \in \mathfrak{g}$

nonzero only
if $x = cH$

$([X, H^\vee], H) + (H^\vee, \underbrace{(-2X)}_{0}) = 0$,

$\Rightarrow [H^\vee, X] = -X$,

• $[H, X^\vee] = -2X^\vee$ similarly to above

• $([X, X^\vee], x) + (\underbrace{X^\vee, [X, x]}_{\neq 0 \text{ only if } x = cH}) = 0$, $x \in \mathfrak{g}$,

$([X, X^\vee], H) + (X^\vee, -2X) = 0$,

$\Rightarrow ([X, X^\vee], H) = 2$

• $([X^\vee, X], x) + (\underbrace{X, [X^\vee, x]}_{\neq 0 \text{ only if } x = cH^\vee}) = 0$, $x \in \mathfrak{g}^*$

$$([X^\vee, X], H^\vee) = 1$$

$\bullet \Rightarrow \quad [X, X^\vee] = 2H^\vee - H$

Lie brackets in $\mathcal{D}(b_+)$:

$$[H, H^\vee] = 0, \quad [H, X^\vee] = -2X^\vee, \quad [H, X] = 2X$$

$$[H^\vee, X] = -X, \quad [H^\vee, X^\vee] = X^\vee,$$

$$[X, X^\vee] = 2H^\vee - H$$

Lie co-brackets:

$$\delta H^\vee = \delta H = 0, \quad \delta X = \tfrac{1}{2} H \wedge X, \quad \delta X^\vee = -H^\vee \wedge X^\vee,$$

$$(\quad b_+ \hookrightarrow \mathcal{D}(b_+) \hookleftarrow b_+^{*op} \quad)$$

$H' = \frac{H}{2} - H^\vee,$	$[H', X'] = 2X', \ [H', Y'] = -2Y'$
$X' = X,$	$[X', Y'] = H', \quad H'' \in$ center
$Y' = -\frac{1}{2} X^\vee,$	$\delta X' = \frac{1}{2} H' \wedge X' + \frac{1}{2} H'' \wedge X',$
$H'' = \frac{H}{2} + H^\vee,$	$\delta Y' = \frac{1}{2} H' \wedge Y' - \frac{1}{2} H'' \wedge Y',$

$$H = H' + H'', \quad H^\vee = \tfrac{1}{2}(H'' - H'), \quad \delta H' = \delta H'' = 0$$

$$r = H \otimes H^\vee + X \otimes X^\vee = \frac{1}{2} H'' \otimes H'' + \frac{1}{2}(H' \otimes H'' - H'' \otimes H')$$

$$- \frac{1}{2}\left(\frac{H' \otimes H'}{4} + X' \otimes Y' \right),$$

$$\Rightarrow \mathcal{D}(b_+) \supset I = \mathbb{C} H'',$$

$\nwarrow$ Lie bialgebra ideal

$$(sl_2, -\delta_2) \simeq \mathcal{D}(b_+) / I$$

h.w. Find $\mathcal{D}(b_+)$, $b_+ \subset \mathfrak{g}$ (as in the example above).

Note Title 2/7/2009

Real forms of Lie bialgebras.

① Real Lie algebra $\mathfrak{g}_{\mathbb{R}}$: $[\cdot,\cdot] : \mathfrak{g}_{\mathbb{R}}^{\otimes 2} \to \mathfrak{g}_{\mathbb{R}}$ real

Its complefixation: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The Lie algebra structure extended by linearity.

② $\mathfrak{g}/\mathbb{C}$, $\sigma : \mathfrak{g} \to \mathfrak{g}$ $\mathbb{C}$-antilinear automorphism of $\mathfrak{g}$:

$$\sigma([a,b]) = [\sigma(a), \sigma(b)]$$

$$\sigma(\lambda a) = \bar{\lambda}\,\sigma(a)$$

Def. The real form of $\mathfrak{g}$ corresponding to σ is the set of σ-fixed points $\mathfrak{g}^{\sigma}$.

It is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (considering $\mathfrak{g}_{\mathbb{C}}$ as a real Lie algebra).

It is clear: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{\sigma} \otimes_{\mathbb{R}} \mathbb{C}$, i.e. $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}^{\sigma}$.

2. Similarly for Lie groups.

(3) $(\mathfrak{g}, \delta)$ is a real Lie bialgebra if $/\mathbb{R}$ and δ is a real linear map.

