**LECTURE 19**

Review

**Problem**: This problem almost made it onto the midterm.

Let $b_n \geq 0$ for all $n$ and suppose $\sum_{n=1}^{\infty} b_n$ converges.

Show that $\sum_{n=1}^{\infty} b_n^2$ converges.

Solution. $\sum_{n=1}^{\infty} b_n$ converges, so $\lim_{n\to\infty} b_n = 0$.

Use the Limit Comparison Test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{b_n^2}{b_n} = \lim_{n\to\infty} b_n = 0$$

Therefore, $\sum_{n=1}^{\infty} b_n^2$ converges.

---

Today's Lecture

Now our intent will be to show some reasons why people study series. One reason is that series give ways to approximate a function with a polynomial. These approximation methods are important in fields such as engineering.

Important applications of Series

Approximating functions using power series:

Basic Ideas.

1. A polynomial is a function of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

   Polynomials are easy to study since it is easy to compute $P'$, $\int P\,dx$, etc.

2. An arbitrary $f$ can be approximated by a sequence of polynomials$s_n(x)$ .

$$s_n(x) = \sum_{i=0}^{n} a_i x_i$$

   If this polynomial is built correctly, then the error should go to zero as you take $n$ to $\infty$.

$$\underbrace{|f(x) - s_n(x)|}_{\text{error}} \to 0 \quad \text{as} \quad n \to \infty.$$

3. In fact, we can (often) write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Here we are using a series that involves a variable $x$ (and its powers) which you may think of as an infinite polynomial. Such series are called **Power Series**. Think of the Power Series as what happens when you have infinitely many terms. Carrying out this process of finding an approximating polynomial gives you a **Taylor Series**, also known as a **Maclaurin Series**.

**Example 1**:(Important examples of power series)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Note that when you plug in $x = 1$ in the last formula, you get $\ln 2 = 1 - 1/2 + 1/3 - \dots$

This method of approximating functions is very powerful and is used by many, including applied mathematicians and engineers. Sometimes, when you run into a terrible function and all else fails, you will want to expand it into a power series. Also, this procedure is easy to do on a computer. For ordinary differential solutions, it allows you to get an approximation to the solution of a problem that would otherwise be unbelievably difficult...

**Main Questions**

a) Given $f(x)$, how do we compute $a_0, a_1, a_2, \dots$?

b) For which $x$ does the series $\sum_{n=0}^{\infty} a_n x^n$ converge?

We will address the theoretical issues first by discussing b).

**Definition**:

i) A series of the form $\sum_{n=0}^{\infty} a_n x^n$ is called a **power series**. (They are called this not because they are "powerful", but because they have powers of $x$.)

ii) A series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$ is called a **power series centered at** $c$.

This is the same as above except that it is shifted from 0 to $c$.

iii) $a_0, a_1, a_2, \ldots$ are the **coefficients**. At this point, assume that the coefficients are given.

The Question of Convergence Remember that just because you can write down a series, this does not mean that it converges. Given $a_0, a_1, a_2, \ldots$ for which $x$ does the power series converge? To answer this, we freeze the value of $x$ and use the convergence tests. (Hint – use Ratio/Root Test.)

**Example 2**:

$$\sum_{n=0}^{\infty} \underbrace{\overbrace{\frac{1}{n!}}^{a_n} x^n}_{u_n} \qquad (0! = 1)$$

Fix a point $x$. Use the Ratio Test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} \\ &= \lim_{n\to\infty} \frac{|x|}{n+1} \\ &= 0 < 1 \end{aligned}$$

So for all $x$, $\sum\limits_{n=0}^{\infty} \frac{1}{n!} x^n$ converges.

**Example 3**:

$$\sum_{n=1}^{\infty} \underbrace{\overbrace{\frac{1}{n}}^{a_n} x^n}_{u_n}$$

Notice that $n$ starts at 1 because you cannot divide by 0

Fix $x$, use the Ratio Test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{n+1}}{\frac{|x|^n}{n}} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)|x| \\ &= |x| = L \end{aligned}$$

If $|x| < 1$, then $\sum\limits_{n=1}^{\infty} \frac{1}{n} x^n$ converges.
If $|x| < 1$, then the series diverges.
If $x = 1$, the series diverges.
If $s = -1$, the series converges by the Alternating Series Test.

This example shows that a power series may converge for some values of $x$, and not for others.

In general, your region of convergence will always be a single, unbroken strip. It may be infinitely long, or it may only be a point.

**Example 4** :

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-6)^n$$

This is the same as the previous example, except that everything is shifted over by 6.

Theorem: Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ be a Power Series centered at $c$. Then there are three possibilities:

a) The series converges only for $x = c$. $(R = 0)$

b) The series converges for all $x$. $(R = \infty)$

c) There exists a number $0 < R < \infty$ such that:

  1) The series converges absolutely if $|x - c| < R$.

  2) The series diverges if $|x - c| > R$.

**Definition** We call $R$ the **radius of convergence** of $\sum_{n=0}^{\infty} a_n(x-c)^n$.

However, the theory gives no information about convergence at the endpoints $x = c + R$, $x = c - R$.

**Example 5** :

$$\sum_{n=0}^{\infty} \underbrace{\overbrace{n!}^{a_n}(x - \overbrace{1}^{c})^n}_{u_n}$$

Compute $R$.

Use the Ratio Test.

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!|x-1|^{n+1}}{n!|x-1|^n} \\ &= \lim_{n\to\infty} (n+1(x-1) \\ &= \infty \quad \text{unless} \quad x = 1. \end{aligned}$$

This series diverges except for $x = 1$.
(Case a: $R = 0$.)