LECTURE 4

Review

An example from the last lecture:

$$\int \frac{1}{(9x^2 6x-8)^{1/2}}dx$$

$$= \int \frac{1}{[(3x+1)^2-9]^{1/2}}dx \qquad \begin{array}{c} u=3x+1 \text{ has} \\ du=3dx \text{ so } \frac{1}{3}du=dx \end{array}$$

$$= \frac{1}{3}\int \frac{du}{(u^2-9)^{1/2}} \qquad \begin{array}{c} u=3\sec\theta \text{ has} \\ du=3\sec\theta\tan\theta \end{array}$$

$$= \frac{1}{3}\int \frac{3\sec\theta\tan\theta}{(9\sec^2\theta-9)^{1/2}}d\theta \text{ 3's cancel}$$

$$= \frac{1}{3}\int \sec\theta d\theta$$

$$= \frac{1}{3}\ln|\sec\theta+\tan\theta|+C \text{ Convert to an expression in } u$$

$$= \frac{1}{3}\ln\Big|\frac{u}{3}+\frac{(u^2-9)^{1/2}}{3}\Big|+C \text{ Convert to an expression in } x$$

$$= \frac{1}{3}\ln\big|3x+1+(9x^2+6x-8)\big|+C'$$

Partial Fractions

Definition: A polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_n \neq 0$ (if $a_n = 0$ then $P(x)$ has degree $m < n$ where a_m is the first non-zero coefficient after a_n).

An expression,$f(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials, is called **rational function**.

Our plan will be to use algebra to rewrite: $f(x) = \frac{P(x)}{Q(x)}$ into a sum of simpler terms which are easy to integrate.

Step One: Reduction (long division with polynomials). If necessary, divide Q into P to get

$f(x) = S(x) + \frac{R(x)}{Q(x)}$ where degree of $R(x) < \deg Q$

- If deg$P$ <deg$Q$ do nothing (such rational functions are called **proper**).
- If deg$P$ >deg$Q$ use long division to bring the rational functionto the form

$$f(x) = S(x) + \frac{R(x)}{Q(x)} \qquad \begin{array}{c}\text{where degree of} \\ R(x) < \deg Q\end{array}$$

here $S(x)$ and $R(x)$ are polynomials,

Example 1

$f(x) = \frac{x^3 \leftarrow P(x)}{x+1 \leftarrow Q(x)}$

$$Q(x) \to x+1)\frac{x^2-x+1 \leftarrow S(x)}{x^3 \qquad \leftarrow P(x)}$$

$$\frac{x^3+x^2}{-x^2}$$

$$\frac{-x^2-x}{x}$$

$$\frac{x+1}{-1} \leftarrow R(x) \text{ remainder}$$

$$\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1} \text{ so}$$

$$\int \frac{x^3}{x+1} dx = \int (x^2 - x + 1 - \frac{1}{x+1}) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$$

Step Two: Factor $Q(x)$ into product of various linear terms $(ax+b)$, and irreducible **quadratic** terms (ax^2+bx+c), (terms that cannot be factored into linear terms, such as, $b^2 - 4ac < 0$).. Any polynomial can be factored into a product of linear and irreducilbe quadratic factors.

Example 2

$Q(x) = x^2 - 1 = (x-1)(x+1)$

$Q(x) = x^2 - 2x + 1 = (x-1)^2$

$Q(x) = x^3 - x^2 + x - 1 = \underbrace{(x^2+1)}_{\text{quadratic}}\underbrace{(x-1)}_{\text{linear}}$

Step Three: Decompose $f(x) = S(x) + \frac{R(x)}{Q(x)}$ into **partial fractions**.

Case A:

If $Q(x) = (ax+b)'$ (times other terms),write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots$$

$$+ \frac{A_r}{(ax+b)^r} + \underbrace{\text{other terms}}_{\cdots}$$

Case B:

If $Q(x) = (\underbrace{\text{irreducible quadratic}}_{(ax^2+bx+c)^r}$ (times other terms),write

$$\frac{R(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + b_2}{(ax^2 + bx + c)^2} \cdots$$

$$+ \frac{A_r x + B_r}{(ax^2 + bx + c)^r} + \text{ other terms}$$

Use Cases A and B for all factors of Q. Solve for A, B, etc.
Example 3
$\int \frac{1}{x^2-3x+2} dx = ?$
Step Two:

$$\frac{1}{x^2 - 3x + 2} = \frac{1}{\underbrace{\text{(x-1)(x-2)}}_{Q(x)}}$$

Apply case A twice:

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

Solve for A, B:

$$\frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2) + B(x-1)}{(x-2)(x-1)} = \frac{1}{(x-2)(x-1)}$$

$\Rightarrow A(x-2) + B(x-1) = 1$
$\Rightarrow (A+B)x - 2A - B = 1 + 0x$

$$\Rightarrow \begin{cases} A + B = 0 \\ \quad A = -1, B = 1 \\ -2A - B = 1 \end{cases}$$

So

$$\begin{aligned} \int \frac{1}{x^2-3x+2} dx &= \int \frac{-1}{x-1} dx + \int \frac{1}{x-2} dx \\ &= -\ln|x-1| + \ln|x-2| + C \\ &= \ln\left|\frac{x-2}{x-1}\right| + C \end{aligned}$$

Note that the same method gives the useful formula:

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right| + C$$

Example 4
$\int \frac{x}{x^2-2x+1} dx = ?$
Case A, $r = 2$:

$$\frac{x}{x^2-2x+1} = \frac{x}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Solve for A, B:

$$\frac{x}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

$\Rightarrow Ax + B - A = x$

$$\Rightarrow \begin{cases} A = 1 \\ A = 1, B = 1 \\ B - A = 0 \end{cases}$$

So

$$\int \frac{x}{(x-1)^2} dx = \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

$$= \ln|x-1| - (x-1)^{-1} + C$$

Example 5

$\int \frac{x+1}{x^3+x} = ?$

$$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Solve for A, B, C

$$\frac{x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$\Rightarrow (A+B)x^2 + Cx + A = x + 1$

$\Rightarrow A + B = 0$, $C = 1$ $A = 1$, and so $B = -1$

$$\begin{aligned} \int \tfrac{x+1}{x^3+x} &= \int \overbrace{\tfrac{1}{x}}^{A} dx + \int \overbrace{\tfrac{-1}{x^2+1}}^{B} x\,dx + \int \overbrace{\tfrac{1}{x^2+1}}^{C} dx \\ &= \ln|x| - 1/2 \ln|x^2+1| + \tan^{-1} x + const \end{aligned}$$

Where we use *const* as our constant on integration to avoid confusion with the C we used in the partial fractions calculation.