# Lecture 3.

Note Title 1/26/2009

Last time:

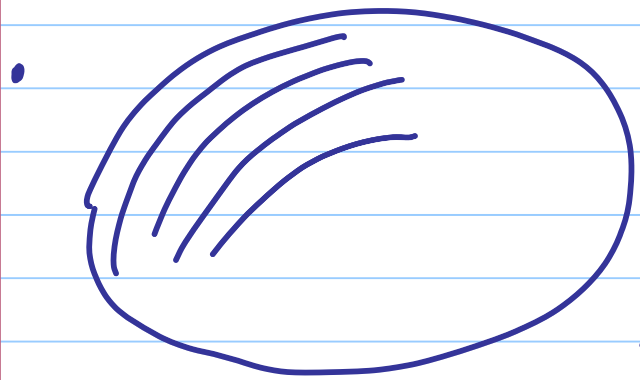
- Poisson algebras
- Poisson manifolds
- Symplectic manifolds
- Symplectic leaves in Poisson manifolds

## Symplectic leaves:

Let $(M, p)$ be a Poisson manifold.

**Def.** A *symplectic leaf* through $x \in M$ is the subspace spanned by p.w. Hamiltonian flow lines (flow lines of Ham. v. f.)

Theorem. A symplectic leave is a smooth submanifold. Poisson structure on $\mathcal{M}$ induces a symplectic structure on it.



Thus any Poisson manifold produces a collection of symplectic submanifolds.

($p: T^*\mathcal{M} \to T\mathcal{M}$, $\operatorname{Im} p \subset T\mathcal{M}$
sympl. leaves are leaves of $\operatorname{Im} p$)

Example 1. $(\mathcal{M}, p)$ is a symplectic manifold. It has unique symplectic leaf, the manifold itself.

**Example 2**. (1) $G$ acts on $\mathfrak{g}$ via the adjoint action:

- $\mathfrak{g} = T_e G$, $\quad Ad_g(x) = dl_g(dr_{g^{-1}}(x))$

$dr_{g^{-1}}(x) \in T_{g^{-1}} G$, $dl_g(dr_{g^{-1}}(x)) \in T_e G$

$dl_g$, $dr_g$ are differentials of left and right translations on $G$

$$l_g(x) = gx \quad , \quad r_g(x) = xg$$

$$dl_g : T_x G \to T_{gx} G$$

$(v \in T_x G, \quad dl_g(v) \cdot f = v \cdot l_g^*(f))$

- When $G$ is a matrix group, $\mathfrak{g} = Lie(G)$ is a matrix Lie algebra ($GL_n$, $SL_n$, $SO_n$, ...)

$$A_g(x) = g x g^{-1}$$

(2) Corresponding action on $\mathfrak{g}^* = (\mathfrak{g} \to \mathbb{C})$ is the co-adjoint action:

$$\langle Ad_g^*(\ell), x \rangle = \langle \ell, Ad_g(x) \rangle$$

**Thm.** A symplectic leaf through $x \in \mathfrak{g}^*$ is identical to the coadjoint orbit through $x$.

**Example:**

1) $SU_2$ - compact real form of $SL_2(\mathbb{C})$ (the group of unitary $2\times 2$ unimodular matrices)

$su_2 = Lie(SU_2)$ - 3-dim real

$su_2^* \simeq \mathbb{R}^3$ ($\simeq su_2$ via the Killing form $tr(xy)$)

Coadjoint orbits in $su_2^*$:

- $\mathcal{O}_x = \{ y \in su_2^* \mid |x| = |y| \} \simeq S^2, x \neq 0$

  $|x|^2 = tr(x^2)$

- $\mathcal{O}_0 = \{0\}$

2) $SL_2(\mathbb{C})$ , $sl_2(\mathbb{C}) = \mathrm{Lie}(SL_2(\mathbb{C}))$

traceless $2\times 2$ complex matrices

$sl_2(\mathbb{C})^*$ is a complex algebraic Poisson manifold

$\mathrm{tr}(xy)$ on $sl_2(\mathbb{C})$ gives a linear isom.

$$sl_2(\mathbb{C})^* \simeq sl_2(\mathbb{C})$$

Coadjoint orbits in $sl_2(\mathbb{C})^* \simeq$

$\simeq$ Conjugation orbits in $sl_2(\mathbb{C})$ (adjoint)

