Lecture 7 $SL_2(\mathbb{C})$

Note Title 2/8/2009

(1) $SL_2(\mathbb{C})$ is an affine algebraic variety

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\}$$

Let $C(SL_2)$ be the algebra of polynomials in coordinate functions a, b, c, d

The comultiplication for $C(SL_2)$ is

$$\Delta a = a \otimes a + b \otimes c, \quad \Delta b = a \otimes b + b \otimes d$$

$$\Delta c = c \otimes a + d \otimes c, \quad \Delta d = c \otimes b + d \otimes d$$

the pull-back of the group multiplication for $SL_2(\mathbb{C})$:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \beta\gamma', & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma', & \gamma\beta' + \delta\delta' \end{pmatrix}$$

In other words:

$$\Delta g_{ij} = \sum_k g_{ik} \otimes g_{kj}$$

② The Poisson Lie structure with the tangent Lie bialgebra (sl_2, δ_2) is given by

$$p(x) = -(Ad_x \otimes Ad_x)(r) + r$$

Let us compute Poisson brackets between coordinate functions a, b, c, d.

(a) If G is a matrix group $(G \subset GL(V))$ its Lie algebra is a matrix Lie algebra $(\mathfrak{g} \subset gl(V))$. Since now $G, \mathfrak{g} \subset End(V)$ we have the matrix multiplication

$$Xg, \quad X \in \mathfrak{g}, \quad g \in G, \quad \mathfrak{g}, G \subset End(V)$$

In this case

$$\frac{d}{dt} f(e^{tX} g)\Big|_{t=0} = \sum_{ij} \frac{d}{dt}(e^{tX} g)_{ij}\Big|_{t=0} \frac{\partial f}{\partial g_{ij}}(g) =$$

$$= (Xg)_{ij} \frac{\partial f}{\partial g_{ij}}$$

Here g_{ij} are matrix elements of g with respect to some basis $\{e_i\}$ in V.

(b) For the Poisson bracket

$$\{f_1, f_2\}(g) = \frac{1}{2} \langle p(g), df_1(g) \wedge df_2(g) \rangle$$

of functions on a matrix group G we have:

$$\{f_1, f_2\}(g) = \sum_{ab} p^{ab}(g) \frac{d}{dt} f_1(e^{te_a} g)\Big|_{t=0} \cdot$$

$$\cdot \frac{d}{ds} f_2(e^{se_a} g)\Big|_{s=0} = \sum_{ab, ij, kl} p^{ab}(g) (e_a g)_{ij} \frac{\partial f_1}{\partial g_{ij}} \cdot$$

$$(e_b g)_{kl} \frac{\partial f_2}{\partial g_{kl}} = [r, g \otimes g]_{ij, kl} \frac{\partial f_1}{\partial g_{ij}} \frac{\partial f_2}{\partial g_{kl}}$$

(c) for coordinate functions $\{g_{ij}\}$ we have:

$$\{g_{ij}, g_{k\ell}\} = [r, g\otimes g]_{ij,k\ell}$$

Here and above we use notations

$$(A\otimes B)_{ij,k\ell} = A_{ij} B_{k\ell}$$

Let us use the tensor product notations

$$(\{A \overset{\otimes}{,} B\})_{ij,k\ell} = \{A_{ij}, B_{k\ell}\}$$

then

$$\{g \overset{\otimes}{,} g\} = [r, g\otimes g] \qquad (*)$$

(d) Now, let us compute the Poisson brackets between a, b, c, d. Choose the basis in $\mathbb{C}^2\otimes\mathbb{C}^2$:

$$f_1 = e_1\otimes e_1, \; f_2 = e_1\otimes e_2, \; f_3 = e_2\otimes e_1, \; f_4 = e_2\otimes e_2$$

$$g\otimes g = \left[\begin{array}{cc|cc} aa & ab & ba & bb \\ ac & ad & bc & bd \\ \hline ca & cb & da & db \\ cc & cd & dc & dd \end{array}\right],$$

The r-matrix for sl_2 considered as an element of $\mathrm{End}(\mathbb{C}^2)^{\otimes 2}$ is

$$r = \frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} =$$

$$= \left[\begin{array}{cc|cc} \frac{1}{4} & 0 & 0 & 0\\ 0 & -\frac{1}{4} & 1 & 0\\ \hline 0 & 0 & -\frac{1}{4} & 0\\ 0 & 0 & 0 & \frac{1}{4}\end{array}\right]$$

and $\{g \overset{\otimes}{,} g\}$ is

$$\left[\begin{array}{cc|cc} \{a,a\} & \{a,b\} & \{b,a\} & \{b,b\}\\ \{a,c\} & \{a,d\} & \{b,c\} & \{b,d\}\\ \hline \{c,a\} & \{c,b\} & \{d,a\} & \{d,b\}\\ \{c,c\} & \{c,d\} & \{d,c\} & \{d,d\}\end{array}\right]$$

From (*) we find:

$\{a,b\} = \frac{1}{2}ab$, $\{a,c\} = -\frac{1}{2}ac$, $\{b,c\} = 0$

$\{a,d\} = bc$, $\{b,d\} = \frac{1}{2}bd$, $\{c,d\} = \frac{1}{2}cd$

(3) Poisson Lie subgroups:

i) $B_+ \subset SL_2$, $B_+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$

The Poisson Hopf algebra $\mathbb{C}(B_+)$:

$\Delta a = a \otimes a$, $\Delta b = a \otimes b + b \otimes a^{-1}$

$\{a, b\} = \frac{1}{2} ab$

$\mathbb{C}(B_+) = \mathbb{C}(SL_2) / \langle c \rangle$

where $\langle c \rangle$ is the ideal generated by c. It is a Poisson Hopf ideal (a Hopf ideal and also a Poisson ideal)

ii) Similarly $B_- = \left\{ \begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix} \right\} \subset SL_2(\mathbb{C})$ is a Poisson Lie subgroup with $\Delta c = a \otimes c + c \otimes a^{-1}$, $\Delta a = a \otimes a$,

$\{a, c\} = -\frac{1}{2} ac$

iii) $H \subset SL_2$, trivial Poisson Lie group