LECTURE 13

Review

$s_n = \sum_{i=1}^{n} a_i = n$th partial sum.

$$\sum_{n=1}^{\infty} a_n = s \text{ means that } \lim_{n\to\infty} s_n = s$$

Example A : Geometric Series.

$$\begin{aligned} \sum_{n=0}^{\infty} r^n &= 1 + r + r^2 + r^3 + \dots \\ &= \tfrac{1}{1-r} \quad \text{for } -1 < r < 1 \end{aligned}$$

So we also know that

$$r + r^2 + r^3 + \cdots = \frac{r}{1-r} \text{ for } -1 < r < 1$$

Example B : Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges to} \quad \infty.$$

Puzzle

$$\begin{aligned} s &= 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots \\ 2s &= 2 - 2/2 + 2/3 - 2/4 + 2/5 - 2/6 + \dots \\ &= 2 - 1 + 2/3 - 1/2 + 2/5 - 1/3 + 2/7 - 1/4 + \dots \\ &= 1 - 1/2 + (2/3 - 1/3) - 1/4 + (2/5 - 1/5) - 1/6 \dots \\ &= 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots = s. \end{aligned}$$

So, $s = 2s$? Something must be wrong. Three possibilities are:

- $s = 0$.
- s not defined.
- Something else is wrong.

The above is an example of what could happen if we just jumped in and started doing calculations with series, without really understanding the theory.

Today's Lecture

Theory of Series

Theorem.(Divergence test) If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. . $s_n = \sum_{i=1}^{n} a_i$.

We know $\lim_{n\to\infty} s_n = s = \sum_{n=1}^{\infty} a_n$ by definition.

Fix $\varepsilon > 0$. I must find N so that $|a_n - \underbrace{0}_{=L}| < \varepsilon$ if $n \geq N$.

Since $\lim_{n\to\infty} s_n = s$, there exists an M so that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \text{if} \quad n \geq M \qquad (^*)$$

(This says that the difference between the series and the nth partial sum is less than $\varepsilon/2$.)

Take $N = M + 1$. Let $n \geq N$.

Now $|a_n| = |s_n - s_{n-1}|$. This says that the difference between the nth partial sum and the $(n-1)$th partial sum is the nth term.

$$\begin{aligned}
|a_n| &= |s_n - s_{n-1}| \\
&= |s_n - s + s - s_{n-1}| && \text{add and subtract } s \\
&\leq |s_n - s| + |s - s_{n-1}| && \text{(by triangle inequality which says } |a+b| \leq |a| + |b|) \\
&\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} && (\text{ since } n, n-1 \geq M \text{ and } ^*) \\
&= \varepsilon
\end{aligned}$$

□

Note that everything in the proof is a finite quantity.

Another, shorter prof of this theorem uses the fact that the limit of the sum of sequences is equal to the sum of limit, when both these limits exits. In our case $a_n = s_n - s_{n-1}$. We know that $\lim_{n\to\infty} s_n = s$, which means $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

This completes the shorter proof.

Example 1: Does $\sum_{n=1}^{\infty} \cos\left(\frac{n^2+6n}{n^2+7}\right)$ converge?

First, check if the $a_n s$ go to 0.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \cos\left(\tfrac{n^2+6n}{n^2+7}\right) \\
&= \cos\left(\lim_{n\to\infty}\left(\tfrac{1+\frac{6}{n}}{1+\frac{7}{n^2}}\right)\right) && \text{since } \cos \text{ is continuous we can move the limit inside the function} \\
&= \cos 1 \\
&\neq 0
\end{aligned}$$

So we know that the series does not converge.

Example 2: Does $\sum_{n=1}^{\infty} \frac{1}{3+2^{-n}}$ converge?

$\lim\limits_{n \to infty} a_n = \frac{1}{3}$, so this series does not converge.

One thing to note about this theorem is that the implication does not work in the other direction.