- Complex 2-dimensional orbits

$$\mathcal{O}_{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}} = \left\{ g \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} g^{-1} \,\middle|\, g \in SL_2(\mathbb{C}) \right\}$$

(compl. algebraic sympl. / $\mathbb{C}$ )

- 2-dimensional $\mathcal{O}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$
- 0-dimensional $\mathcal{O}_0 = \{0\}$

## Poisson Lie groups.

**Def.** A Lie group $G$ is a Poisson Lie group if it is a Poisson manifold and $m: G\times G \to G$ is a Poisson mapping and $i: G \to G$, $g \mapsto g^{-1}$ is an anti-Poisson mapping ($\{i^*(f_1), i^*(f_2)\} = -i^*\{f_1, f_2\}$).

Consider the Lie algebra of $G$, $\mathfrak{g} = \mathrm{Lie}(G)$. Trivialize $TG$ by left translations:

$$TG \simeq \mathfrak{g} \times G,$$

$p \in \Gamma(\Lambda^2 TG)$,

$p: G \longrightarrow \Lambda^2 \mathfrak{g}$

Poisson Lie:

$$p(gh) = (Ad_g \otimes Ad_g)(p(h)) + p(h)$$

$l_g : G \to G$
$h \mapsto gh$

$dl_g : T_h G \to T_{gh} G$

$dl_h^{-1} : T_h G \xrightarrow{\sim} T_e G$

trivialization:

$dl\circ i : TG \xrightarrow{\sim} \mathfrak{g}\times G$

Notice: $p(e)=0$ $\left(p(e^2)=p(e)+p(e)\right)$

Define $\delta = dp(e) : T_e G \to T_0\mathfrak{g} \wedge T_0\mathfrak{g}$

i.e. $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$

Theorem. The pair $(\mathfrak{g}, \delta : \mathfrak{g} \to \Lambda^2\mathfrak{g})$ is a Lie bialgebra

Proof. 1. Let $f_1, f_2 \in C(G)$, $\overset{\xi_1}{df_1(e)}, \overset{\xi_2}{df_2(e)} \in$ $\in T_e^*G = \mathfrak{g}^*$. The bracket on $\mathfrak{g}^*$ dual to $\delta$:

$$[\xi_1, \xi_2]_{\mathfrak{g}^*}(x) = \langle dp(e)(x), \xi_1 \wedge \xi_2 \rangle =$$

$x \in \mathfrak{g}$
$$= \frac{d}{dt}\{f_1, f_2\}(e^{tx})\Big|_{t=0}$$

$$0 = \frac{d^2}{dt^2}\left(\{f_1\{f_2 f_3\}\} + \text{cycle}\right)(e^{tx}) =$$

$$= \left([\xi_1[\xi_2\xi_3]] + \text{cycle}\right)(x), \Rightarrow \text{Jacobi for } [\cdot,\cdot]_{\mathfrak{g}^*}$$

2. Compatibility:

$$\rho(xy) = (Ad_x \otimes Ad_x)\,\rho(y) + \rho(x)$$

$\Rightarrow$ • $0 = (Ad_{y^{-1}} \otimes Ad_{y^{-1}})(\rho(y)) + \rho(y^{-1})$,

• $\rho(y^{-1}zy) = (Ad_{y^{-1}z} \otimes Ad_{y^{-1}z})\,\rho(y) + $

$\qquad + (Ad_{y^{-1}} \otimes Ad_{y^{-1}})\,\rho(z) + \rho(y^{-1})$

• $z = e^{tX}$, $t \to 0$, order 1 elements:

$$\rho(Ad_{y^{-1}}(X)) = (Ad_{y^{-1}} \otimes Ad_{y^{-1}})\,[X, \rho(y)] +$$

$$+ (Ad_{y^{-1}} \otimes Ad_{y^{-1}})\,\delta(X)$$

where $\delta = d\rho(1) : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$, $[X, Y \wedge Z] =$

$= [X,Y] \wedge Z + Y \wedge [X, Z]$

• $y = e^{tY}$, $t \to 0$, order 1 elements:

$$\delta[X,Y] = [X, \delta Y] - [Y, \delta X]$$