LECTURE 6

This lecture is meant to be a consolidation of the methods of integration that have been learned so far.

• Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du$$

• Integration by parts

$$\int fg'dx = fg - \int fgdx$$

Partial fractions

The method for integrating partial functions.

Chart of substitutions

Expression	Substitution	
$\sin^m x \cos^n x$	$u = \sin x, \cos x$	trigonometric
$\tan^m x \sec^m x$	$u = \tan x, \sec x$	
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	implicit trig
$\sqrt{x^2 - a^2}$	$x = a\tan\theta$	
$\sqrt{a^2 + x^2}$	$x = a\sec\theta$	

continues on next column

Expression	Substitution
Terms with fractional powers	$u = g^{1/n}$where g is some expression
$------$	$-----$
Rational function of sin and cos	$t = \tan(\frac{x}{2})$ Wierstrass

Implicit trig substitutions

These substitutions are called implicit because you are defining your original variable in terms of a new variable, and not the other way around.

The direct (explicit) substitution is $u = f(x)$. In this case

$$g(f(x))f'(x)dx = g(u)du$$

The inverse (implicit) substitution is $x = f(u)$ (or, as in the case of trigonometric substitutions $x = f(\theta)$) and is useful when the integral of $F(f(\theta)f'(\theta)d\theta$ is easier the integral of $F(x)dx$.

Memorizing equations

Although it is possible to derive equations, it is faster to memorize them. This way, you will save time on a test and not have to worry if you derived them correctly. Three important ones to memorize are those involved in the Weierstrass substitution.

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

Advice

• First of all, **don't panic** .

• Second, realize that most people always try substitution first, because that is how they've learned to do integrations. Many times, it would be much easier to integrate by parts.

• Simplify the integrand. A complicated looking integrand may actually be much simpler after some algebraic manipulation.

• Look for easy substitutions. Sometimes this is much preferred over using the other two methods, in which even though you know that you will eventually get an answer, it may take much longer in the long run. When in doubt, dont't be greedy and try to do all at once with a complicated substitution. Even a simple substitution can greatly simplify a messy integral.

• Use a substitution table for non-obvious substitutions.

• Use integration by parts.

• Use integration tables.

The key is that if you get stuck trying to use substitutions try to use integration by parts.

Example 1 (an example of integration by parts)

$$\int \sin^2 x \cos x \ln(\sin x) dx = \left\{ \begin{array}{c} u = \sin x \\ du = \cos x dx \end{array} \right\} = \int u^2 \ln(u) du$$

Notice that this integrand two incompatible functions. This is a sign that we should try to integrate by parts. We want to get rid of one of those functions, namely the $\ln(u)$ by taking it's derivative. To do this, we remember the formula for integration by parts.

$$\int fg' \, dx = fg - \int f'g \, dx$$

Now we write our integral as

$$= \int (\frac{u^3}{3})' \ln u \, du$$

Using this formula and the integration by parts move the prime from one function to the other:

$$= (\frac{u^3}{3}) \ln u - \int (\frac{u^3}{3}) \ln(u)' du$$

$$= \frac{u^3}{3} \ln u - \int \frac{u^2}{3} du$$

The rest is easy...

Example 2

$$\int \frac{x}{(x^2-4x)^{1/2}}dx$$

This integral looks almost like the one in the chart except that there's a $4x$ linear term inside the radical. In this case, when something that has the form ax^2+bx+c, it is a sign that we need to "massage" the radicand into the appropriate form by completing the square.

$$= \int \frac{x}{(x^2-4x+4-4)^{1/2}}dx = \left\{ \begin{array}{r} u = x-2 \\ du = dx \end{array} \right\} = \int \frac{x}{((x-2)^2-4)^{1/2}}dx$$

$$= \int \frac{u+2}{(u^2-4)^{1/2}}du$$

The chart suggests the implicit trigonometric substitution of $u = 2\sec\theta$ with $du = 2\sec\theta\tan\theta$, therefore we have

$$= \int \frac{(2\sec\theta+2)\cdot 2\sec\theta\tan\theta}{4(\sec^2\theta-1)}^{1/2} d\theta =< \sec^2\theta - 1 = \tan^2\theta >$$

$$= 2\int (\sec\theta+1)\sec\theta\, d\theta$$

$$= 2\int (\sec^2\theta+\sec\theta)\, d\theta$$

$$= 2(\tan\theta + \ln|\sec\theta+\tan\theta|) + C$$

Draw the triangle and rewrite the answer in terms of u, then x.

