LECTURE 5

Review of Partial Fractions

This technique is useful for converting rational functions into forms that are easily integrable.

If you want to integrate a rational function $f(x) = \frac{P(x)}{Q(x)}$ to the following.

Step 1. If $degP \geq degQ$ do the long division:

$$f = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are polynomials and $degR < degQ$

Step 2. Factor the denominator:

$$Q(x) = \ldots (ax+b)^r \ldots (cx^2+dx+e)^s$$

Step 3. If $\frac{P(x)}{Q(x)}$ is proper, decompose it into partial fractions

$$\frac{P(x)}{Q(x)} = \cdots \frac{A}{ax+b} + \frac{A_r}{(ax+b)^r} + \cdots \frac{B_1x+C_1}{cx^2+dx+e} + \cdots + \frac{B_sx+C_5}{(cx^2+dx+e)^s}$$

Useful Formulae

a)

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + C$$

b)

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Example:

$$\int \frac{3x+2}{x^2+2x+5} dx = \int \frac{3(x+1)}{(x+1)^2+4} dx - \int \frac{1}{(x+1)^2+4} dx$$

Use the substitution $u = (x+1)^2 + 4$ in the first integral.

Use the substitution $u = x + 1$ in the second integral and formula (b) from above.

Merely knowing the techniques for integration does not mean that you will be able to do every problem (many people know how to drive, but there are many bad drivers).

Gaining experience (especially from the homework) is extremely helpful.

Overview

Rationalizing Substitution

Suppose that you are given an awful looking integral. With the right substitution, you can some convert it into a **rational function**, which in principle, can always be integrated. This method is called making a rationalizing substitution

Substitutions to try

1) $u = g^{1/n}$ where g is some expression

2) $t = \tan(\frac{x}{2})$ Weierstrass substitution, a.k.a. the "nuclear bomb" of substitution

Example 1: (using the substitution in (1)).

$$\int \frac{dx}{\sqrt{1+\sqrt{x}}}$$

try $u = x^{1/2} \Rightarrow u^2 = x$
$\Rightarrow 2udu = dx$

$$= \int \frac{2u}{\sqrt{1+u}} du$$

This didn't work. Try $v = (1+u)^{1/2}$
$\Rightarrow v^2 = 1+u$
$\Rightarrow 2vdv = du$

$$= \int \frac{2(v^2-1)\cdot 2}{v} dv$$

Now this is in the form of a rational function

$$= 4\int (v^2-1)dv = 4(v^3/3 - v) + C$$

$$4\big(\frac{(1+x^{1/2})^{3/2}}{3} - (1+x^{1/2})^{1/2}\big) + C$$

Example 2: (using substitution (2)).

$$\int \frac{dx}{x^{1/2}+x^{1/3}}$$

How do we apply this when x has both 1/2 and 1/3 powers? We use a substitution that is a product of those powers.

$$u = x^{1/6} \Rightarrow u^6 = x \Rightarrow 6u^5 du = dx$$

$$= \int \frac{6u^5}{u^3+u^2} du$$

Now this is rational function cancel out the u^2 and divide.

$$= 6\int \frac{u^3}{u+1} du = 6\int \big(u^2 - u + 1 - \frac{1}{u+1}\big) du$$

$$= 6(u^3/3 - u^2/2 + u - \ln|u+1|) + C$$

Now just plug in $u = x^{1/6}$.

Weierstrass substitution

The Weierstrass substitution is sometimes called the "nuclear bomb" because it is a very powerful tool for converting a rational function of sines and cosines into a rational function.

Theory : Given a right triangle with these properties

$\tan(\frac{x}{2}) = \frac{opp}{adj} = t$

$\cos(\frac{x}{2}) = \frac{adj}{hyp} = \frac{1}{(t^2+1)^{1/2}}$

$\sin(\frac{x}{2}) = \frac{opp}{hyp} = \frac{t}{(t^2+1)^{1/2}}$

Now we use some half angle identities:

$$\sin x = \sin 2(\frac{x}{2}) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$$

$$\cos x = \cos 2(\frac{x}{2}) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})$$

When you plug in the respective half angles, the multiplication will cause the square roots in the denominator to drop out. These formulae are important to know.

$$x = 2\tan^{-1} t \Rightarrow \quad \boxed{\begin{array}{c}\sin x = \frac{2t^2}{1+t}\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2}{1+t^2}dt\end{array}}$$

These equations convert a rational function of sin and cos into a rational function of t.

Example 4

$$\int \frac{dx}{3\sin x + 4\cos x} \qquad \text{Using the above formulae. this integral becomes}$$

$$= \int \frac{2}{3(2t) + 4(1-t^2)}dt = \int \frac{2}{6t + 4 - 4r^2}dt$$

$$= -\int \frac{dt}{2r^2 - 3t - 2} = -\int \frac{dt}{(2t+1)(t-2)}$$

Using partial fractions,

$$\frac{-1}{(2t+1)(t-2)} = \frac{A}{2t+1} + \frac{B}{t-2} = \frac{A(t-2) + B(2t-1)}{(2t+1)(t-2)}$$

$$\Rightarrow \underset{=0}{\rightarrow} (A+2B)t + (\underset{=-1}{\rightarrow} (B-2A) = -1 \Rightarrow = -1/5\ ,\ A = ?$$

Now the integral becomes

$$2/5\int \frac{dt}{2t+1} - 1/5\int \frac{dt}{t-2} = 2/5 \cdot 1/2\ln|2t+1| - 1/5\ln|t-2| + C$$

$$= 1/5\ln|\frac{2t+1}{t-2}| + C = 1/5\ln|\frac{2\tan(\frac{x}{2}+1}{\tan(\frac{x}{2}-2}| + C$$