LECTURE 29

Review

Separable ODE: $y' = g(x)f(y)$

Homogeneous ODE: $y' = g\left(\frac{y}{x}\right)$

Geometric method for seeing the solution of the differential equation: $y' = F(x, y)$

Draw a picture with little line segments, each which has a slope $F(x, y)$ corresponding to its position.

Then draw one or more solution curves, which are tangent to the direction field.

With this method, even if you can't explicitly solve the ODE, you can at least qualitatively see the solution.

Example A (Physical Example) This is the problem from the end of the last lecture. In this case, once you have the physics straightened out, the rest is just math.

Today's Lecture

Keep in mind that there is no one method that solves all differential equations. However,there are methods for solving certain classes of ODE's. The last type of first order differential equation that we will study is called **linear**.

Definition A first order ODE is **linear** if it has the form

$$y' + P(x)y = Q(x)$$

Again in these abstract definitions it is important to be clear about which of the objects are given to you and which are unknown. Here we have

Given	Unkown
$P(x)\,Q(x)$	$y = y(x)$

As with all first order ODE's,you could be asked to find the **general solution**, which involves an arbitrary constant. Then you could be asked to find a **specific** solution, where you are given an initial condition, and you solve for the constant. What we usually did was perform algebra until we got it into one of the forms. With this case, we also need to develop a procedure to get it into a form we can solve.

Idea (of solutions to linear ODEs)

The equation of this form does not look easy to solve, since y appears in two different terms. However, we could try to combine these into one term involving a derivative. By itself, this idea will not work, but what if we multiplied both terms by some function I, which we call in **integrating factor**? We choose it in such a way that the two terms of y can be combined into one. In other words,

$$\begin{array}{rcl} y' + P(x)y & = & Q(x) \\ \underbrace{Iy' + IP(x)y}_{\text{mult. by } I} & = & IQ(x) \end{array}$$

This looks worse, but since we get to pick the function I, we can try to pick I so that the left-hand side becomes a total derivative, namely $(Iy)'$. So we want to be able to pick a function I so that:

$$\begin{array}{rcl} Iy' + IPy & = & (Iy)' \\ & = & Iy' + I'y \end{array}$$

Now notice that we have an Iy' on both sides of the equation: and we get:

$$\begin{array}{rcl} IPy & = & I'y \\ & \text{so} & \\ I' & = & IP \end{array}$$

How do we figure out an I so that this works? We solve the equation $I' = PI$. This equation is separable, so we can solve it by moving I to the left side and integrating. After some algebra, the answer we get is

$$I = Ae^{\int P\,dx}$$

This will work for any A, since it satisfies the condition $I' = PI$. To make things easier, we take $A = 1$. So

$$I = e^{\int P\,dx}$$

Now go back to the equation we originally wanted to solve: $y' + Py = Q$

Multiplied by I, we get: $\underbrace{Iy' + IPy}_{=(Iy)'} = IQ$.

With the function I that we've chosen, we know that $(Iy)' = IQ$ is true. Now we can integrate.

$$\begin{array}{rcll} Iy & = & \int IQdx + C & \qquad \text{Solve for } y \\ y & = & I^{-1}\big(\int IQdx + C\big) & \end{array}$$

Summary

How to solve linear first order ODE:

Step 1: Compute integrating factor.

$$I = e^{\int P\,dx}$$

Step 2: Multiply $y' + Py = Q$ by I.

$$(Iy)' = Iy' + IPy = IQ$$

Step 3: Integrate: $Iy = \int IQdx + C$

Step 4: Solve for y: $y = I^{-1}\big(\int IQdx + C\big)$

Example 1 Solve $xy' + 2y = e^{x^2}$

First, divide by x to get it into the right form.

$$y' + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{\frac{e^{x^2}}{x}}_{Q(x)}$$

Step 1:

$$\int P(x)dx = \int \frac{2}{x}dx = 2\ln|x| = \ln(x^2)$$

Note that when we integrate P, we don't need to add a constant, because that constant, when it becomes a power of e, will become part of A.

So $I = e^{\int P\,dx} = e^{\ln(x^2)} = x^2$

Step 2: Go back to the original equation $y' + \frac{2}{x}y = \frac{e^{x^2}}{x}$ and multiply it by $I = x^2$.

$\underbrace{x^2y' + 2xy}_{(x^2y)'} = xe^{x^2}$

This looks bad, but the left side is a derivative.

