LECTURE 33

Review

Constant coefficient homogeneous ODE: is a second order equation that may be put in the form:

$$ay'' + by' + cy = 0.$$

idea Try a solution of the form $y = e^{rx}$ and compute an r which satisfies the **characteristic equation**,

$$ar^2 + br + c = 0.$$

When computing the roots of the characteristic equation, there are three cases:

1) $b^2 - 4ac > 0 \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\begin{aligned} y_1 &= e^{r_1 x} \\ y_2 &= e^{r_2 x} \end{aligned}$$

2) $b^2 - 4ac > 0 \qquad r = \frac{-b}{2a}$

$$\begin{aligned} y_1 &= e^{rx} \\ y_2 &= xe^{rx} \end{aligned}$$

3) $b^2 - 4ac < 0 \qquad r_1, r_2 = \alpha + i\beta$ where $\alpha = \frac{-b}{2a} \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a} \qquad i^2 = -1$

$$\begin{aligned} y_1 &= e^{\alpha x} \cos \beta x \\ y_1 &= e^{\alpha x} \sin \beta x \end{aligned}$$

General solution: In all three cases the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

To obtain the third case, we used **Euler's Formula**:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

Here is another way to derive Euler's formula from series:

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \end{aligned}$$

(This series converges absolutely, so we are allowed to rearrange the terms).

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{\sin\theta}$$

Today's Lecture (Non-homogeneous 2nd order linear ODE)

Now we go back and solve the more difficult case of

$$(^*) \qquad ay'' + by' + cy = G(x).$$

Given	Unkown
a, b, c $G(x)$	$y = y(x)$

We will need to study the corresponding homogeneous equation, called the **complimentary equation:**

$$(^{**}) \qquad ay'' + by' + cy = 0.$$

We are reminded by theory that the general solution of $(^{**})$ is $y(x) = c_1y_1(x) + c_2y_2(x)$, where y_1 and y_2 are **linearly independent** solutions of $(^{**})$. But now we evoke a new theorem:

Theorem. Let y_p be any **particular solution** of the non-homogeneous ODE$(^*)$. Then the general solution of $(^*)$ has the form

$$y(x) = y_p(x) + \underbrace{c_1y_1(x) + c_2y_2(x)}_{y_g(x)=\text{general solution of}(^{**})}$$

Proof If y solves $(^*)$, then $y - y_p$ solves $(^{**})$. (You can check this for yourself.)

So the main issue is to find any particular solution y_p of the non-homogeneous ODE $(^*)$. There are two methods to doing this. We will discuss the following method today.

Method of Undetermined Coefficients

To solve $(^*)$ $ay'' + by' + cy = G(x)$, we try to find the particular solution $y = y_p$ with a form **similar to the form of** $G(x)$. The following is a chart with the form of $G(x)$ and the form of the particular solution that you should try.

	$G(x)$	Try y of this form
1)	Ce^{kx}	Ae^{kx}
2)	$C\cos kx + D\sin kx$	$A\cos kx + B\sin kx$
3)	$C_nx^n + \cdots + C_1x + C_0$	$A_nx^n + \cdots + A_1x + A_0$

(*Note*: The chart in the textbook contains typos).

Example 1: Find the general solution of

$$y'' + 4y = \underbrace{x + 4}_{G(x)}$$

Step 1 : First, study (**), the homogeneous (complimentary) equation:

$$y'' + 4y = 0 \quad r^2 + 4 = 0 \quad r = \pm 2i$$

Solutions for this are $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$. Thus the general solution to this complimentary equation is

$$y_c = c_1\cos 2x + c_2\sin 2x.$$

Step 2 : Next, we find a particular solution $y = y_p$ of the non-homogeneous equation (*). Here we use the method of undetermined coefficients.

$$G(x) = x + 4.$$

$$\text{We look for} \quad \begin{array}{c} y = A_1x + A_0 \\ \nwarrow\!\nearrow \\ \text{adjust these} \end{array}$$

$$\begin{aligned} y &= A_1x + A_0 \\ y' &= A_1 \\ y'' &= 0 \end{aligned}$$

We now plug these into our differential equation:

$$\begin{array}{rcl} y'' + 4y & = & 4(A_1x + A_0) + 4A_0 \\ \rightarrow & = & x + 4 \\ \uparrow & & \\ \text{want} & & \end{array}$$

After doing some algebra, we can make this equation true if $A_0 = 1$ and $A_0 = 1/4$. Thus, a particular solution is

$$y = A_1 x + A_0 = 1/4x + 1 = y_p.$$

Step 3 : So the general solution of $y'' + 4y = x + 4$ is

$$\begin{aligned} y &= y_p + y_c \\ &= 1/4x + 1 + c_1 \cos 2x + c_2 \sin 2x \end{aligned}$$

Example 2: Find the general solution of

$$y'' - 7y' + 12y = \underbrace{170 \sin x}_{G(x)}.$$

Step 1 : First, solve for the homogeneous equation.

$$\begin{aligned} y'' - 7y' + 12y &= 0 \\ r^2 - 7r + 12 &= 0 \qquad \text{look at the charateristic equation} \\ (r-3)(r-4) &= 0 \\ r_1 = 3 \quad r_2 &= 4 \end{aligned}$$

$$\text{so our sol. is:} \qquad y_c = c_1 e^{3x} + c_2 e^{4x}$$

Step 2 : Find a particular solution of the form

$$y = y_p = A \sin x + B \cos x.$$

(Note that even though $G(x)$ only has a sin in it, the chart still tells to guess a particular solution containing **both** sin and cos).

$$\begin{aligned} y &= A \sin x + B \cos x \\ y' &= A \cos x - B \sin x \\ y'' &= -A \sin x - B \cos x. \end{aligned}$$

Thus,

$$\begin{aligned} y'' - 7y' + 12y \quad &= (-A \sin x - B \cos x) \\ &\quad -7(A \cos x - B \sin x) \\ &\quad +12(A \sin x + B \cos x) \end{aligned}$$

$$\begin{aligned} &= (-A + 7B + 12A) \sin x + (-B - 7A + 12B) \cos x \\ &= (11A + 7B) \sin x + (11B - 7A) \cos x \quad \underset{\substack{\uparrow \\ \text{want}}}{\rightarrow} \quad = 170 \sin x \end{aligned}$$

So we want

$$\left.\begin{aligned} 11A + 7B &= 170 \\ 11B - 7A &= 0 \end{aligned}\right\} \Rightarrow A = 11, \quad B = 7$$

Step 3 : Thus $y_p = 11\sin x + 7\cos x$ and the general solutions is

$$y = 11\sin x + 7\cos x + c_1e^{3x} + c_2e^{4x}.$$