

## LECTURE 24

### Review

**Taylor Series:** If there exists a power series expansion in  $|x - c| < R$ , then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

Important Special Case: Binomial Series.

$$f(x) = (1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad |x| < 1$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}, \quad \binom{k}{0} = 1$$

where  $k$  is not necessarily a positive integer.

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### Today's Lecture

We have learned that we could get an approximate answer by generating a Taylor series and cutting it off after a finite number of terms. Now we need to know how big this error will be. First, we need to introduce some terminology.

**Definition:** The  $n$ th partial sum in the Taylor series is

$$\begin{aligned} T_n(x) &= \sum_{i=0}^n \frac{f^{(i)}(c)}{i!} (x - c)^i \\ &= f(c) + f'(c)(x - c) + \frac{f''(c)(x - c)^2}{2!} + \dots + \frac{f^{(n)}(c)}{n!} (x - c)^n \end{aligned}$$

In the case of Taylor series, the  $n^{\text{th}}$ -partial sum is called the  $n^{\text{th}}$  **Taylor polynomial**

### Geometric Interpretation of Taylor Polynomials

Suppose you are given a function  $f(x)$

What, geometrically, does each Taylor polynomial look like? They are supposed to approximate the curve near the point  $c$ . The Taylor polynomial for  $n = 0$  is a constant.

How about  $n = 1$ ? Now we will get a straight line approximation. Differential calculus is based upon this picture.

How about  $n = 2$ ?  $n = 3$ ? We are now approximating with parabolas and cubic equations. We saw something like this before in Simpson's Rule; we saw

that a parabola gave a better fit than a line. A cubic equation should fit even better.

When you take higher degree polynomials, you get better approximations. However as in all situations where we approximate we must also consider how far we are off by, which in this case involves finding the difference between the function and the Taylor polynomial,

### Finding the Formula for the Remainder

**Definition:** The **remainder** of the  $n^{th}$  Taylor polynomial is

$$R_n(x) = f(x) - T_n(x)$$

We would like to figure out how big the error (or remainder) is, just as in previous approximation theories (such as Simpson's Rule). How we approach this topic here is different than how it is done in the book. The basic idea of our method will be to generate Taylor polynomials by using integration by parts.

What is  $f(x)$  near  $c$ ?

Recall the formula for integration by parts:

$$\int_a^b fg' dx = fg|_a^b - \int_a^b f'g dx$$

where  $fg|_a^b = f(b)g(b) - f(a)g(a)$

We start off by writing  $f(x)$  as

$$f(x) = \underbrace{f(c)}_{T_0} + \underbrace{\int_c^x f'(t) dt}_{=f(x)-f(c)} \quad \begin{array}{l} \text{Fundamental Theorem} \\ \text{of Calculus.} \end{array}$$

The first term is the constant term of the Taylor polynomial,  $T_0(x)$ . This can also be written as

$$f(x) = f(c) - \int_c^x (x-t)' f'(t) dt$$

where the prime ( $'$ ) means  $d/dt$ .

Now we do an integration by parts on the integral:

$$\begin{aligned} f(x) &= f(c) - (x-t)f'(t)|_c^x + \int_c^x (x-t)f''(t) dt \\ &= \underbrace{f(c) + (x-c)f'(c)}_{T_1(x)} + \int_c^x (x-t)f''(t) dt \end{aligned}$$

Now we have the next term of the Taylor polynomial. Keep on integrating...

$$= f(c) + f'(c)(x - c) - \int_c^x \left[ \frac{(x-t)^2}{2} \right]' f''(t) dt$$

Keep doing this and a term will pop out every time.

