1 LECTURE 2

Review

Product and Chain rules: repackage these and you will get the integration by parts method and the substitution method. These methods can be summarized in identities

$$\int f(g(x))g'(x)dx = F(g(x)) + C,$$

where $F' = f$, and

$$\int fg'dx = fg - \int f'gdx.$$

Equivalently a substitution can be written as

$$\int f(g)g'dx = \int fdu$$

where $u = g$, "$du = g'dx$" and the integration by parts as

$$\int udv = uv - \int vdu.$$

where $u = f, v = g$

Example:

$$\int \sin^{-1} xdx = \left\{ \begin{array}{cc} u = \sin^{-1} x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}}dx & v = x \end{array} \right\} \quad (1)$$

$$= x\sin^{-1} x - \int x\left(\frac{1}{\sqrt{1-x^2}}\right) dx = \left\{u = 1 - x^2,\ du = -2xdx\right\} \quad (2)$$

$$= x\sin^{-1} x + \frac{1}{2}\int \frac{du}{u^{1/2}} = x\sin^{-1} x + u^{1/2} + C \quad (3)$$

$$= x\sin^{-1} x + (1 - x^2)^{1/2} + C. \quad (4)$$

Trigonometric Integrals

These are integrals involving products of sin and cos only

$$\int \sin^m x \cos^n xdx = ?$$

A method of solving them is to use the identity $\sin^2 x + \cos^2 x = 1$ along with substitutions. Recall the trigonometric circle.

Fig. trig. circle

Case A: If $n = 2k + 1$ is odd,

$$\int \sin^m x \cos^{2k+1} xdx = \text{take out a cos} = \int \sin^m x \cos^{2k} x \cos xdx$$

$$= \int \underbrace{\sin^m x}_{u}(1-\sin^2 x)^k \underbrace{\cos x dx}_{du} = \left\{(\cos^2 x = 1 - \sin^2 x),\ u = \sin x,\ du = \cos x dx\right\} = \int u^m (1-u^2)^k du$$

The integral is now in the form of a polynomial, which you can easily solve.

Case B: If $m = 2k+1$ is odd,

$$\int \sin^{2k+1} x \cos^n x dx = \{\text{take out a sin}\} = \int \sin^{2k} x \sin x \cos^n x\, dx$$
$$= \int (1-\cos^2 x)^k \cos x \sin x\, dx = \{u = \cos x,\ du = -\sin x dx\} \quad (5)$$

(The rest is similar to Case A.)

Case C: If both m, and n are even, use the *Half-Angle trigonometric identities.*

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Use these to convert to Case A or B.

Example:

$$\int \underbrace{\sin^3 x}_{\sin^2 x \sin x} \cos^4 x dx = \int \underbrace{(1-\cos^2 x)}_{1-u^2} \underbrace{\cos^4 x}_{u^4} \underbrace{\sin x dx}_{-du} \quad (6)$$
$$= \{u = \cos x,\ du = -\sin x dx\} = -\int (1-u^2) u^4 du = \int (u^6 - u^4) du \quad (7)$$
$$= \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C \quad (8)$$

Example:

$$\int \cos^2 x dx = \frac{1}{2}\int (1 + \cos 2x) dx = \frac{x}{2} + \frac{\sin 2x}{4} + C. \quad (9)$$

Example:

$$\int \cos^4 x dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx = \dots$$

Recall: $(a+b)^2 = a^2 + 2ab + b^2$. For the integral this gives

$$\dots = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx =$$

$$\frac{x}{4}+\frac{\sin 2x}{4}+\frac{1}{4}\int \cos^2 2x dx=\{u=2x,\ du/2=dx\}=$$

$$\frac{x}{4}+\frac{\sin 2x}{4}+\frac{1}{8}\int \cos^2 u du=\{\text{from an earlier example.}\}$$

$$\frac{x}{4}+\frac{\sin 2x}{4}+\frac{1}{8}\left(\frac{u}{2}+\frac{\sin 2u}{4}\right)+C\ =$$

$$\frac{3}{8}x+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$$

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Integrals Involving tan and sec.

