LECTURE 22

Overall orientation of the course

There are three big parts to this course. The first is techniques for integration, the middle part is what we are on now, sequences and series, and starting next week, we will be doing ordinary differential equations.

Sequences and series. Two parts were involved when studying sequences and series. The first was using tricks such as the Root and Ratio Tests to see if a series of numbers converged. Then instead of looking at just a series of numbers, we looked at a series with powers of x, the power series. This gave powerful methods for taking "crummy" functions and systematically finding ways to approximate them using polynomials. Or, we could obtain a perfect fit by taking "infinite polynomials."

Review

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

converges absolutely if $|x-c| < R$, where R is the radius of convergence. If you go outside of R, the series diverges. What happens at the endpoints is questionable.

You can differentiate and integrate the power series form by term.

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C$$

Taylor Series. Before, we were given the coefficients a_0, a_1, a_2, $a_3, \ldots$ and were asked to define $f(x)$. In practice, it is more important go go the other way. So how do we compute a_0, a_1, a_2, $a_3 \ldots$ in terms of $f(x)$?

$a_n = \frac{f^{(n)}(c)}{n!}$ 0!=1

So $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

What we get is the formula for the **Taylor Series**, an extremely important formula.

Note: When $c = 0$, the series is called a Maclaurin series.

Important Examples:

a) Power series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad R = \infty$$

b) Power series for $= \sin x$

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad R = \infty \end{aligned}$$

What is the Taylor series for $\cos x$? There are two ways to compute it. One is by using the methods above, taking the derivatives of cosine. An easy trick would be to differentiate the series for $\sin x$ since the derivative of $\sin x$ is $\cos x$.

c) Power series for $\cos x$

$$\begin{aligned} \cos x &= \frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{Bring derivative inside} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad R = \infty \end{aligned}$$

d) Power series for $\ln(1 + x)$

$$\begin{aligned} \ln(1 + x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad R = 1 \end{aligned}$$

Today's Lecture: (Methods of Generating Taylor Series)

a) Use $a_n = \frac{f^{(n)}(c)}{n!}$

b) Differentiate another Taylor series

c) Integrate another Taylor series

d) Substitution

e) Multiply Taylor series

f) Divide Taylor series

Example 1 : (Substitution). Find the Taylor series of $f(x) = e^{x^2}$ $(c = 0)$. You don't want to do this problem by a), because as you start differentiating, the function will get more complicated. Instead, we will use substitution with

$$y = x^2$$

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

So $e^{x^2} = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$

Now, how do we find $\int e^{x^2} dx$? The answer cannot be expressed as a closed function, but by using the above rules, we can find an approximation by integrating the series term by term...

Since $e^{x^2} dx = x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \dots + C$ we know

$$\int_0^1 e^{x^2} dx = 1 + \frac{1}{3} + \frac{1}{5 \cdot 2!} + \frac{1}{7 \cdot 3!} + \dots$$

This infinite series gives the **exact** answer. To actually **compute** this, you can do out as many terms as you like.

Note: Since we are inside the endpoints, the series is **absolutely** convergent. This means that the series is more robust and that we can do operations on it without worrying like we had to when dealing with **conditionally** convergent series.

Example 2: (Multiplying Taylor series).

Find the Taylor series $(c = 0)$ of $f(x) = \cos x \sin x$. First, we will do this without trignometric identities.

$$\begin{aligned} f(x) &= (\cos x)(\sin x) \\ &= (1 - x^2/2! + x^4/4! - \dots)(x - x^3/3! + x^5/5! - \dots) \end{aligned}$$

When we multiply the terms out, the theory says that the answer can be expressed as $c_0 + c_1 x + c_2 x^2 + \dots$ We group the terms according to their powers and ask ourselves how many times do each terms show up. "How many times will the constant term occur?" (never) "How many times will x occur?", etc. We then get out a big piece of paper and multiply the terms by arithmetic.

$$
\begin{array}{cccccc}
 & 1 & 1\frac{1}{2}x^2 & +\frac{1}{4}x^4 & -\frac{1}{6}x^6 & \ldots \\
\times & x & -\frac{1}{3}x^3 & +\frac{1}{5}x^5 & -\frac{1}{7}x^7 & \ldots \\
\hline
 & x & -\frac{1}{3!}x^4 & +\frac{1}{4!}x^5 & -\frac{1}{6!}x^7 & \\
+ & -\frac{1}{3!}x^3 & +\frac{1}{(2!)(3!)}x^5 & -\frac{1}{3!}4!x^7 & +\frac{1}{(3!)(6!)}x^9 & \\
+ & \ldots & & & & \\
\hline
 & x & -\frac{4}{3!}x^3 & +\frac{16}{5!}x^5 & \ldots &
\end{array}
$$

Note that these are all power series, and that we are only writing the first few terms. This method is tedious, but we have computers for that. We can now check this result by using the trig identity.

$f(x) = \cos x \sin x = 1/2 \sin 2x$. Let $y = 2x$

$$\sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$$

So we see that

$$
\begin{aligned}
\cos x \sin x &= 1/2 \sin 2x \\
&= \tfrac{1}{2} \sum_{n=0}^{\infty} (-1)^n \tfrac{(3c)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \tfrac{2^{2n} x^{2n+1}}{(2n+1)!} \\
&= x - \tfrac{4}{3!}x^3 + \ldots
\end{aligned}
$$

The answers match!

Example 3 (Dividing Taylor Series).

Find the Taylor series for $f(x) = \tan x = \frac{\sin x}{\cos x}$.

For this, we need an even bigger piece of paper.

$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/3! + x^5/5! \ldots}{1 - x^2/2! + x^4/4! \ldots} = \frac{x - x^3/6 + x^5/120 \ldots}{1 - x^2/2 + x^4/24 \ldots}$

$-1/2x^2 + 1/24x^4 \frac{x + 1/3x^3 + 2/15x^5}{x - 1/6x^3 + 1/120x^5 \ldots}$

$\frac{x - 1/2x^3 + 1/24x^5 \ldots}{1/3x^3 - 1/30x^5 \ldots}$

$\frac{1/3x^3 - 1/6x^5}{2/15x^5 \ldots}$

Therefore, $\tan x = x + 1/3x^3 + 2/15x^5 \ldots$

Again, this is tedious calculation. Here, the moral is that there are many ways to generate Taylor series, and so this is a powerful and useful technique.