LECTURE 35

Review

$$(^{*}) ay'' + by' + cy = G(x) \qquad \text{Non-homogeneous ODE}$$

$$(^{**}) ay'' + by' + cy = 0 \qquad \text{Homogeneous ODE}$$

The general solution of $(^{*})$ is

$$y = \underbrace{y_p(x)}_{\text{particular solution to }^{*}} + \underbrace{c_1y_1(x) + c_2y_2(x)}_{\text{general solution to }^{**}}$$

There are two methods for finding a particular solution to *:

A Undetermined coefficients:

Guess that y_p has a similar form to G. Multiply by x or x^2 if necessary. Then select the constants. (This method does not always work.)

B Variation of parameters (constants):

Look for $y_p = u_1y_1 + u_2y_2$ where y_1, u_2 are functions.

This will always work if these conditions hold:

$$u_1'y_1 + u_2'y_2 = 0$$

$$a(u_1'y_1' + u_2'y_2') = G$$

Note After using either **A** or **B** you will have to use the general solution to the homogeneous equation to get the general solution to *.

Example A : $y'' + y = \tan x$

The form of $G(x)$ is not on the chart, so we can't use the method of undetermined coefficients to guess the form of the solution. We will have to use variation of parameters.

a) Find y_1, y_2 by looking at the homogeneous equation

$$y'' = y = 0$$

The characteristic equation is$r^2 + 1 = 0$, with roots $r = \pm i$

Two linearly independent solutions are

$y_1 = \sin x, \quad y_2 = \cos x$

b) Find the particular solution.

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \quad (a = 1, G = \tan x) \\ &= u_1 \sin x + u_2 \cos x \end{aligned}$$

Conditions for u_1, u_2:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= \tan x \end{aligned}$$

Plug in y_1, y_2:

$(i) \quad u_1' \sin x + u_2' \cos x = 0$

$(ii) \quad u_1' \cos x - u_2' \sin x = \tan x$

Now we want to solve for u_1', u_2'. Multiply (i) by $\sin x$, (ii) by $\cos x$, and add.

$u_1'(\sin^2 x + \cos^2 x) = \tan x \cos x = \sin x$

So $u_1' = \sin x$ and $u_1 = -\cos x$.

(Remember, there is no need to add a constant of integration.) Go back to (i) and solve for u_2'.

$$u_2' = \frac{-u_1' \sin x}{\cos x} = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

Hence

$$\begin{aligned} u_1 &= \int (\cos x - \sec x) dx \\ &= \sin x - \ln |\sec x + \tan x| \end{aligned}$$

Thus

$$\begin{aligned} y_p &= u_1 \sin x + u_2 \cos x \\ &= (-cosx) \sin x \\ &+ (\sin x - \ln |\sec x + \tan x|) \cos x \\ &= -\cos x \ln |\sec x + \tan x| \end{aligned}$$

c) Thus the general solution is

$$\begin{aligned} y &= y_p + c_1 y_1 + c_2 y_2 \\ &= -\cos x \ln |\sec x + \tan x| + C_1 \sin x + c_2 \cos x \end{aligned}$$

Remember that this method **always** works. However, you may not get an explicit answer if you can't work the integrals.

Today's Lecture(Applications of 2nd Order ODE)

In this section we will use the theory of solving 2nd order ODEs, to solve 2 types of application questions:

a) Mechanical systems

b) Electrical circuits

a) Vibrating spring. (You are not responsible for knowing the laws of physics.)

1 No damping.

Suppose we more the weight to a non-equilibrium position and release it. It will oscillate. We want to know how fast it will oscillate, and other factors. To begin to model this system, we first need to choose a reference point.

$m =$ mass of the weight
$x(t) =$ distance from rest position at time t (can be $+$ or $-$)
Newton's Law:

$$\begin{array}{c} mx'' = F^{\nearrow \text{force}} \\ \swarrow \quad \searrow \\ \text{mass} \quad \text{acceleration} \end{array}$$

Hooke' Law: $F = -kx$

$$k = \begin{array}{l} \text{spring constant} \\ \text{(how strong the spring is)} \end{array}$$

The negative sign in this law says that when you move the weight away from its equilibrium position, the spring exhibits a restoring force that tries to move the weight back.

So $mx'' = F = -kx$, and hence $mx'' + kz = 0$

Now that we have written the ODE for the notion of a weight (without damping), the rest is just math-not physics.

Unknown: $x = x(t)$

The characteristic equation is $mr^2 + k = 0$

$$r = \pm i\sqrt{\frac{k}{m}}$$

This is case 3, where the
solution involves sin and cos.
that is, it supports oscillatory
behavior.

General solution: $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$

$\omega =$ omega $= \sqrt{\frac{k}{m}} =$ frequency (in radians/second)

(You can also be given initial conditions, such as how much the weight was initially displaced, and be asked to find the particular solution.)

Remark: We can use various trigonometric substitutions to rewrite the general solution as

$x(t) = A\cos(\omega + \delta)$

$A = \sqrt{c_1^2 + c_2^2} =$ amplitude

$\delta = \cos^{-1}\left(\frac{c_1}{A}\right) = \sin^{-1}\left(\frac{c_2}{A}\right) =$ phase shift

(You don't need to know this.)

2 Damping

Now that we have damping, there will be a new term in the ODE. The faster the mass moves, the more damping there is, so the new term will be depend on the velocity of the mass. We write down the laws of physics again.

Newton's Law: $mx'' = F$

But due to the damping force, we have to add another term into the force equation:

$$F = kx \quad \underset{\substack{\text{restoring}\\\text{force}}}{-cx} \quad - \quad \underset{\substack{\text{damping}\\\text{force}}}{cx'}$$

$c =$ damping constant

This is the ODE for the situation with damping

$$mx'' + cx' + kx = 0$$

To find the solution, look at the characteristic equation.

$mr^2 + cr + k = 0$

Roots:

$$r_1\,,\, r_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Note that the quantity inside the square root can be positive, negative, or zero. This corresponds to the three types (cases of solutions for 2nd order linear ODEs. The solution may not always be oscillatory, especially if the damping is strong.

Case 1 $c^2 - 4mk > 0$

r_1, r_2 are real and distinct.

This is called **overdamped** because the large amount of damping prevents any oscillatory motion.

General solution: $x(t) = c_1 e^{r_t} + c_2 e^{r_2 t}$

Note: Since c, m, $k > 0$ in the quadratic equation, r_1, $r_2 < 0$.

Picture of $x(t)$, $(c_1, c_2 > 0)$

This has exponential decay. There is no oscillatory behavior.

Picture of $x(t)$. $(c_1, c_2$ have different signs)

This corresponds to the case where the mass is given a strong downward push. It will pass the equilibrium point and then slowly go back towards it.

Cases 2 and 3 will be covered in the next lecture.