Note Title 1/29/2009

Lecture 5. Poisson Lie groups (3)

Example of a classical r-matrix for sl_2:

$$r = \frac{H\otimes H}{4} + X\otimes Y.$$

H, X, Y - generators (and linear basis) for sl_2

$[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$.

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Let $r \in \mathfrak{g}\otimes\mathfrak{g}$ be a solution to the classical YBE:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

Consider linear maps

$$r_\pm : \mathfrak{g}^* \to \mathfrak{g},$$

$r_+(\ell) = (\mathrm{id}\otimes \ell)(r)$, $\quad r_-(\ell) = -(\ell\otimes \mathrm{id})(r)$

Denote $\mathfrak{g}_\mp = \ker(r_\pm)^\perp \subset \mathfrak{g}$, linear subspaces.

<u>Proposition</u> $\mathfrak{g}$, $\tilde{\mathfrak{g}} = \mathfrak{g}_+ + \mathfrak{g}_- \subset \mathfrak{g}$ are Lie subalgebras.

Proof. ① Choose splitting $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}'_-$ and a basis in $\mathfrak{g}$ where the subset of basis vectors is a basis in $\mathfrak{g}$. Write

$$r = \sum_{ij} r^{ij} e_i \otimes e_j + \sum_{ij} s_{ij} e'_i \otimes e_j + t\, e \otimes e' + u\, e' \otimes e'$$

Let $\{f^i\}$ be a basis in $\ker(r_+)$, $\{g^i\}$ – the basis in a complement $\langle e_i, f^j \rangle = 0$,

Since $\mathfrak{g}_- = \ker(r_+)^\perp$, $\{\langle e'_i, f^j \rangle\}$ is nondegen.

$$(id \otimes f^k) r = \sum_{ij} t_{ij} e_i \otimes \langle e'_j, f^k \rangle + \sum_{ij} u_{ij} e'_i \otimes \langle e'_j, f^k \rangle = 0$$

$\forall k$, $\quad t, u = 0$ if $\{\langle e'_j, f^k \rangle\}$ is non-degenerate

$\Rightarrow \quad r \in \mathfrak{g} \otimes \mathfrak{g}_-$

Similarly, $\quad r \in \mathfrak{g}_+ \otimes \mathfrak{g}$ and $\Rightarrow$

$r \in \mathfrak{g}_+ \otimes \mathfrak{g}_-$

② Consider the classical YBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

3 terms in this identity belong to subspaces:

$$[\mathfrak{g}_+, \mathfrak{g}_+] \otimes \mathfrak{g}_- \otimes \mathfrak{g}_-, \quad \mathfrak{g}_+ \otimes [\mathfrak{g}_-, \mathfrak{g}_+] \otimes \mathfrak{g}_-,$$

$$\mathfrak{g}_+ \otimes \mathfrak{g}_+ \otimes [\mathfrak{g}_-, \mathfrak{g}_-]$$

Therefore $[\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+$, $[\mathfrak{g}_-, \mathfrak{g}_+] \subset \mathfrak{g}_+ + \mathfrak{g}_-$

and $[\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_-$.

This proves the proposition. $\blacksquare$

Consider $t = r + \sigma(r) \in S^2(\tilde{\mathfrak{g}}) \subset S^2(\mathfrak{g}) \subset \mathfrak{g}^{\otimes 2}$

Lemma. $\cdot\ [t_{12}, r_{13} + r_{23}] = 0$

$\cdot\ [t_{12}, r_{31} + r_{32}] = 0$

$\cdot\ [t_{12}, t_{13} + t_{23}] = 0$

Proof. Apply the permutation $\sigma \otimes id$ $(1 \leftrightarrow 2)$ to the cYBE:

$$[r_{21}, r_{13}] + [r_{21}, r_{23}] + [r_{23}, r_{13}] = 0$$

add the result to the cYBE:

$$[r_{12} + r_{21}, r_{13}] + [r_{21} + r_{12}, r_{23}] = 0$$

The proof of the second and third identities is similar. ∎

Corollary. $\cdot\ t \in S^2(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$

$\cdot\ [r, x \otimes 1 + 1 \otimes x] \in \Lambda^2 \tilde{\mathfrak{g}}$

Thm. $\left(\tilde{\mathfrak{g}}, \delta_r(x) = [r, x\otimes 1 + 1\otimes x]\right)$ is a Lie bialgebra.

