LECTURE 16

Review

Comparison Test.

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with a_n, $b_n \geq 0$.

(Typically, your b_n will be a p-series or a geometric series).

a) If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

b) If $a_n \geq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

We don't have to worry about the first few terms, only what happens as $n \to \infty$.

Limit Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $a_n, b_n \geq 0$.

a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either **both** converge, or **both** diverge.

b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and if $\sum_{n=1}^{\infty} b_n$ converges,

then $\sum_{n=1}^{\infty} a_n$ converges.

c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and if $\sum_{n=1}^{\infty} b_n$ diverges,

then $\sum_{n=1}^{\infty} a_n$ diverges.

If the case you get is not one of the above, then the test gives no information.

Example A : Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converge?

Use the **Comparison Test**, trying these b_n:

a) $b_n = \frac{1}{n^3}$, which converges, but $a_n \geq b_n$ for $n \geq 3$, so this comparison won't give any information.

b) $b_n = \frac{1}{n^2}$. In n increases more slowly than n, so $a_n \leq b_n$. If we are not sure of this, then we can now use the Limit Comparison Test.

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\ln n/n^3}{1/n^2} = \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ from Math 1A.

Therefore, $\sum\limits_{n=1}^{\infty} a_n$ converges, from part b) of the Limit Comparison Test.

When using this test, b_n does not necessarily have to be greater than a_n. All we have to do is look at the ratio of those two when we take the limit.

Today's lecture

Proof of the Limit Comparison Test. (part a) only)

Let $\varepsilon = c/2$ in the definition of the limit. So then there exists N such that $\left|\frac{a_n}{b_n} < x\right|\varepsilon = c/2$ if $n \geq N$.

So

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2} \Rightarrow \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$$

Thus $\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$ if $n \geq N$.

This shows that even if the a_n are greater than the b_n, at least they are less than a fixed number times b_n. Now we can use the ordinary Comparison Test to say that if $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges. Similarly, a_n is greater than a fixed number times b_n so if $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges as well.

Where did the $\varepsilon = c/2$ come from? This was a number that we chose to make the algebra work out.

Before, we studied the type of series where all the terms were > 0, and we found that it converged if the sequence of partial sums is bounded. Now we study series where the terms switch sign series like this come up often.

Definition: A series of the from

$\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$, where each $a_n > 0$, is an **alternating**

series. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 \ldots$

We feel it is easier for this series to converge because the + and − signs will tend to cancel each other out. The way to find out is to use the...

Alternating Series Test

Theorem. Let $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ be an alternating series with

a) $a_1 \geq a_2 \geq a_3 \cdots > 0$

b) $\lim\limits_{n\to\infty} a_n = 0$. The terms get smaller and go to zero.

Then the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

The Alternating Series test does not apply if the signs are not strictly alternating, that is, if there are, for example, three +'s, then two −'s.

Example 1 :

$1 - 1/2 + 1/3 - 1/4 + \cdots = s$ This was the puzzle presented earlier where we "calculated" that $s = 2s$. One possibility was that the series did not converge, but the Alternating Series Test says that is does. So something else is wrong.

Example 2 :

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ converges.

Example 3:

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{1/2}} = (-1) \sum_{n=1}^{\infty} \frac{\overbrace{-\cos(\pi n)}^{(-1)^{n-1}}}{n^{1/2}}$$

This step was to get the series into the required from.

Notice that the first and third series could have diverged if there weren't any alternating signs. The Alternating Series Test is very subtle, but everything is wired into the theory, so it is easy to use. Still, we should know how it works.

Proof. First look at the even partial sums.

$$s_n = \sum_{i=1}^{n} (-1)^{i-1} a_i$$

$s_2 = a_1 - a_2 \geq 0$ because $a_2 < a_1$

$$s_4 = \underbrace{a_1 - a_2}_{s_2} + \underbrace{a_3 - a_4}_{\geq 0} \geq s_2$$

$$s_6 = \underbrace{a_1 - a_2 + a_3 - a_4}_{s_4} + \underbrace{a_5 - a_6}_{\geq 0} \geq s_4$$

Note that the even partial sums are monatonic and increasing. Now look at the odd partial sums.

$$s_1 = a_1$$

$$s_3 = \underbrace{a_1}_{s_1} - \underbrace{a_2 + a_3}_{\leq 0} \leq s_1$$

$$s_5 = \underbrace{a_1 - a_2 + a_3}_{s_3} - \underbrace{a_4 + a_5}_{\leq 0}$$

This is also monatonic. So even though s_n is not monatonic, the even and odd partial sums are.

Consider the sequence s_{2n}. These are increasing: $s_2 \leq s_4 \leq s_6 \ldots$ and are also bounded by $M = s_1$, because s_1 is greater than all the even partial sums. Any bounded monotonic sequence has a limit, so $\lim_{n\to\infty} s_{2n} = s$ exists. We study the limit of the odd partial sums by writing:

$$s_{2n+1} = s_{2n} + a_{2n+1}$$

$$\lim_{n\to\infty} s_{2n+1} = \underbrace{\lim_{n\to\infty} s_{2n}}_{s} + \underbrace{\lim_{n\to\infty} a_{2n+1}}_{=0} = s \text{ provided these limits exist.}$$

The two partial sums converge to the same limit so $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = s$.

Error Estimation

This method can also let you make an error estimate since the error in the partial sum is always less than the next term omitted.

$$\underbrace{|s - s_n|}_{\text{error}} \leq a_{n+1}$$

Pictire.

Example 4: How large should n be to compute $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ to within 0.001?

First, we should note that this series converges.
$|s - s_n| \leq a_{n+1} = \frac{1}{(n+1)^3} \leq 0.001 = 10^{-3} \Rightarrow n = 9$

Example 5:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = s$$

How large should n be to compute $\pi/4$ to within 0.001?

$$|s - s_n| \leq \frac{1}{\underbrace{(2n+1)}_{(n+1)\text{th term}}} \leq 0.001 \Rightarrow \; n = 500$$

This is a terrible method to compute π because it takes many steps to get within a small error.