Remark It is a good idea to memorize the formula:

$$\int \frac{dx}{\cos x} = \ln|\sec x + \tan x| + C$$

Example 3

$$\int \sec^3 x\, dx = ?$$

We notice that this is a rational function (of trigonometric functions) so we can, in principle, use the Weierstrass substitution. But in practice, it is probably long and tedious. This is why the "nuclear bomb" of substitutions should be used only as a last resort. The main lesson of today is that.

If substitutions don't work (or are really hard), then try integration by parts instead. In our case we get:

$$\int \sec x(\tan x)' dx$$

$$= \sec x \tan x - \int \sec' x \tan x \, dx$$

Integration by parts puts the integral in a new form, but we still may not know how to solve it.

$$\sec x \tan x - \int \sec x \tan^2 x \, dx$$

Now notice that $\tan^2 x = \sec^2 x - 1$ so that our integral becomes:

$$\sec x \tan x + \int \sec x \, dx - \underline{\int \sec^3 x \, dx}$$

The original integral reappeared, but in this case the negative sign is good because we can move the underlined term to the other side of the equal sign. So we know

$$\int \sec x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

Challenge problem

$$\int \frac{dx}{x^7 - x} = ?$$

Hint: This is a rational function, but don't use partial fractions!

We will return to this problems later.

Example 4

$\int (2 + 3\cos x)^{1/2} \tan x \, dx$ Think: $\tan x = \frac{\sin x}{\cos x}$.

What this shows us is that there is a $\sin x \, dx$ on top, and that we should substitute for cos.

$$u = \cos x \qquad du = -\sin x \, dx$$

Our integral becomes:

$$= -\int \frac{(2 + 3u)^{1/2}}{u} du$$

Now when we are done with the trig. functions we should address the square root:

$$v = (2 + 3u)^{1/2} \text{ so } v^2 = 2 + 3u \text{ and } 2vdv = 3du$$

When you make this kind of a substitution, be sure to raise the powers until there are integer exponents.

$$= \int \frac{2v^2}{v^2 - 2} dv = \quad < \text{ Long divide } > =$$

$$= 2\int 1 + \frac{2}{v^2 - 2} dv$$

Recall:

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

and now you can finish this problem.

Using Tables of Integrals

Many times, it would be a lot easier to look up the answer to an integral rather than try to solve it from scratch. There are many books with tables of solved integrals. All the integrals are classified by subgroups to make them easier to look up. For example, they may be classified as integrals involving trigonometric functions, logarithms, exponentials, etc. Of course, not every integral is exactly in those forms, so you will need to convert your integral into one of those listed.

References for tables of integrals

Example of using a table of integrals

$$\int \sec^5 x \, dx = ?$$

#77 in appendix G says:

$$\int \sec^n u \, du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \sec^{n-2} u \, du$$

setting $n = 5$, the integral becomes

$$= \frac{1}{4} \tan x \sec^3 x + \frac{3}{4} \underline{\int \sec^3 x \, dx}$$

which is something we've already solved.

Using Computers

So how do you solve

$$\int \frac{dx}{x^7 - x} ?$$

Multiply both top and bottom by x^5.

$$= \int \frac{x^5}{x^{12} - x^6} dx = \frac{1}{6} \int \frac{du}{u^2 - u}$$

Now you can use the substitution

$$u = x^6 \text{ with } du = 6x^5 dx$$

and you can now finish the example.

It look us a long time to figure out, but the Mathematica program was able to do it in about a second!

All things are not perfect in computer-land, though. First of all, the computer does not add the constant of integration. Second, it may not express the answer in the simplest form. For example, number 15 on the handout shows the computer's answer to $\int \sec x \, dx$, as

$$'' - \text{Log}\big[\text{Cos}\big[\frac{x}{2}\big] - \text{Sin}\big[\frac{x}{2}\big]\big] + \text{Log}\big[\text{Cos}\big[\frac{x}{2}\big] + \text{Sin}\big[\frac{x}{2}\big]\big]''$$

even though we know it as $\ln \sec x + \tan x| + C$.

There's no guarantee that the computer can do every integral. It could not do $\int \sqrt{\tan x} dx$, which showed up on one the TA's worksheets.

Even though we have access to tables of integrations and computers, we should still learn to do integrals by hand. The analogy to think of is learning long division as a child. While we have calculators to do that, we will need the *principles* of long division in calculus. In short, we need to know the theory in order to understand what is going on, so that we may apply it in other situations.

There are various programs which can compute integrals analytically. Mathematica package was already mentioned. The other well known are Maple and Mat-Lab.

few references