LECTURE 31

The rest of this semester will go into studying second order differential equations. That is, there will be two derivatives on the unknown function $y(x)$. Remember that the **order** of a differential equation is the order of the highest derivative that occurs in it. In general, second order differential equations are more difficult to solve than first order ODE.

Today's Lecture (Second Order ODEs)

To make things easier, we will only study second order **linear** equations.

Definition:

a) A second order ODE is called **linear** if it can be put in the form:

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

b) This linear second order ODE is called

homogeneous if $G(x) \equiv 0$.

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

(In this context, the word "homogeneous" has a completely different meaning than when it was used to describe a first order ODE $\left(y' = g(\frac{y}{x})\right)$)

Again it is important to keep track of the following:

Given	Unknown
$P(x), Q(x)$ $R(x)G(x)$	$y(x)$

Why Study 2nd Order ODEs?

The basic reason is that many of the laws of science give rise to ODE's, particularly 2nd order. That's because many systems (such as a mass on a spring) have oscillatory behavior, which can only be modeled by a 2nd order ODE.

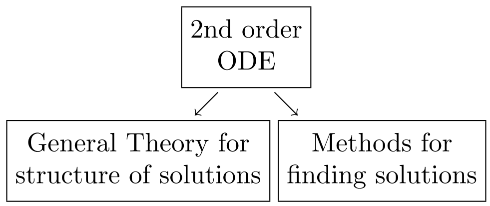
Example 1: (Physics example)
$y(t)$ = position of particle at time t
$y'(t)$ = velocity function
$y''(t)$ = acceleration function

One of Newton's Laws state that force equals mass times acceleration.

$$\underbrace{m}_{\text{mass}} y'' = \underbrace{F}_{\text{force}}$$

You don't need to know this. This is just to show that 2nd order equations show up naturally in nature. Fortunately, 3rd order and higher equations are not common.

The presentation will be divided into two parts.



Theory for 2nd order linear homogeneous ODE

$$(^*) \quad P(x)y'' + Q(x)y' + R(x)y = 0$$

We will see that the solution to this equation will depend on two constants:

Theorem

Suppose that y_1 and y_2 are 2 solution functions to the differential equation *. Then (the new function $y(x)$ is also a solution to *.

What this theorem says is that if you find a solution, then you find another, you can combine those two solutions together, and that combination will also be a solution.

Proof

$$Py'' + Qy' + Ry$$

Lets plug in the new function $y = c_1y_1 + c_2y_2$ to see if it is also a solution (keep in mind that we will have to use that both y_1 and y_2 are solutions:

$$P(c_1y_1'' + c_2y_2'') + Q(c_1y_1' + c_2y_2') + R(c_1y_1 + c_2y_1)$$

$$= c_1 \underbrace{(Py_1'' + Qy_1'Ry_1)}_{=0} + c_2 \underbrace{(Py_2'' + Qy_2' + Ry_2}_{=0}$$

$$= 0$$

Since y_1 and y_2 are solutions, then the expressions inside the parenthesis equal 0. Thus, the linear combination $y = c_1y_1 + c_2y_2$ is a solution.

□

Example 2: $y'' + y = 0$.

We do not know how to solve this yet, but let's try to guess the solutions. By the way, we do not want to guess trivial solutions such as 0. What kind of function, when differentiated twice, will give its negative?

Guess:

$$\begin{aligned} y_1(x) &= \sin x && \text{Notice that these are} \\ y_2(x) &= \cos x && \text{oscillatory solutions.} \end{aligned}$$

Then another solution is $y = c_1 \sin x + c_2 \cos x$. For example, $y = 6\sin x - 15\cos x$ is a solution. In fact, there are infinitely many solutions if we adjust the constants.

Example 3: $y'' - y = 0$.

What's a function, when differentiated twice, that will give itself back?

First guess: $\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= 3e^x. \end{aligned}$

These solutions, when combined will give us another solution, but not one very different from either y_1 or y_2. In order for the solutions y_1 and y_2 to be really useful, they need to be "essentially different", to give good information about the solutions to the differential equation.

Better guess: $\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= e^{-x} \end{aligned}$ These solutions are essentially different.

Thus, $y = c_1 \underbrace{e^x}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$.

Since we are doing math here we need to be more precise; "Essentially different" means that the solutions are linearly independent.

Definition: We call two solutions y_1, y_2 of $(^*)$ **linearly independent** if neither is just a multiple of the other.

Example 3: (continued). $y'' - y = 0$.

$$\left.\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= 3e^x \end{aligned}\right\} \text{These are not linearly independent because one is a multiple of the other}$$

$$\left.\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= e^{-x} \end{aligned}\right\} \text{These are linearly independent.}$$

Why is it so important for the solutions to be **linearly independent**?

Theorem Consider the homogeneous ODE $(^*)$ $Py'' + Qy' + Ry = 0$. Assume y_1 and y_2 are two linearly independent solutions of $(^*)$. Then every solution y of $(^*)$ has the form $y = c_1y_1 + c_2y_2$ for appropriate c_1, c_2.

Proof. Take Math 54.

Methods for solving 2nd order ODEs

Remember that you will have to find two linearly independent solutions. To make things easier, we will only consider an important special case.

Definition: A second order linear homogeneous ODE has **constant coefficients** if it can be put into the form

$$(^{**}) \quad ay'' + by' + cy = 0$$

where a, b, c are constants.

Key idea: We guess that a solution of $(^{**})$ has the form $y = e^{rx}$ where we select the r. How do we choose an r so that it solves the equation? We plug it into the equation.

Plug these in:

$$y' = re^{rx} \qquad y'' = r^2 e^{rx}$$

$$\begin{aligned} ay'' + by' + cy &= ar^2 e^{rx} + bre^{rx} + ce^{rx} \\ &= \underbrace{(ar^2 + br + c)}_{\uparrow} e^{rx} \end{aligned}$$

We want this to be equal to 0 (here we are using 2 facts: (1) the only way this product will be zero is if one of the factors is zero, and (2) $e^x \neq 0$ for any x)

Definition:

$$ar^2 + br + c = 0$$

is called the **characteristic equation** of the differential equation **. If we compute roots that solve this, we get from the quadratic equation:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

So $y(x) = e^{r_1 x0}$ and $e^{r_2 x}$ solve $(^{**})$.

Solving an ODE has been simplified into solving algebraic equation.

When solving for r_1 and r_2, there are 3 cases that will lead to different types of solutions:

Case 1 (Two real roots)

$$b^2 - 4ac > 0$$

Then r_1 and r_2 are real and distinct (different). Thus $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$ are linearly independent solutions of $(^{**})$ $ay'' + by' + cy = 0$. The general solution has the form

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Example 4: Find the general solution of

$$y'' + 6y' + 5y = 0$$

You can factor this (there is no need for the quadratic equation, you should be able to do it by inspection):

$$(r + 5)(r + 1) = 0$$

So the roots are $r_1 = -5$, $r_2 = -1$ and two linearly independent solutions are $y_1(x) = e^{-5x}$, $y_2(x) = e^{-x}$.

Therefore, the general solution is

$$y(x) = c_1 e^{-5x} + c_2 e^{-x}$$

The other two cases will be covered in the next lecture.