1 LECTURE 1

From *The Road Less Travelled*, by S. Peck:
The secret of life is... "Life is difficult."
If you expect life always to be easy, you will be continually disappointed. However, if you expect it to be difficult, you will be able to handle it.

The secret of calculus is..."Calculus is difficult."
If you expect calculus to be difficult and work at it, you'll be able to handle it.

Basic idea of Calculus

- Approximate a "curved object by a piece-wise "flat" object.
- Pass to the limit in the approximation when "it is not distinguishable" from the "curved object".

Differentiation.
The slope of the line passing through points $(x, f(x))$ and $(a, f(a))$ is

$$\frac{f(x) - f(a)}{x - a}.$$

When $x \to a$ this line becomes the tangent line to the graph of $f(x)$ at $x = a$. The limit of its slope becomes the slope of the tangent line at $x = a$, also known as the *derivative* of $f(x)$ at $x = a$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

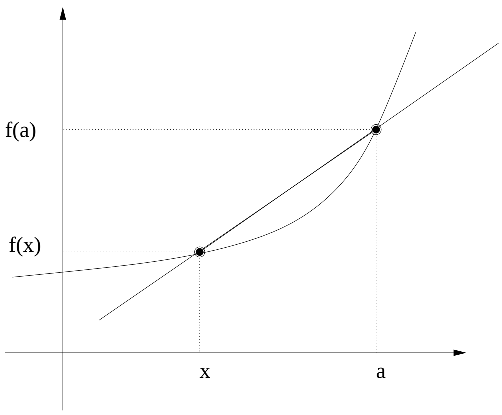

Figure 1: An "almost tangent" line

Remark 1. *The limit may not exist. This means that the tangent line to the graph $y = f(x)$ at $x = a$ does not exist. In other words, the function f is not differentiable at the point $x = a$ (see Fig.* **??**

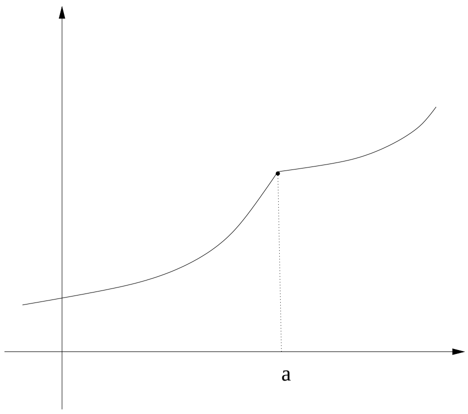

Figure 2: The function is not differentiable at $x = a$.

Recall two important properties of differentiation:

- Product rule: $(fg)' = f'g + fg'$.
- Chain rule: $f(g(x))' = f'(g(x))g'(x)$.

Integration

We want to find the area "between the function $f(x)$ and the real line". Assume the function is positive (for general discussion see lectures on numerical integration). To do this we first replace $f(x)$ by a piecewise flat function (see Fig.**??**:

- Partition the interval into n segments $[x_{i-1}, x_i]$ with $x_0 = a, x_n = b$.
- Choose a point x_i^* in each interval $[x_{i-1}, x_i]$.
- Replace the original function by a piecewise constant function $f_n(x)$ defined as:

$$f_n(x) = f(x_1^*) \text{ when } a \leq x < x_1$$
$$f_n(x) = f(x_i^*) \text{ when } x_{i-1} \leq x < x_i$$
$$f_n(x) = f(x_n^*) \text{ when } x_{n-1} \leq x < b$$

Its graph for $f(x) = x^2$ and $x_i^* = x_{i-1}$, is shown on Fig. **??**. The area between the function $f_n(x)$ and the real line is the sum:

$$\sum_{i=1}^{n} \underbrace{f(x_i^*)\Delta x_i}_{\text{Area of rectangle}}$$

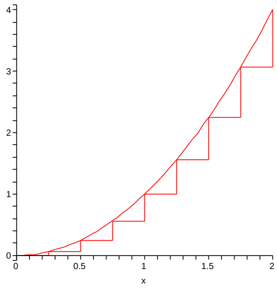

Figure 3: The approximation of x^2 by a piece-wise constant function

where $\Delta x_i = x_i - x_{i-1}$. The limit of this sum when $\Delta x \to 0$

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is the area between the function and the real line, also known as the *integral* of $f(x)$ over the interval $[a, b]$. It is also known as *definite integral.*

Remark 2. *Just as with differentiation not every function can be integrated. Only those for which the limit exists for all partitions and does not depend how to take the limit are integarble. Continuous functions are integrable.*

The Fundamental Theorems of Calculus

1) $\int_a^b f'(x)dx = f(b) - f(a)$

2) $\frac{d}{dx} \underbrace{\int_a^x f(t)dt}_{F(x)} = f(x)$

The function $F(x)$ such that $f(x) = \frac{dF(x)}{dx}$ is called an *antiderivative* of $f(x)$, or *indefinite integral of* $f(x)$. Clearly it is defined up to an arbitrary constant.

Remark 3. *The philosopher Berkeley, after whom this city is named, criticized the "infinitesimal" ideas of calculus.*

- $\underline{df}$ → *"infinitesimal small change in* f*"*
- $\overline{dx}$ → *"infinitesimal small change in* x*"*
- $\int_a^b f(x)dx$ ← *"sum of areas of infinitely many infinitely thin rectangles."*

Perhaps his main problem with calculus was that he never had a chance to study it systematically.