$$\underbrace{f(c) + f'(c)(x - c) + 1/2 f''(c)(x - c)^2}_{T_2(x)} + \int_c^x \frac{(x-t)^2}{2} f'''(t) dt$$

$$\vdots$$

$$f(c) + f'(c)(x - c) + \dots + \frac{f^{(n)}(c)(x-c)^n}{n!} \leftarrow T_n(x)$$

$$+ \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \leftarrow R_n(x)$$

Thus  $f(x) = T_n(x) + R_n(x)$ . So

$$R_n(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$$

According to a version of the Mean Value Theorem (which we will not prove here as its proof is beyond the scope of this class), there exists a point  $z$  between  $c$  and  $x$  such that you can pop  $f^{(n+1)}(z)$  out in front of the integral.

$$R_n(x) = f^{(n+1)}(z) \int_c^x \frac{(x-t)^n}{n!} dt$$

So  $R_n(x) = \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$  for some point  $z$  between  $c$  and  $x$ .

This is an important formula which allows us to estimate the error of replacing  $f(x)$  by  $T_n(x)$ .

Now we will do some concrete examples with this formula.

**Example 1:** Use Taylor polynomials to estimate  $e^{\frac{1}{4}}$  to within 0.0001.

$$f(x) = e^x \quad f(1/4) = ?$$

We want to expand this function about a value which we know, and then “jiggle” it over to  $1/4$ . This is called a “perturbation method”.

We start with noting that we know the value of  $e^x$  at a point near  $\frac{1}{4}$

$$f(0) = e^0 = 1 \quad c = 0$$

Now from the theory that we have developed today we know:

$$\underbrace{f(x)}_{e^x} = T_n(x) + R_n(x) \Rightarrow f(x) - T_n(x) = R_n(x)$$

When  $x = 1/4$ , how big should  $n$  be? That is, how big should  $n$  be so that

$$|R_n(1/4)| < 0.0001?$$

We know that there exists a  $z$  such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

Fortunately, this function is easy to differentiate.

$$f(x) = e^x \text{ always differentiates to itself so } f^{(n+1)}(x) = e^x$$

$$R_n(1/4) = \frac{e^z}{(n+1)!} (1/4)^{n+1} < 0.0001, \text{ when}$$

$0 \leq z \leq 1/4$ . Lets compute  $z$  for the worst case.

$$e^z \leq e^{1/4} \leq 2 \text{ for sure, because } (e^{1/4})^4 = e \leq 2^4.$$

$$\text{So } |R_n(1/4)| \leq \frac{2}{(n+1)!} \cdot \frac{1}{4^{n+1}} < 0.0001.$$

If you work this on a calculator, then you will find out that this works for  $n = 4$ . This would be a fair test question to be worked by hand, if  $n$  is a small number.

$$\text{So } |e^{1/4} - T_4| < 0.0001$$

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\begin{aligned} T_4(1/4) &= 1 + 1/4 + \frac{1}{2!} \left(\frac{1}{4}\right)^2 + \frac{1}{3!} \left(\frac{1}{4}\right)^3 + \frac{1}{4!} \left(\frac{1}{4}\right)^4 \\ &= 1.28402\dots \text{ The last digit may be off} \end{aligned}$$

The purpose of doing this is to know exactly what the biggest error could possibly be, and therefore reduce it accordingly by taking enough terms.

**Example 2:** Approximate  $f(x) = x^{1/3}$  by  $T_2(x)$  around  $c = 8$ .

The reason we chose  $c = 8$  is because it is easy to calculate  $f(c) = 8^{1/3} = 2$ .

$$T_2(x) = f(8) + f'(8)(x-8) + \frac{1}{2!} f''(8)(x-8)^2$$

$$\begin{array}{ll} f(x) = x^{1/3} & f(8) = 2 \\ f'(x) = \frac{1}{3}x^{-2/3} & f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12} \\ f''(x) = (1/3)(-2/3)x^{-5/3} & f''(8) = -\frac{1}{144} \end{array}$$

Hence  $T_2(x) = 2 + 1/12(x - 8) - 1/288(x - 8)^2$ .

This example will be continued in the next lecture.