# Lecture 8. Poisson Lie group $SL_2^*$

Note Title 2/8/2009

① Recall the Lie algebra structure on $sl_2^*$:

$$[H^\vee, X^\vee] = X^\vee, \quad [H^\vee, Y^\vee] = Y^\vee,$$

$$[X^\vee, Y^\vee] = 0$$

The Lie co-algebra structure

$$\langle \delta H^\vee, x \wedge y \rangle = \langle H^\vee, [x, y] \rangle$$

$$\Rightarrow \quad \underline{\delta H^\vee = \tfrac{1}{2} X^\vee \wedge Y^\vee}$$

$$\langle \delta X^\vee, x \wedge y \rangle = \langle X^\vee, [x, y] \rangle$$

$$\underline{\delta X^\vee = \quad H^\vee \wedge X^\vee}$$

$$\langle \delta Y^\vee, x \wedge y \rangle = \langle Y^\vee, [x, y] \rangle$$

$$\underline{\delta Y^\vee = - H^\vee \wedge Y^\vee},$$

The representation

$X^\vee \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I, \quad Y^\vee \mapsto I \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$H^\vee \mapsto \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I + \frac{1}{2} I \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

is a linear representation.
Exponentiating $sl_2^*$ in this representation
we obtain $SL_2^*$ as a matrix group

$SL_2^* = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix} \middle| \text{ with pairwise multipl.} \right\}$

2. Lie algebra homomorphisms

$\tau_I : sl_2^* \longrightarrow sl_2$

lift to Lie group homomorphisms

$$R_\pm : SL_2^* \to SL_2$$

$$R_+\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ c & a\end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix},$$

$$R_-\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ c & a\end{pmatrix}\right) = \begin{pmatrix} a^{-1} & 0 \\ c & a\end{pmatrix},$$

The factorization linear isomorphism

$$t : sl_2^* \to sl_2$$

lifts to factorization mappings

$$I : SL_2^* \to SL_2, \quad x \mapsto R_+(x) R_-(x)^{-1} \text{ or } R_-(x)^{-1} R_+(x)$$

$$\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ c & a\end{pmatrix}\right) \mapsto \begin{pmatrix} a^2 - bc, & b a^{-1} \\ -a^{-1}c, & a^{-2}\end{pmatrix}$$

$$\text{or } \begin{pmatrix} a^2 & ab \\ -ca & a^{-2} - cb\end{pmatrix},$$

The following thm describes the Poisson Lie structure on $SL_2^*$ but does not explain where how it was obtained. We will see it later

<u>Thm</u> (i) The comultiplication on $C(SL_2^*)$:

$\Delta a = a \otimes a$, $\Delta b = a \otimes b + b \otimes a^{-1}$,

$\Delta c = a \otimes c + c \otimes a^{-1}$,

ii) the following brackets define a Poisson Hopf structure on $C(SL_2^*)$:

$\{a, c\} = -\frac{1}{2} ac$, $\{a, b\} = \frac{1}{2} ab$

$\{b, c\} = a^2 - a^{-2}$

iii) The tangent Lie bialgebra to the Poisson Lie group $SL_2^*$ is $sl_2^*$ with the Lie cobracket dual to $[\cdot,\cdot] : sl_2^{\otimes 2} \to sl_2$

<u>Proof</u>. i) and ii) is a simple algebra.

iii) h.w.

$\square$