Remark 4. *The name "Calculus" is probably the legacy of Newton. Another name for calculus is "Mathematical Analysis". These two different names taken together give better picture of the subject:*

The theory of Calculus, and especially the careful study of limits, makes the ideas outlined above clear and precise.

| Theory | → | Practical Methods |

The three main topics of this course are:
1) **Techniques to computing integrals.** These require computational and manipulative skills, but no new theory.
Example.

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

2) **Sequences and Series.** These involve much more theory.
Examples.
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$
$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$
$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2} + \ldots$ Taylor Series
3) **Ordinary Differential Equations.**
This subject will have some new theory and lots of applications.
Example. Mass on a spring with damping.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0, \; x = x(t)$$

We will learn how to solve this differential equation for $x(t)$, and so be able to predict the motion of the physical system.

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Here we begin part one of the course: techniques for computing integrals.

Review

Recall that if $F' = f$, we say F is an **antiderivative of** f. It is clear that if $F(x)$ and $G(x)$ are two aniderivatives of f then

$$\frac{dF(x)}{dx} - \frac{dG(x)}{dx} = f(x) - f(x) = 0$$

Therefore $F(x) - G(x) = C$ is a constant and thus, all antiderivatives of $f(x)$ has the form $F(x) + C$. The *indefinite integral* of f is an antiderivative of f up to an arbitrary constant:

$$\int f dx = F + C$$

Here C is an arbitrary constant and $F(x)$ is an aniderivative of f.

The Fundamental Theorem of Calculus relates definite integrals which are important in applications and indefinite integrals. It states.

$$\int_a^b f\,dx = F(x)|_a^b = F(b) - F(a)$$

Thus, if we can find an antiderivative (compute indefinite integral) we can compute definite integrals. Computation of indefinite integrals will be our goal for the next few lectures where we will focus on techniques of computing indefinite integrals.

Examples: 1. Let $f = \sec^2 x$, then $F = \tan x$ is an antiderivative of f. We have

$$\int \sec^2 x\,dx = \tan x + C$$

and

$$\int_0^{\pi/4} \sec^2 x\,dx = \tan x|_0^{\pi/4} = 1 - 0 = 1.$$

2. For many simple functions antiderivatives can be found by easy guessing:

$$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = x^n$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}e^x = e^x$$

etc. Many of such integrals can be found on the inner cover of the textbook.

Substitution Method

This is, essentially, a repackaging of the Chain Rule:

$$F(g(x))' = F'(g(x))g'(x)$$

Remember that F is an antiderivative of f, thus:

$$F(g(x))' = f(g(x))g'(x)$$

By the definition of an autiderivative

$$\int F' dx = F + C$$

combining this with the Chain rule we have

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

This is the Substitution Rule. Symbolically, we can write

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$, $\frac{du}{dx} = g'(x)$, and $du = g'(x)dx$. **Examples:**

1.

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

Do change of variables $u = \cos x$, "$du = -\sin x\, dx$" Recall: $(\ln|u|)' = \frac{1}{u}$ for $u \neq 0$. Therefore

$$\int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du = -\ln|u| + C = -\ln|\cos x| + C = \ln\sec x| + C$$

2. $\int e' \cos x\, dx$

$= e' \cos x + \int e' \sin s\, dx$ Integrate by parts again.

$= e'(\cos x + \sin x) - \int e' \cos x\, dx$

$\Rightarrow \int e' \cos x\, dx = \frac{e'}{2}(\cos x + \sin x) + C$

Integration By Parts

This is essentially, a repackaging of the Product Rule (or Leibnits rule) for differentiation

$$(fg)' = f'g + fg'$$

Integrating this identity we obtain

$$\int f'g + fg'dx = \int (fg)'dx = fg + C$$

Therefore

$$\int fg'dx = fg - \int f'gdx$$

Symbolically the integration by parts is usually written as

$$\int udv = uv - \int vdu$$

where

$$u = f,\ dv = g'dx,\ du = f'dx,\ v = g$$

Examples

1.

$$\int \underbrace{\overbrace{x}^{f}}_{u} \underbrace{\overbrace{\sec^2 x}^{g'} dx}_{dv} = \left\{ \begin{array}{cc} dv = \sec^2 x & u = x \\ v = \tan x & du = dx \end{array} \right\}$$

$$= x\tan x - \int \tan x\, dx = x\tan x - \ln|\sec x| + C$$

2.

$$\int e^{x^{1/2}} dx \quad = \quad \left\{ \begin{array}{cc} u = x^{1/2} & du = \frac{1}{2} x^{-1/2} dx \\ 2x^{1/2} du = dx & 2udu = dx \end{array} \right\} = \int ue^u du$$

Integrating by parts, we find that:

$$\int ye^y dy = ye^y - e^y + C.$$

Thus

$$\int e^{x^{1/2}} dx = 2 \int ue^u du == 2e^u(u-1) + C = 2e^{x^{1/2}}(x^{1/2} - 1) + C.$$

3.

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$$

Integrate by parts again:

$$= e^x(cosx + \sin x) - \int e^x \cos x dx$$

The unknown integral appeared on the right side with negative sign. Solving this equation for the initial integral we obtain:

$$\int e^x \cos x dx = \frac{e^x}{2}(\cos x + \sin x) + C$$