Step 3: Integrate:

$$\begin{aligned} x^2y &= \int xe^{x^2}dx + C \quad \text{Here, we need the constant.} \\ &= \frac{e^{x^2}}{2} + C \end{aligned}$$

Step 4: Solve for $y : y = \frac{e^{x^2}}{2x^2} + \frac{C}{x^2}$

Example 2

Solve $y' + \underbrace{y}_{P(x)=1} = \underbrace{x + e^x}_{Q(x)}$ with the initial condition $y(0) = 0$

We are asked to find the general solution, and the particular solution.

Step 1: Compute the integrating factor.

$$\int P(x)dx = x \quad I = e^{\int P(x)dx} = e^x$$

Step 2 and 3 Multiply and integrate:

$\underbrace{e^x y' + e^x y}_{(e^x y)'} = xe^x + e^{2x}$

$\Rightarrow e^x y = \int (xe^x + e^{2x})dx + C \leftarrow$ integrate by parts

$= xe^x - e^x + \frac{e^{2x}}{2} + C$

Step 4: Solve for y:

$y = x - 1 + \frac{e^x}{2} + Ce^{-x}$ General solution

Plug in the initial condition to find the particular solution.

$$0 = y(0) = 0 - 1 + \frac{e^0}{2} + Ce^0 \quad = -1 + 1/2 + C$$
$$\Rightarrow C = 1/2$$

Therefore, $y = x - 1 + \frac{e^x}{2} + 1/2e^{-x}$

Example 3(Application: Electric circuit)

. Here is another example of how differential equations show up in science. Once we get through the physics of setting up the equations, solving them is the same as solving a regular linear ODE.

The **resistor** (zig zag) "resists" current flow. The **inductor** (coil) resists changes in the current. Closing the switch starts the current.

We want to find $I(t)$, the current as a function of time. According to Kirchoff's Laws (This is stated; you don't need to know how to derive it),

$L\frac{dl}{dt} + RI = E$

Example. $R = 12$ $L = 4$ $E = 60$ (Don't worry about the usits)

The only difference is that the unknown is $I(t)$ instead of $y(x)$.

Suppose you close the switch at time 0. $I(0) = 0$. Find $I(t)$.

$$4I' \quad = \quad 12I \quad = 60$$

$$I' \quad + \quad \underbrace{3I}_{P} \quad = \underbrace{15}_{Q}$$

We won't use the symbol I again for the integrating factor. We will just say that the integrating factor $= r^{\int Pdt} = e^{3t}$.

Thus

$$\begin{array}{rcll}
\underbrace{e^{3t}I' + 3t^{3t}I}_{(e^{3t}I)'} & = & 15e^{3t} & \\
e^{3t}I & = & \int 15e^{3t}dt + C & \text{Integrate both sides} \\
 & = & 5e^{3t} + C & \\
1 & = & 5 + Ce^{-3t} & \text{evaluate at } t = 0 \\
I(0) = 0 & = & 5 + C & \\
C & = & -5 &
\end{array}$$

Thus

$$I(t) = 5 - 5e^{-3t}$$

This solution makes sense. At the beginning, the current I is 0. The current then rises, with the inductor slowing down the rise. At time infinity, the circuit has reached a steady state with a constant current of 5 (amps). The inductor doesn't do anything because the current is now constant.

Example 4 Do the same problem, except that the voltage source is an alternating current.

$E(t) = 60\sin(30t)$

Go back to the previous example and plug in the new value of E.

$4I' + 12I = 60\sin(30t)$

So then we have:

$$I' + 4\underbrace{15\sin(30t)}_{Q}$$

Integrating factor $= e^{3t}$

$$\begin{array}{rcll}
\underbrace{e^{3t}I' + 3t^{3t}I}_{(e^{3t}I)'} & = & 15\sin(30t)e^{3t} & \\
e^{3t}O & = & \int 15e^{3t}\sin(30t)dt + C & \\
e^{3t}I & = & \frac{15}{909}e^{3t}(30\sin(30t) - 30\cos(30t)) + C & \text{Integrate by parts} \\
I(t) & = & \frac{15}{909}e^{3t}(30\sin(30t) - 30\cos(30t)) + \frac{C}{e^{3t}} &
\end{array}$$

In this case, the current responds to the alternating voltage source $E(t)$ by oscillating.