Note Title 1/23/2009

Lecture 2. Poisson Lie groups.

1. Poisson algebras.

Def. A pair $(A, \{\cdot,\cdot\})$ where A is a commutative algebra and $\{\cdot,\cdot\}: A^{\otimes 2} \to A$ is a Lie bracket on A is a Poisson algebra if

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

- $\{\cdot,\cdot\}$ = Poisson bracket

<u>One motivation</u>: Consider a family of associative multiplications $*_h$ on a vector space A

$$a *_h b \quad , \qquad a *_h (b *_h c) = (a *_h b) *_h c$$

s.t. $*_0$ is a commutative multiplication ($ab := a *_0 b$).

Assume $a *_h b = ab + m_1(a,b)h + m_2(a,b)h^2 + O(h^3)$

<u>Proposition</u> $\{a, b\} = m_1(a,b) - m_1(b,a)$ is a Poisson bracket.

2. Poisson manifolds.

Let $A = C(\mathcal{M})$ (if $\mathcal{M}$ is smooth, $C(\mathcal{M})$ is the algebra of smooth functions, if $\mathcal{M}$ is affine algebraic, $C(\mathcal{M})$ is the algebra of polynomial functions).

Let $p \in \Gamma(\Lambda^2 T\mathcal{M})$ be a bivector field

In local coordinates $\{x^i\}$:

$$p(x) = \sum_{ij} p^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

The operation

$$\{f, g\} = p(df \wedge dg) \qquad (*)$$

satisfies the Leibnitz identity:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

<u>Def</u>. A pair $(\mathcal{M}, p \in \Gamma(\Lambda^2 T\mathcal{M}))$ is a Poisson manifold if $(*)$ satisfies the Jacobi identity.

Example. $\mathfrak{g}$ = Lie algebra, $\mathfrak{g}^*$ = its dual vector space

$f, g \in C(\mathfrak{g}^*)$, $df(x) \in T_x^* \mathfrak{g}^* \simeq \mathfrak{g}$

$\Rightarrow$ we have $[df(x), dg(x)] \in \mathfrak{g}$

Thm. $\{f, g\}(x) = x([df(x), dg(x)])$ is a Poisson bracket on $C(\mathfrak{g}^*)$.

Proof. hw.

Choose a basis $\{e_i\}$ in $\mathfrak{g}$ and let

$[e_i, e_j] = f_{ij}^k e_k$

Let x_i be coordinate functions on $\mathfrak{g}^*$ corresponding to the dual basis $\{e^i\}$ in $\mathfrak{g}^*$.

Then
$$\{x_i, x_j\} = \sum_k f_{ij}^k x_k .$$

- Product of Poisson manifolds

$$(M_1, p_1) \times (M_2, p_2) = (M_1 \times M_2, p_{12})$$

where $p_{12}(x,y) \in \Lambda^2(T_{(x,y)}(M_1 \times M_2)) =$

$$= \Lambda^2(T_x M_1 \oplus T_y M_2) =$$

$$= \Lambda^2 T_x M_1 \oplus T_x M_1 \otimes T_x M_2 \oplus \Lambda^2 T_y M_2$$

and

$$p_{12}(x,y) = p_1(x) \oplus 0 \oplus p_2(y)$$

- Morphisms of Poisson manifolds:

$$(M_1, p_1) \xrightarrow{\varphi} (M_2, p_2) \quad \begin{pmatrix} \text{Poisson} \\ \text{mapping} \end{pmatrix}$$

if $\varphi : M_1 \to M_2$ is a morphism in the appropriate category of manifolds (smooth, algebraic, etc.) and

$$\varphi^* : C(M_2) \to C(M_1)$$

is a homomorphism of Poisson algebras (of Lie algebras)

Symplectic manifolds

Def. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a closed nondegenerate 2-form.

- In local coordinates non-degeneracy means $\omega = \sum_{ij} \omega_{ij}\, dx^i \wedge dx^j$, ($\det(\omega_{ij})$ does not vanish on M)

Symplectic manifolds are even-dimensional

Or, (ω_{ij}) is invertible,

Or, $\omega : TM \to T^*M$ is an isomorphism of bundles ($\omega(v) = \sum_i \omega_{ij}\, dx^i v^j$, where $v = \sum_j v^j \frac{\partial}{\partial x^j}$)

- nondegeneracy also means

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \quad \text{non-vanishing } (\dim M = 2n)$$

(symplectic volume form)

Symplectic manifolds are known as phase spaces in classical Hamiltonian mechanics.

Example 1. $M_{2n} = k^n \oplus (k^n)^*$ $(k = \mathbb{R}, \mathbb{C})$

$\{q^i\}$ $\{p_i\}$

$$\omega = \sum_i dp_i \wedge dq^i \quad (D)$$

Theorem. Any symplectic manifold has an atlas of local coordinates in which ω has the form (D). These are Darboux coordinates.

Example 2. M - smooth manifold, T^*M has a natural symplectic structure

. In local coordinates:

$\{q^i\}$ - coord. on $U \subset M$, $T^*U \simeq \mathbb{R}^n \times U$

coordinates $\{p_i\}$ in basis $\frac{\partial}{\partial q^i}$

$$\omega = \sum_i dp_i \wedge dq^i$$

Global description: $\omega = d\theta$

(h.w.) define θ.