$\sum\limits_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$.

but

$\lim\limits_{n\to\infty} a_n = 0$ **does not mean that** $\sum\limits_{n=1}^{\infty} a_n$ converges.

Example 3: $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges even though

$$\lim_{n\to\infty} a_n = 0.$$

We will need more tests for convergence.

Test Problems

Which of the following series converge:

- $\sum\limits_{n=1}^{\infty} \frac{-1^{n-1}}{n}$
- $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$
- $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$
- $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

We will later develop tools to handle these examples.

Rules for Series

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge, then

- $\sum\limits_{n=1}^{\infty} ca_n = c \sum\limits_{n=1}^{\infty} a_n$ here c is just some constant.
- $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$
- $\sum\limits_{n=1}^{\infty} (a_n - b_n) = \sum\limits_{n=1}^{\infty} a_n - \sum\limits_{n=1}^{\infty} b_n$

The way we've solved series so far is to take the partial sum, find a formula for it using algebraic tricks, and then take the limit. Here is a special example.

Decimal Expansion:

$$\pi = 3.141592653\ldots$$

Notation: If $0 \leq s \leq 1$, we write

$s = a_1a_2a_3a_4\ldots$

where each $a_n \in \{0, 1, \ldots, 9\}$.

This means $s = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$.

Note: This series always converges.

However, just because we are able to write something down, doesn't automatically mean that it converges. Why does this particular sequence converge?

$$s_n = \sum_{i=1}^{n} \frac{a_i}{10^i}$$

Notice that $s_1 \leq s_2 \leq s_3 \leq s_4 \leq \dots$ is monotonic.

I'll show the $\{s_n\}$ are bounded.

$$\begin{aligned}
0 &\leq s_n \\
&= \sum_{i=1}^{n} \frac{a_i}{10^i} \\
&\leq 9 \sum_{i=1}^{n} \left(\frac{1}{10}\right) && \text{the worst case scenario is that all the } a_i\text{'s are 9} \\
&= \frac{9}{10} \sum_{i=1}^{n} r^{i-1} && \text{with } r = \frac{1}{10} \\
&= \frac{9}{10} \left(\frac{1-(1/10)^n)}{1-1/10} \right)
\end{aligned}$$

By the Theorem from last time, since the sequence is bounded and monotonic, we known that $\lim_{n \to \infty} s_n = s$ exists.

<u>Repeated Decimal Expansions</u>

Example 4 : $0.3171717 \cdots = 0.3\overline{17}$

The bar means that that segment is repeated. Repeating segments show that the decimal is a rational number. We solve for that number by using this trick:

$$\begin{aligned}
1000s &= 317.171717\dots \\
\underline{-10s} &= \underline{\quad 3.171717\dots} \\
990s &= 314
\end{aligned}$$

Hence we see that $s = \frac{314}{900}$

$s = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$. The $a'_n s$ were $3, 1, 7, 1, 7, \dots$

Example 5: $s = 0.999 \cdots = 0.\bar{9} = 1$.

This number is **equal** to one, not "a little bit less".

$$\begin{aligned}
s &= \sum_{n=1}^{\infty} \frac{9}{10^n} \\
&= \frac{9}{10} \sum_{n=1}^{\infty} (1/10)^{n-1} \\
&= \frac{9}{10} \cdot \frac{1}{1-1/10} \\
&= 1
\end{aligned}$$

Return to the Puzzle

$$\begin{aligned} s &= \sum_{n=1}^{\infty} \tfrac{(-1)^{n+1}}{n} \\ &= \underbrace{1 - 1/2}_{>0} + \underbrace{1/3 - 1/4}_{>0} + \dots \end{aligned}$$

Recall that we "computed" $s = 2s$.

• Consider the first possibility that $s = 0$. However, each pair of terms is greater than zero, so we cannot have $s = 0$.

• The second possibility says that s does not converge, so writing down that $s = 2s$ does not mean anything. However, that series **does** converge (to $\ln 2$), so this is not the problem either.

• The only explanation left is that something wrong was done in the calculation.