Proof. We already know $\delta_r(x) \in \Lambda^2 \tilde{\mathfrak{g}}$

- $\delta_r([X,Y]) = [r, [X,Y]\otimes 1 + 1\otimes [X,Y]] =$
$= (\text{Jacobi id}) = [X, \delta_r(Y)] + [\delta_r(X), Y]$

- $\mathrm{Alt}((\delta_r \otimes id)\circ \delta_r) = 0$ follows from the cl YBE.

$\blacksquare$

Such Lie bialgebras are called quasitriangular.

Now, assume that $\tilde{\mathfrak{g}} = \mathfrak{g}$, i.e. that $t = r + \sigma(r) \in S^2(\mathfrak{g})^{\mathfrak{g}}$

Def. The Lie bialgebra $(\mathfrak{g}, \delta_r)$ is factorizable if $t = r + \sigma(r)$ defines a nondegenerate bilinear form on $\mathfrak{g}^*$.

For factorizable Lie bialgebras t defines a linear isomorphism

$$t : \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g} , \quad t(\ell) = r_+(\ell) - r_-(\ell)$$

An imediate corollary of this definition is

<u>Proposition</u> Any element $x \in \mathfrak{g}$ admits unique factorization

$$x = x_+ - x_-$$

where $x_\pm = r_\pm(\ell)$ for some $\ell \in \mathfrak{g}^*$.

The cobracket $\delta_2 : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ defines a Lie algebra structure on $\mathfrak{g}^*$.

<u>Thm</u>. The linear maps $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ are Lie algebra homomorphisms.

<u>Proof</u>.

$$r_-([\ell_1, \ell_2]) = -([\ell_1, \ell_2] \otimes id)(r) =$$

$$= -(\ell_1 \wedge \ell_2 \otimes id)([r_{12}, r_{13} + r_{23}]) =$$

$$= (\ell_1 \wedge \ell_2 \otimes \mathrm{id})([r_{13}, r_{23}]) = [r_-(\ell_1), r_-(\ell_2)]$$

and similarly for r_+ $\blacksquare$

We will return to the discussion of the structure of factorizable Lie bialgebras.

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Poisson Lie groups with quasitriangular tangent Lie bialgebras.

Recall that we choose the trivialization of $TG \simeq \mathfrak{g} \times G$, $\mathfrak{g} = T_eG$ by right translations. Let $(\mathfrak{g}, \delta_r)$ be a quasitriangular Lie bialgebra.

Thm. The Lie group G with $\mathfrak{g} = \mathrm{Lie}(G)$ is a Poisson Lie group with

$$p(x) = -(\mathrm{Ad}_x \otimes \mathrm{Ad}_x)(r) + r$$

and $\delta_r = dp(e)$.

<u>Proof</u>. • the proof of the cocycle property is a simple algebra

• $f_1, f_2 \in C(G)$

$$\{f_1, f_2\} = \langle r, df_1 \wedge df_2 - d'f_1 \wedge d'f_2 \rangle$$

The Jacobi identity follows from the classical YBE. Here (after the trivialization of the tangent bundle by right translations):

$$\langle X, df(g)\rangle = \frac{d}{dt} f(e^{tX} g)\Big|_{t=0}, \quad \langle X, d'f(g)\rangle = \frac{d}{dt} f(g e^{tX})\Big|_{t=0}$$

∎

Let G^* be a Lie group (connected) with $\mathfrak{g}^* = \mathrm{Lie}(G^*)$. Lie algebra homomorphisms $r_\pm : \mathfrak{g}^* \to \mathfrak{g}$ lift to Lie group homomorphisms

$$R_\pm : G^* \to G$$

If $(\mathfrak{g}, \delta_r)$ is factorizable, $G^* \to G$

$$x \mapsto R_+(x) R_-(x)^{-1}$$

$$x \mapsto R_-(x)^{-1} R_+(x)$$

are diffeom. in a nbd of $e \in G$