For a nondegenerate 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ define the bivector field $\omega^{-1} \in \Gamma(\Lambda^2 TM)$

$(\omega : TM \to T^*M, \quad \omega^{-1} : T^*M \to TM)$

$$\omega = \sum_{ij} \omega_{ij}\, dx^i \wedge dx^j, \quad \omega^{-1} = \sum_{ij} (\omega^{-1})^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

Theorem A symplectic manifold is a Poisson manifold with $p = \omega^{-1}$

Proof. $d\omega = 0$ implies the Jacobi identity for $\{f, g\} = \langle \omega^{-1}, df \wedge dg \rangle$

(h.w. prove this)

Def. A Hamiltonian vector field generated by $H \in C(M)$ on a Poisson manifold (M, p) is

$$v_H = p(dH) \in \Gamma(TM)$$

where $p \in \Gamma(\Lambda^2 TM)$ is considered as a linear map $T^*M \to TM$.

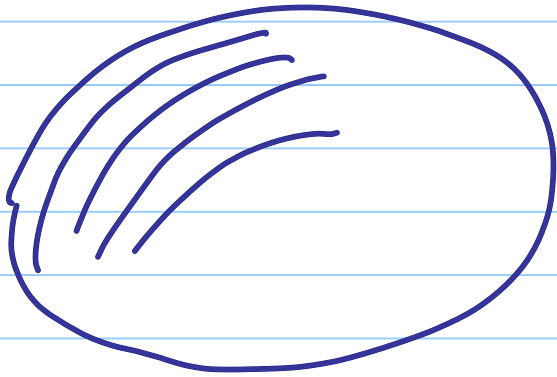
Remark. Classical Hamiltonian mechanics studies Hamiltonian vector fields on symplectic manifolds.

Symplectic leaves:

Let (M, p) be a Poisson manifold.

Def. A *symplectic leaf* through $x \in M$ is the subspace spanned by p.w. Hamiltonian flow lines (flow lines of Ham. v. f.)

Theorem. A symplectic leave is a smooth submanifold. Poisson structure on M induces a symplectic structure on it.



Thus any Poisson manifold produces a collection of symplectic submanifolds.

($p: T^*M \to TM$, $\operatorname{Im} p \subset TM$
sympl. leaves are leaves of $\operatorname{Im} p$)

Example 1. (M, p) is a symplectic manifold. It has unique symplectic leaf, the manifold itself.

Example 2. (1) G acts on $\mathfrak{g}$ via the adjoint action:

- $\mathfrak{g} = T_e G$, $\quad Ad_g(x) = dl_g(dr_{g^{-1}}(x))$

$dr_{g^{-1}}(x) \in T_{g^{-1}} G$, $dl_g(dr_{g^{-1}}(x)) \in T_e G$

dl_g, dr_g are differentials of left and right translations on G

$$l_g(x) = gx \quad , \quad r_g(x) = xg$$

$$dl_g : T_x G \to T_{gx} G$$

$$\left(v \in T_x G , \quad dl_g(v) \cdot f = v \cdot l_g^*(f) \right)$$

- When G is a matrix group, $\mathfrak{g} = Lie(G)$ is a matrix Lie algebra (GL_n, SL_n, SO_n, ...)

$$A_g(x) = g x g^{-1}$$

(2) Corresponding action on $\mathfrak{g}^* = (\mathfrak{g} \to \mathbb{C})$ is the co-adjoint action:

$$\langle Ad_g^*(\ell), x \rangle = \langle \ell , Ad_g(x) \rangle$$

Thm. A symplectic leaf through $x \in \mathfrak{g}^*$ is identical to the coadjoint orbit through x.

Example:

1) SU_2 - compact real form of $SL_2(\mathbb{C})$
(the group of unitary 2×2 unimodular matrices)

$su_2 = Lie(SU_2)$ - 3-dim real

$su_2^* \simeq \mathbb{R}^3$ ($\simeq su_2$ via the Killing form $tr(xy)$)

Coadjoint orbits in su_2^*:

- $\mathcal{O}_x = \{ y \in su_2^* \mid |x| = |y| \} \simeq S^2, x \neq 0$
  $|x|^2 = tr(x^2)$
- $\mathcal{O}_0 = \{0\}$

2) $SL_2(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(SL_2(\mathbb{C}))$

traceless 2×2 complex matrices

$\mathfrak{sl}_2(\mathbb{C})^*$ is a complex algebraic Poisson manifold

$\mathrm{tr}(xy)$ on $\mathfrak{sl}_2(\mathbb{C})$ gives a linear isom.

$$\mathfrak{sl}_2(\mathbb{C})^* \simeq \mathfrak{sl}_2(\mathbb{C})$$

Coadjoint orbits in $\mathfrak{sl}_2(\mathbb{C})^* \simeq$

$\simeq$ Conjugation orbits in $\mathfrak{sl}_2(\mathbb{C})$ (adjoint)

- Complex 2-dimensional orbits

$$\mathcal{O}_{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}} = \left\{ g \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} g^{-1} \,\middle|\, g \in SL_2(\mathbb{C}) \right\}$$

(compl. algebraic sympl. / $\mathbb{C}$)

- 2-dimensional $\mathcal{O}_{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}}$
- 0-dimensional $\mathcal{O}_0 = \{0\}$