$$\int \tan^m x \sec^n\ x dx = ?$$

The idea here will be to use $1+\tan^2 x=\sec^2 x$ along with substitutions.
Case A: If $n=2k$ is even,

$$\int \tan^m x \sec^n\ x dx =$$

$$\int \tan^m x(1+\tan^2 x)^{k-1}\sec^2 x\, dx=\left\{u=\tan x du=\sec^2 x dx\right\}$$

$$=\int u^m(1+u^2)^{k-1}du$$

This integrant in the last integral is a polynomial. It can be written as a sum of monomials and therefore the integral is easy to compute.
Case B: If $m=2k+1$ is odd,

$$\int \tan^{2k+1} x \sec^n x dx =$$

$$\int \underbrace{(sec^2 x-1)^k}_{(\tan^2 x)^k}\sec^{n-1} x \underbrace{\sec x\tan x dx}_{du} =$$

$$\int (u^2-1)^k u^{n-1}du=\{u=\sec x,\ du=\sec x\tan x\}$$

Example:

$$\int \sec^6 x dx=\int \sec^4 x\sec^2 x dx=$$

$$\int (\sec^2 x)^2\sec^2 x dx=$$

$$\int (1+\tan^2 x)^2\sec^2 x dx=\left\{u=\tan x,\ du=\sec^2 x dx\right\}=$$

$$\int (1+u^2)^2 du = \int (1+2u^2+u^4) du$$
$$u + 2/3u^3 + 1/5u^5 + C =$$
$$\tan x + 2/3\tan^3 x + 1/5\tan^5 x + C\ .$$

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Integrals of the form

$$\int \sin(nx)\cos(mx)dx \, m \neq \pm n\ .$$

Can be easily computed using the identity $\sin A\cos B = \frac{1}{2}[\sin(A-B) - \sin(A+B)]$:

$$\int \sin(nx)\cos(mx)dx = \frac{1}{2}\int [\sin(n-m)x - \sin(n+m)x]dx =$$
$$\frac{1}{2}\left[\frac{-\cos((m-n)x)}{m-n} + \frac{\cos((m+n)x)}{m+n}\right] + C$$

Similar identities for $\sin A\cos B, \cos A\cos B$ give other integrals of this type. For example

$$\int \sin(mx)\sin(nx)dx =$$
$$\frac{1}{2}\int [\cos((m-n)x) - \cos((m+n)x)]dx =$$
$$\frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right] + C\ .$$

As an exercise compute the integral

$$\int \cos(mx)\cos(nx)dx =$$

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Answers for trigonometrical integrals can look quite different being nevertheless identical. This can happen due to trigonometric identities. Here is an example.

Compute the integral

$$\int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx =$$

$$\left\{u = \sin x,\ du = \cos x dx,\ cos^2 x = 1 - \sin^2 x = 1 - u^2\right\} = \int \frac{du}{1-u^2} = \dots$$

Using the identity

$$\frac{1}{1-u^2} = \frac{1}{2}\frac{1}{1-u} + \frac{1}{2}\frac{1}{1+u}$$

we can evaluate the integral as

$$\dots = \frac{1}{2}\int \frac{du}{1-u} + \frac{1}{2}\int \frac{du}{1+u} =$$

$$\frac{1}{2}\ln(\frac{1+u}{1-u}) = \frac{1}{2}\ln(\frac{1+\cos x}{1-\cos x}) + C$$

On the other hand we can multiply the numerator and the denominator in the integral by $\sec x + \tan x$. Then we can notice that

$$\frac{\sec x + \tan x}{\cos x} = (\sec x + \tan x)'$$

For the integral this means

$$\int \frac{1}{\cos x} dx = \int \frac{\sec x + \tan x}{\cos x(\sec x + \tan x)} dx =$$

$$= \int \frac{(\sec x + \tan x)'}{\sec x + \tan x} dx = \{y = \sec x + \tan x\} =$$

$$\int \frac{dy}{y} = \ln y + C = \ln(\sec x + \tan x) + C$$

The two answer for the integral look quite different but they represent the same function. Indeed, it is a nice exercise to check that there is an identity

$$\frac{1}{2}\ln(\frac{1+cosx}{1-cosx}) = \ln(\sec x + \tan x)$$