

LECTURE 20

Review

What we did last time was to change a series of numbers into a series with powers of x . The issue for any series is *Does this converge?*

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{Power Series}$$

$$\sum_{n=0}^{\infty} a_n (x - c)^n \quad \text{Power Series centered at } c$$

For any real number x that you plug in, the power series becomes a regular series. So for which x does a given power series converge?

Theorem: Let $\sum_{n=0}^{\infty} a_n (x - c)^n$ be a Power Series centered at c . Then there are three possibilities:

- a) The series converges only for $x = c$. ($R = 0$)
- b) The series converges for all x . ($R = \infty$)
- c) There exists a number $0 < R < \infty$ such that:
 - 1) The series converges absolutely if $|x - c| < R$.
 - 2) The series diverges if $|x - c| > R$.

R is called the **radius of convergence**. We use the Ratio or Root Tests to compute R .

However, this Theorem does not say what happens at the endpoints.

Example A: $\sum_{n=0}^{\infty} \underbrace{\frac{n}{4^n} (2x - 1)^n}_{u_n}$ Converges for what x ?

First, convert it to the power series form:

$$\begin{aligned} u_n &= \underbrace{n \frac{2^n}{4^n}}_{a_n} \underbrace{(x - 1/2)^n}_c \\ &= n(1/2)^n (x - 1/2)^n \end{aligned}$$

Use the Ratio Test and fix x .

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} &= \lim_{n \rightarrow \infty} \frac{(n+1)(1/2)^{n+1} |x - 1/2|^{n+1}}{n(1/2)^n |x - 1/2|^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) \frac{1}{2} |x - 1/2| \\ &= 1/2 |x - 1/2| \\ &< 1 \end{aligned}$$

When this is true, the series will converge.

$$1/2|x - 1/2| < 1 \text{ if and only if } \begin{array}{ccc} |x - 1/2| < 2 & & \\ \uparrow & & \uparrow \\ c & & R \end{array}$$

Example B :

$$\sum_{n=0}^{\infty} \underbrace{\frac{n}{4^n} (x + 10)^n}_{u_n} \quad u_n = \underbrace{\frac{n}{4^n}}_{a_n} (x - \underbrace{(-10)}_c)^n$$

For variety, we will now use the **Root Test**.

$$\begin{aligned} \lim_{n \rightarrow \infty} |u_n|^{1/n} &= \lim_{n \rightarrow \infty} \frac{n^{1/4}}{4} |x + 10| \\ &= 1/4 |x + 10| \\ &< 1 \end{aligned}$$

Recall that $\lim_{n \rightarrow \infty} n^{1/4} = 1$ (Use logs if you forget.)

$$\frac{|x + 10|}{4} < 1 \quad \text{if} \quad |x - \underbrace{(-10)}_c| < \underbrace{4}_R$$

When asked to compute the radius of convergence, we don't have to worry about the endpoints unless we are specifically asked to consider them.

Once we get past the theoretical baggage, these problems are straightforward. We now use the theory to make stronger connections with calculus.

Today's Lecture(Manipulating Power Series)

Main idea: Let $\sum_{n=0}^{\infty} a_n(x - c)^n$ be a power series with $0 < R \leq \infty$. We will ignore the trivial case where $R = 0$. We will *define* and then *study* the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n \quad \text{for} \quad |x - c| < R \quad (*)$$

$f(x)$ can be thought of as an "infinite polynomial", defined over a certain region.

Suppose you were asked to differentiate or integrate $f(x)$. Normally, you should be cautious about doing operations on infinite things, but in this case, *you can differentiate and integrate a power series "term by term"*.

Theorem: Define $f(x)$ by (*). Then

$$1) f'(x) = \sum_{n=0}^{\infty} n a_n (x - c)^{n-1}$$

$$2) \int f(x) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1} + C$$

In particular, the new power series in 1) and 2) converges absolutely in the same region, that is, if $|x-c| < R$.

Example 1: Recall the Geometric series.

$$f(x) = \underbrace{\sum_{n=0}^{\infty} x^n}_{\text{power series}} = \frac{1}{1-x}$$

$c = 0$

Note that $f(x)$ is not a polynomial. On the other hand, it is expressed as an “infinite polynomial”. Suppose we integrate both sides.

$$\int f(x) dx = \int \frac{dx}{1-x} = -\ln(1-x) + C_1,$$

We do not need the absolute value signs because $|x| < 1$. Also by the theorem,

$$\int f(x) dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C_2.$$

$$\text{So, } -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \quad |x| < 1$$

(We’ve combined C_1 and C_2 into C .)

What is C ? Let $x = 0$ then we see that $0 = -\ln 1 = C$.

Hence we now can express $\ln(1-x)$ as a power series centered around 0

$$\begin{aligned} \ln(1-x) &= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots |x| < 1 \end{aligned}$$

Replace x by $-x$ and we get the following expression:

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad |x| < 1 \end{aligned}$$

Now let $x = 1$. (Not justified since $|1| = 1$. However, the resulting series converges by the Alternating Series Test.)

$$\ln 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots$$

The method is an example of what is called a **Taylor series**, where a function is broken down into a polynomial. This is useful in the applied sciences, where a terrible looking function is changed into a power series, and the higher order terms can be discarded while still maintaining a high level of accuracy.

Example 2 :

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= \sum_{n=0}^{\infty} (-x^2)^n |x| < 1 \\ &= 1 - x^2 + x^4 - x^6 + \dots\end{aligned}$$

Now integrate both sides term by term:

$$\begin{aligned}\int \frac{dx}{1+x^2} &= \tan^{-1} x + C \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

Compute C by taking $x = 0$ which gives us that $C = 0$.

Hence we now have a power series expression for $\tan^{-1} x$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad |x| < 1$$

If you plug in $x = 1$, this will give you the formula for $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$