Lecture 6. The sl_2-example

Note Title 2/2/2009

① Linear basis H, X, Y, brackets as in (...)

(a) The classical r-matrix (we will see why later), i.e. a solution to the cl. YBE:

$$r = \frac{H\otimes H}{4} + X\otimes Y,$$

$$t = r + \sigma(r) = \frac{H\otimes H}{2} + X\otimes Y + Y\otimes X, \text{ nondeg.}$$

$$\mathfrak{g}_+ = \mathbb{C}H \oplus \mathbb{C}X, \quad \mathfrak{g}_- = \mathbb{C}H \oplus \mathbb{C}Y,$$

(b) The Lie bialgebra:

$$\delta_r H = [r, H\otimes 1 + 1\otimes H] = 0$$

$$\delta_r X = \frac{H}{4}\otimes [H, X] + \frac{[H,X]}{4}\otimes H + X\otimes [Y, X] =$$

$$= \frac{1}{2}(H\otimes X - X\otimes H) = \frac{1}{2} H\wedge X$$

$$\delta_r Y = \frac{1}{2} H\otimes Y$$

(2) The dual Lie algebra:

$\mathfrak{g}^* = \mathbb{C}H^\vee \oplus \mathbb{C}X^\vee \oplus \mathbb{C}Y^\vee$, $H^\vee, Y^\vee, X^\vee$ is the basis dual to H, Y, X.

$[H^\vee, X^\vee](a) = (H^\vee \wedge X^\vee)(\delta a)$, $\Rightarrow$ it is nonzero only if $a = cX$, $c \in \mathbb{C}$

$$[H^\vee, X^\vee](X) = (H^\vee \wedge X^\vee)(\tfrac{1}{2} H \wedge X) = \tfrac{1}{2}\langle H^\vee \wedge \hat{X}, H \wedge X \rangle =$$
$$= \tfrac{1}{2}(\langle H^\vee, H\rangle \langle X^\vee, X\rangle + \langle X^\vee, X\rangle \langle H^\vee, H\rangle) = 1$$

i.e. $\quad [H^\vee, X^\vee] = X$

Similarly

$$[H^\vee, Y^\vee] = Y^\vee, \quad [X^\vee, Y^*] = 0$$

(3) The factorization: $t = \frac{H \otimes H}{2} + X \otimes Y + Y \otimes X$

$t : \mathfrak{g}^* \to \mathfrak{g}$, $t(H^\vee) = \frac{H}{2}$, $t(X) = Y$, $t(Y) = X$

For factorizable Lie bialgebras

$\mathfrak{g}_+ = \ker(r_-)^\perp = \mathrm{Im}\, r_+$, $\quad \mathfrak{g}_- = \ker(r_+)^\perp = \mathrm{Im}\, r_-$

For $\mathfrak{sl}_2(\mathbb{C})$:

$$\mathfrak{g}_+ = \mathbb{C}H + \mathbb{C}X, \quad \mathfrak{g}_- = \mathbb{C}H + \mathbb{C}Y$$

The factorization in $\mathfrak{sl}_2$, induced by $r_\pm$:

$$x = \alpha H + \beta X + \gamma Y = \underbrace{\left(\tfrac{\alpha}{2} H + \beta X\right)}_{x_+} + \underbrace{\left(\tfrac{\alpha}{2} H + \gamma Y\right)}_{-x_-}$$

$$x_\pm = r_\pm(\ell), \quad \ell = 2\alpha H^\vee + \beta Y^\vee + \gamma X^\vee = t^{-1}(x)$$

② The Poisson Lie group $SL_2(\mathbb{C})$.

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\}$$

a, b, c, d – functions on $SL_2(\mathbb{C})$

The Poisson structure with $p(x) = Ad_x \otimes Ad_x (r) - r$

$$\{ g \overset{\otimes}{,} g \} = [r, g \otimes g].$$