Lecture 4. Poisson Lie groups

Note Title 1/27/2009

<u>Def</u>. The Lie bialgebra structure induced by the Poisson Lie structure on $\mathfrak{g} = \mathrm{Lie}(G)$ is called the tangent Lie algebra to a Poisson Lie group.

<u>Theorem</u> Let $(\mathfrak{g}, \delta)$ be Lie bialgebra and G be a connected Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. There exists unique Poisson Lie structure p on G s.t. $(\mathfrak{g}, \delta)$ is the tangent Lie bialgebra to (G, p).

<u>Proof</u>. h.w.

Let (G, p), (H, q) be two Poisson Lie groups

<u>Def.</u> A Lie group homomorphism $\varphi: G \to H$ which is also a Poisson mapping ($\varphi^*: C(H) \to C(G)$ is a homom. of Poisson algebras) is called a homom. of Poisson Lie groups.

<u>Def</u> Let $(\mathfrak{g}_1, \delta_1)$ and $(\mathfrak{g}_2, \delta_2)$ be two Lie bialgebras. A linear map $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie bialgebra homomorphism if φ and φ^* are Lie algebra homom.

<u>Thm.</u> A Poisson Lie group homom. induces a homom. of tangent Lie bialgebras.

Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra

Def. $(\mathfrak{g}^*, \delta_*)$ where $\delta_* : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$ is dual to the Lie bracket on $\mathfrak{g}$, is called a dual Lie bialgebra to $(\mathfrak{g}, \delta)$.

Let
$$(\mathfrak{g}, \delta) = T(G, p)$$
$$(\mathfrak{g}^*, \delta_*) = T(G^*, p_*)$$

The pair (G, p), (G^*, p_*) is called a dual pair of Poisson Lie groups.

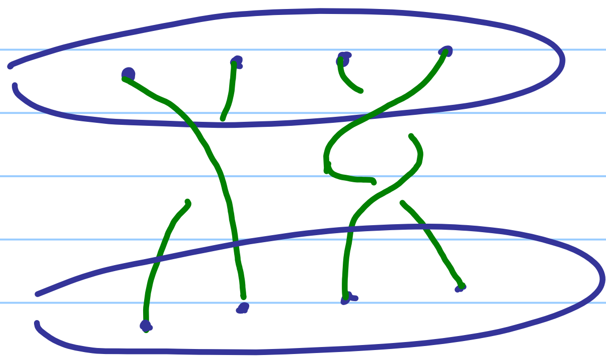
Remark. $\exists!$ Poisson Lie dual to any connected simply connected Poisson Lie group.

Braid groups

$X_n = \{(z_1, .., z_n) \mid z_i \in \mathbb{C},\ z_i \neq z_j\}$

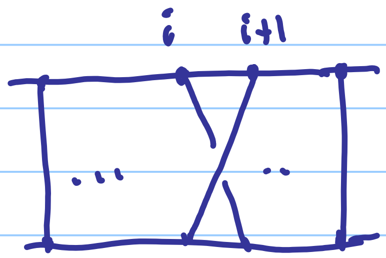
S_n acts on X_n permuting points.

Def. $B_n = \pi_1(X_n / S_n)$



Thm. $B_n \simeq \left\langle s_i,\ i = 1, 2, .., n-1 \;\middle|\; \begin{array}{l} s_i s_j = s_j s_i \\ |i-j| \neq \pm 1 \\ s_i s_{i\pm 1} s_i = \\ = s_{i\pm1} s_i s_{i\pm 1} \end{array} \right\rangle$

Proof....

$s_i \longleftrightarrow$  , Tensor product representations of B_n:

$$\pi : B_n \longrightarrow \mathrm{Aut}(V^{\otimes n}), \quad V = \text{vector space}$$

$$\pi(s_i) = 1 \otimes \cdots \otimes \underset{i\,i+1}{S} \otimes \cdots \otimes 1$$

is a representation if

$$(S\otimes 1)(1\otimes S)(S\otimes 1) = (1\otimes S)(S\otimes 1)(1\otimes S)$$

the (quantum) Yang-Baxter equation.

an over-determined system ($(\dim V)^6$ equations for $(\dim V)^2$ unknowns). But there are many solutions.

Example: $\quad S = P, \quad P(x\otimes y) = y \otimes x$

Assume we have a family of such solutions $S(h)$, s.t.

$$S(h) = P(1 + h r + O(h^2)), \quad h \to 0$$

Thm. Then

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0$$

where $r_{12} = r \otimes 1$, $r_{23} = 1 \otimes r$, $r_{13} = \dots$

(classical) Yang-Baxter equation. (cYBE)

Proof h.w.

In the cYBE we have only commutators.
This motivates the definition:

Def. $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the classical Yang-Baxter equation if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

in $U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}$, assuming $\mathfrak{g} \hookrightarrow U\mathfrak{g}$ as order 1 elements.