(4) Let $(\mathfrak{g}, \delta)$ be a complex Lie bialgebra and $\sigma : \mathfrak{g} \circlearrowleft$ be an automorphism of the Lie bialgebra $\mathfrak{g}$ which is $\mathbb{C}$-antilinear:

$$\sigma[a,b] = [\sigma a, \sigma b]$$

$$\sigma \otimes \sigma(\delta a) = \delta \sigma a$$

$$\sigma(\lambda a) = \bar{\lambda} \sigma a,$$

$\mathfrak{g}^\sigma$ = fixed points of σ is a real form of the Lie bialgebra $\mathfrak{g}$.

It is clear: $\mathfrak{g} = \mathfrak{g}^\sigma \otimes_{\mathbb{R}} \mathbb{C}$, i.e. $\mathfrak{g}$ is the complexification of $\mathfrak{g}^\sigma$.

(5) Real forms of the Lie algebra

$$sl_2(\mathbb{C}) = \underset{s}{\mathbb{C}H} \oplus \underset{t}{\mathbb{C}X} \oplus \underset{u}{\mathbb{C}Y},$$

The Killing form

$$(A, A) = s^2 + tu \neq \tfrac{1}{2} \operatorname{tr}(A^2)$$

(i) The compact real form:

$$\sigma(X) = -Y, \quad \sigma(Y) = -X, \quad \sigma(H) = -H$$

$$sl_2^{\sigma}(\mathbb{C}) = su_2 = \underset{x}{\mathbb{R}(X-Y)} \oplus \underset{y}{\mathbb{R}i(X+Y)} \oplus \underset{z}{\mathbb{R}iH},$$

$$c = -(x^2+y^2+z^2) \leq 0$$

$$\delta(iH) = 0, \quad \delta(X-Y) = \tfrac{1}{2} iH \wedge (X-Y), \quad \delta(i(X+Y)) = \tfrac{1}{2} iH \wedge i(X+Y)$$

(ii) The split real form:

$$\sigma(X) = X, \quad \sigma(Y) = Y, \quad \sigma(H) = H$$

$$sl_2(\mathbb{R}) = \underset{\alpha}{\mathbb{R}X} \oplus \underset{\beta}{\mathbb{R}Y} \oplus \underset{x}{\mathbb{R}H}$$

$$c = x^2 + \alpha\beta = x^2 + \left(\frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2 \; (++-) \text{ type}$$

$$\delta H = 0, \quad \delta X = \tfrac{1}{2} H \wedge X, \quad \delta Y = \tfrac{1}{2} H \wedge Y$$

(iii) $\sigma(H) = -H, \quad \sigma(X) = Y, \quad \sigma(Y) = X,$

$$sl_2^{\sigma} = su(1,1) = \underset{x}{\mathbb{R}iH} \oplus \underset{y}{\mathbb{R}i(X-Y)} \oplus \underset{z}{\mathbb{R}(X+Y)}$$

$c = -x^2 + y^2 + z^2$, $(+,+,-)$ type

$\delta H = 0$, $\delta(X+Y) = \frac{1}{2} iH \wedge (X+Y)$, $\delta(i(X-Y)) = \frac{iH}{2} \wedge i(X-Y)$

- We have two non-isomorphic real forms of $sl_2(\mathbb{C})$:

  su_2 and $sl_2(\mathbb{R}) \simeq su_{1,1}$

- Complexifications of (su_2, δ), $(su_{1,1}, \delta)$ gives $sl_2(\mathbb{C})$ with

  $\delta(H) = 0$, $\delta X = \frac{i}{2} H \wedge X$, $\delta Y = \frac{i}{2} H \wedge Y$

- Complexification of $(sl_2(\mathbb{R}), \delta)$ gives $sl_2(\mathbb{C})$ with

  $\delta H = 0$, $\delta X = \frac{1}{2} H \wedge X$, $\delta Y = \frac{1}{2} H \wedge Y$

These Lie bialgebras are not isomorphic.

Thus, we have three non-isomorphic real Lie bialgebras, real forms of $sl_2(\mathbb{C})$.

$$(su_2, \delta), \quad (sl_2(\mathbb{R}), \delta), \quad (su_{1,1}, \delta)$$

Similar to other simple Lie algebras.