LECTURE 11

Review

Call S the curved surface area, then we have:

$$S = \int_a^b 2\pi f(x)(1 + f'(x)^2)^{1/2} dx$$

From Math 1A

Volume $= V \int_a^b \pi f^2(x) dx$

You can find out more about surfaces and volumes if you take Math 53.

Today's Lecture

We will slow down for the abrupt transition to sequences and series. This course will seem different and unrelated until we reach differential equations. Then everything will come together. Today, we will address the theoretical issues for sequences and series. The most difficult idea is that of the **limit** of a sequence. These limits are **almost** the same as those in Math 1A.

Sequences and Series

Definition: A **sequence** is a listing of infintely many numbers written in an order (*i.e.*, we refer to a number in the list by a natural number)

$$a_1, a_2, a_3 \ldots$$

Definition: A **series** is the sum (**if** it exists) of a sequence of numbers:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

Main Problems

- Does a given sequence **converge**?

  That is, does $\lim_{n\to\infty} a_n$ exist?

- Does a given series **converge**?

  That is, does the sum $\sum_{n=1}^{\infty} a_n$ exist?

When talking about sequences and series, we need to be careful with our mathematical terminology. For example, many people will refer to a"series of numbers" when in fact, a series refers to the **sum** of those numbers.

Why study this?

- Approximation in science. Usually in science and engineering, you will need to use a computer to generate approximations to answers. Can you trust the approximation?

Computer generated approximations	$\longrightarrow \underbrace{\ldots, a_3, a_2, a_1}$
Can you trust the algorithm?	Can you trust what is coming out? That is, does the sequence converge?

- Approximation in Math

  1) Approximating numbers

$$\pi = 4(1 - 1/3, +1/5 - 1/7 + 1/9 \ldots)$$

  (By the way, this is a bad approximation to use because it converges too slowly.)

  2) Approximating functions

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \ldots \\ &= \sum_{n=0}^{\infty} a_n x^n \qquad \text{Power Series} \end{aligned}$$

  For example you will eventually learn that you may approximate the exponential function e^x by :

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{\cdot 2} + \frac{x^4}{4 \cdot 3 \cdot 2} + \ldots$$

Uses for approximating functions by power series include working with integrals that have functions that are difficult to integrate. We can break the function apart into a series, integrate term by term, and put it back together. Another use is for solving ordinary differential equations, as we will see later.

Sequences

In studying sequences we will first have to develop terminology and precise definitions (for sequences and what it means for a sequence to converge). Then we will want to analyze sequences so that we may develop practical methods.

Notation: $\{a_n\}$ means $a_1, a_2, a_3 \ldots$

The following definition is central, fundamental, and subtle.

Definition: We say the sequence $\{a_n\}$ has limit L, written

$$\lim_{n\to\infty} a_n = L$$

provided,

for each $\varepsilon > 0$ (no matter how small)

there exists and integer N

so that $|a_n - L| < \varepsilon$ if $n \geq N$.

It helps to remember the interpretation: L =limit and ε =“error tolerance”

What this definition says is that “Eventually, all the a_n’s are within distance ε of L.”

The following two pictures will better help us understand the definition.

Picture 1

If the limit is L, then **eventually** all the points lie within the strip. And if this is true for **every** tolerance $\varepsilon > 0$, then the limit is L. However, it if fails for an ε small enough, then we know that L is not the limit.

Picture 2. Think of this as a machine that spits out a number in a sequence every second. Eventually, all the points will begin falling within the banded lines. If this is true for every error tolerance, then the limit exists (and we know the limit is L).

If $\lim_{n\to\infty} a_n$ exists, the sequence **converges**. Otherwise it **diverges**. Ways this could happen are if the sequence keeps oscillating without setting down or it could go to infinity.

Now we are ready to study limits. First, we will learn some tools, then we will apply those tools to some examples.

<u>Rules for Limits</u>

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then:

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

  Provided we do not ever divide by zero.

These are intuitive, so we will not bother proving them.

- If f is continuous, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$$

This says that we can move limits past continuous functions *i.e.,* we can move the limit in, or take it out of a continuous function.

<u>Squeeze Theorem</u>

Suppose $a_n \le b_n \le c_n$ for all $n \ge N_0$ (N_0 is just some fixed integer).

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

How do you know if a weird sequence converges? If you can "squeeze" it between two nice sequences converging to the same limit, then the middle sequence must also converge.

<u>Functional Limit Theorem</u>

Suppose $a_n = f(n)$ for some function f. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Here we define the nth term of a sequence with a function. So far, we have taken the limit of **sequences**. This theorem relates us to Math 1A, where we took the limit of **functions**.

<u>Practical uses</u>

Now we will use these rules in examples. The first example will be obvious, but we will do it carefully using the definitions.

Example 1: (using the definition)

$a_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} a_n = 0 = L$

Proof: when we do this proof, we think of ε as frozen, and try to find an explicit N.

Fix any number $\varepsilon > 0$. We must find N so large that $|\underbrace{a_n}_{1/n} - \underbrace{L}_{0}| < \varepsilon$ if $n \ge N$. That is, $|\frac{1}{n}| < \varepsilon$ if $n \ge N$.

Let N be the smallest integer $> 1/\varepsilon$

Suppose $n \ge N$. So,

$$n \ge N > 1/\varepsilon \text{ implies } n > 1/\varepsilon \Rightarrow 1/n < \varepsilon$$

Example 2: (using the Squeeze Theorem)

$$b_n = \frac{(-1)^n}{n^3}$$

$$\underbrace{\frac{-1}{n}}_{a_n} \le \underbrace{\frac{(-1)^n}{n^3}}_{b_n} \le \underbrace{\frac{-1}{n}}_{c_n} \quad n = 1, 2, \ldots$$

$$\left.\begin{array}{r}\lim_{n\to\infty} a_n = 0\\ \lim_{n\to\infty} c_n = 0\end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n^3} = 0$$

Example 3: (of a diverging sequence)

$$a_n = (-1)^n$$

There is no limit because this sequence keeps jumping back and forth between -1 and 1.

Example 4: (using Limit Rules)

$$a_n = \frac{n^3 + 17n^2 - n}{6n^3 + n + 12}.$$

Divide top and bottom by n^3.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1+17/n-1/n^2}{6+1/n^2+12/n^3} \\ &= \frac{\lim_{n\to\infty}(1+17/n-1/n^2)}{\lim_{n\to\infty}(6+1/n^2+12/n^3)} \\ &= \frac{1}{6} \end{aligned}$$

Example 5: (using the Functional Limit Theorem)

$$a_n = n^{1/n}.$$

As n gets larger, its exponent gets smaller. Recall the formula for natural logs,

$$a^b = e^{b \ln a}.$$

We put the limit inside the function (note this is only ok since the function e^x is continuous) and get

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} e^{1/n \ln n} \\ &= e^{\lim_{n\to\infty} \frac{\ln n}{n}} \quad \leftarrow \begin{cases} \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \\ \text{By L'Hospital's rule} \\ = \lim_{x\to\infty} \frac{(\frac{1}{x})}{1} = 0 \end{cases} \\ &= e^0 = 1 \end{aligned}$$