LECTURE 14

Review

- Each bounded, monotonic sequence converges.
- $$\sum_{n=1}^{\infty} a_n = s \text{ , i.e. the series converges to } s \text{ means } \lim_{n\to\infty} s_n = s$$

  Here $s_n = \sum_{i=1}^{n} a_i$ is the sequence of partial sums.
- The divergenve tets: **If** $\sum_{n=1}^{\infty} a_n$ converges, **then** $\lim_{n\to\infty} a_n = 0$,

  but $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Today's Lecture

We need to develop various **tests** for the convergence of series. However, you should know that:

- There is no single test which works for all series.
- These tests, even when they apply, only say that a series $\sum_{n=1}^{\infty} a_n$ converges (but does **not** let you compute $\sum_{n=1}^{\infty} a_n = s$.)
- There are two basic cases for series:
  1) $\sum_{n=1}^{\infty} a_n$, each term $a_n \geq 0$. In this case the sequence of partial sums is monotonically increasing

     $$s_1 \leq s_2 \leq s_3 \ldots$$

     The only problem is that the s_n may go to ∞.
  2) $\sum_{n=1}^{\infty} a_n$, some terms are > 0, other terms < 0.
     This is harder. Maybe the s_n oscillate as $n \to \infty$.

Today and next time we will focus on the first case. The tool we use involves calculus and is called the Integral Text.

Integral Test

Theorem. (The integral test) Left f be a **positive, decreasing** function defined on $[1, \infty)$. Suppose $a_n = f(n)$.

a) If $\int\limits_1^\infty f(x)dx$ converges, then $\sum\limits_{n=1}^\infty a_n$ converges.

b) If $\int\limits_1^\infty f(x)dx$ diverges, so does $\sum\limits_{n=1}^\infty a_n$.

These two pictures will help us understand the test.

Picture 1

Think of the nth partial sum $\sum\limits_{i=2}^n a_i$ as the area of the rectangles under the graph, which is **less** than the area under the graph.

Picture 2

Note that all the rectangles are above the graph. The partial sum $\sum\limits_{i=1}^{n-1} a_i$ is **greater** than the area underneath the graph.

From these pictures, we get

$$\underbrace{\sum_{i=2}^{n} a_1}_{s_n - a_1} \leq \int\limits_1^n f(x)dx \leq \underbrace{\sum_{i=1}^{n-1} a_1}_{s_{n-1}} .$$

The integral is "wedged" between the partial sums. Don't worry about the first term on the left and the last term on the right. They are only finite terms, so they won't make a difference as to whether the series blows up or not.

The proof of the Integral Test.

Proof. Recall:

$$\int\limits_a^\infty f(x)dx = \lim_{t\to\infty} \int\limits_a^t f(x)dx$$

assuming this limit exists.

Assume $\int\limits_1^\infty f(x)dx = R$ converges.

Note: Since $a_n \geq 0$,

$0 \leq s_1 \leq s_2 \leq s_3 \leq \dots$ the s_n are **monotonic**.

Let $M = R + a_1$.

Then

$$0 \leq s_n \leq a_1 + \int\limits_1^n f(x)dx \leq a_1 + \underbrace{\int\limits_1^\infty f(x)dx}_{=R} = M$$

Note that the s_n are also **bounded**.

Thus, $\lim\limits_{n\to\infty} s_n = s = \sum\limits_{n=1}^\infty a_n$ exists.

Similarly, if $\int\limits_1^{\infty} f(x)dx$ diverges, the sequence $\{\int\limits_1^{n} f(x)dx\}_{n=1}^{\infty}$ diverges to $+\infty$ and since $\int\limits_1^{n} f(x)dx \leq s_{n-1}$, we have $\lim_{n\to\infty} s_{n-1}$ diverges. Which means the series diverges.

$\square$

Example 1: For what values of p does $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ $p > 1$ converge?

$$f(x) = \frac{1}{x^p} \quad x \geq 1$$

Before we can apply the integral test, we need to check for two conditions.

Note: $f > 0$, $f'(x) = -\frac{p}{x^{p+1}} < 0$ since the derivative is always negative we know that f is decreasing.

So it's okay to use the integral test.

$$\begin{aligned} \int\limits_1^{t} f(x)dx &= \int\limits_1^{t} \tfrac{1}{x^p} dx \\ &= \tfrac{x^{1-p}}{1-p}\Big|_1^t \\ &= \tfrac{1}{p-1}\left(1 - \tfrac{1}{t^{p-1}}\right) \end{aligned}$$

Let $t \to \infty$

$$\begin{aligned} \int\limits_1^{\infty} f(x)dx &= \lim_{t\to\infty}\left[\tfrac{1}{p-1}\left(1 - \tfrac{1}{t^{p-1}}\right)\right] \\ &= \tfrac{1}{p-1} \end{aligned}$$

So we see, since the integral converges, the series converges.

Warning: Integral test $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, but it does **not** say what value the series converges to!

<u>Remark</u>: $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (difficult to calculate)

$\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $0 \leq p \leq 1$.

Example 2 :

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

This seems to be an in-between case, where the denominator is slightly bigger than n. Does this converge? Use the integral test to find out.

$$f(x) = \frac{1}{x \ln x} \quad x \geq 2$$

Check necessary conditions:

- $f > 0$,
- $f'(x) = \frac{-(x\ln x)'}{(x\ln x)^2} = \frac{-\ln x - 1}{(x\ln x)^2} < 0$, therefore $f(x)$ is decreasing.
- $\int\limits_2^t f(x)dx = \int_2^t \frac{dx}{x\ln x} \quad u = \ln x$ has $du = \frac{dx}{x}$

  Let $t \to \infty$

  $$= \int\limits_{\ln 2}^{\ln t} \frac{du}{u} = \ln u|_{\ln 2}^{\ln 1} = \ln(\ln t) - \ln(\ln 2)$$

  So $\lim\limits_{t\to\infty} \frac{dx}{x\ln x} = +\infty$ diverges.

Thus, by the integral test, the series is divergent.

Example 3 :

$\sum\limits_{n=2}^{\infty} \frac{1}{n\ln^2 n}$. Does this converge?

$$f(x) = \frac{1}{x\ln^2 x} = \frac{1}{x(\ln x)^2} \quad x \geq 2$$

Check to see if the conditions are met. (They are).

$$\int\limits_2^t f(x)dx = \int\limits_2^t \frac{dx}{x\ln^2 x} \qquad u = \ln x \text{ has } du = \frac{dx}{x}$$

$$= \int\limits_{\ln 2}^{\ln t} \frac{du}{u^2} \qquad \text{Let } t \to \infty$$

The limit is

$$\int\limits_{\ln 2}^{\infty} \frac{du}{u^2}$$

Converges because the exponent
is greater than one (p-integral with p greater then 1).

So the series converges.