

Lectures on motivic Donaldson-Thomas invariants and wall-crossing formulas

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December 9, 2011

Contents

1	Introduction	4
2	Motivations	4
2.1	Overview of Lectures	4
2.2	Calabi-Yau manifolds	5
2.3	Calabi-Yau manifolds and mirror symmetry	7
2.4	Calabi-Yau categories	8
2.5	Geometric example of DT-invariant: holomorphic Casson invariant	9
2.6	Algebraic example: representations of quivers	10
2.7	DT vs GW	12
3	Wall-crossing formulas	13
3.1	A -model wall-crossing: example in the Fukaya category	13
3.2	Same example but from the algebraic perspective	15
3.3	Stability data and wall-crossing formulas for graded Lie algebras	16
3.4	An example of the wall-crossing formula	19
3.5	Another example: stability on $\mathfrak{gl}(n)$ and $2d$ wall-crossing of Cecotti and Vafa	21
4	Cohomological Hall algebra of a quiver without potential	22
4.1	Why quivers?	22
4.2	Semistable representations of quivers	23

4.3	Counting of stable objects: the case of finite fields	25
4.4	Counting of stable objects: arbitrary field case	28
4.5	COHA: definition	31
4.6	Explicit formula for the product	33
4.7	Example: quivers with one vertex	34
4.8	Non-symmetric example: quiver A_2	37
5	Generalization of quivers: smooth algebras	38
6	COHA of a quiver with potential	39
6.1	Cohomology theories for a space with a function	39
6.1.1	Betti realization	40
6.1.2	De Rham and étale realizations	41
6.2	Example	42
6.3	Twisted graded algebras in tensor categories	43
6.4	Cohomology theories	44
6.5	Category of exponential mixed Hodge structures	45
6.6	A generalization to smooth algebras	46
7	Motivic COHA and motivic DT-series	48
7.1	Motivic DT-series and quantum tori	48
7.2	An application: finite order birational maps	50
7.3	Potentials linear in a group of variables	52
8	Stability conditions and motivic DT-invariants	53
8.1	Stability and Harder-Narasimhan filtration	54
9	Admissible series and motivic DT-invariants	56
9.1	Admissible series	56
9.2	Admissibility of DT-series and motivic DT-invariants	59
10	Motivic DT-invariants for 3CY categories	62
10.1	Stability conditions	62
10.2	Critical COHA and 3CY categories	65
10.3	Line operators, framed objects and positivity	69
11	Motivic DT-invariants, mutations and cluster transformations	73
11.1	Mutations of quivers with potentials	73

11.2	Motivic DT-series and mutations	74
11.3	Categorical meaning	76
12	Collapsing Calabi-Yau manifolds, SYZ torus fibrations and Mirror Symmetry	78
12.1	Collapsing Calabi-Yau manifolds	78
12.2	Semiflat spaces	79
12.3	Modification of semiflat structure	82
12.4	Hyperkähler case	86
13	Twistor families and Stokes data	90
13.1	Complex integrable systems and families of complex symplectic manifolds	90
13.2	Irregular singularities and Stokes phenomenon	92
13.2.1	Formal classification	93
13.2.2	Non-formal classification	94
14	Application to Hitchin systems	96
14.1	Reminder on complex integrable systems of Seiberg-Witten type	96
14.2	Hitchin systems	97
15	Black holes moduli spaces and attractor flows	99
15.1	Local Calabi-Yau manifolds vs compact ones	99
15.2	Moduli spaces of supersymmetric black holes	100
16	Wall-crossing formulas for the Hall algebra of lattices	104
17	Some open problems and speculations	108
17.1	Chern-Simons theory with complex gauge group and new invariant of $3d$ manifolds	108
17.2	Holomorphic Chern-Simons and manifolds with G_2 -holonomy .	109
17.3	Categorification of critical COHA	113
17.4	Donaldson 4d theory, Borchers automorphic forms	114
17.5	Stability conditions on curves and moduli of abelian differentials	115
17.6	Resurgence and WCF	115

1 Introduction

These notes originate in two lecture courses: the master class on wall-crossing given by M.K. at the Centre for Quantum Geometry of Moduli Spaces, Aarhus University in August 2010 and the Chern-Simons master class on motivic Donaldson-Thomas invariants given by Y.S. at the University of California, Berkeley in October 2010. Aim of present notes of the lecture courses is to give an exposition of our approach to the subject avoiding technical details. The latter can be found in our original papers. Such an informal style allowed us to cover a lot of material in a relatively short period of time (one week for each lecture course, which included 20 lectures in Aarhus and 10 lectures in Berkeley). We tried to preserve that lecture style in present notes. Nevertheless, on several occasions we decided to put on the paper precise definitions and theorems instead of their informal descriptions given at the lectures.

We thank to Jorgen Andersen and Nikolai Reshetikhin who convinced us to make our notes available not only to the participants of master-classes but also to a wider audience.

2 Motivations

2.1 Overview of Lectures

Aim of these lectures is to give an introduction to and overview of the joint project of M.K. and Y.S. on algebraic and geometric structures underlying two closely related topics:

- a) Donaldson-Thomas invariants for Calabi-Yau 3-folds.
- b) BPS invariants in string theory and gauge theory.

Putting them in the appropriate framework of $3CY$ categories we will find a relationship of our theory with so different topics as representation theory of quivers, matrix integrals, complex integrable systems, cluster algebras and topological invariants of 3-dimensional manifolds. We are not going to speak much about the relationship to physics. A lot of interesting work was done in order to understand and interpret our results in physics terms. We just mention here papers by Gaiotto-Moore-Neitzke, Cecotti-Vafa, Aganagic, Ooguri and many others. This lecture course is intended for mathematical audience. Hence we will use physics only for motivations and analogies.

Here are some arXiv references to our works, which are relevant to the lectures:

arXiv:math/0011041

arXiv:math/0406564

arXiv:math/0606241

arXiv:0811.2435

arXiv:0910.4315

arXiv:1006.2706

Technically, only last three papers are devoted to motivic DT-invariants. First two were motivated by our approach to Homological Mirror Symmetry, which in the end clarified how one should think about wall-crossing formulas. The 2006 paper contains our approach to A_∞ -algebras and A_∞ -categories via non-commutative algebraic geometry. It gives a very clear understanding that invariants of 3CY categories should be understood in terms of so-called (super)potential, which is a sort of generalized Chern-Simons functional. In order to make the relations to categories more transparent, we will spend first lecture on motivations, starting with the classical work of Richard Thomas on DT-invariants. But first we recall the notion of Calabi-Yau manifold and Calabi-Yau category.

2.2 Calabi-Yau manifolds

Although many results of our lecture course are valid for any ground field, We will assume for most of the time that we work over the field \mathbf{C} of complex numbers. Let us first recall a definition of Calabi-Yau manifold.

Definition 2.2.1 *A CY manifold is a connected Riemannian manifold X , $\dim_{\mathbf{R}} X = 2d$ with the holonomy group reduced to $SU(d) \subset O(2d, \mathbf{R})$.*

Clearly X is a complex manifold of dimension d , which carries a Kähler form $\omega^{1,1}$ and holomorphic non-vanishing form $\Omega^{d,0}$ such that $(\omega^{1,1})^d = |\Omega^{d,0} \wedge \overline{\Omega^{d,0}}|$. It follows from the definition that the Ricci curvature of X vanishes. If X is projective then the holomorphic volume form gives a trivialization of the determinant bundle, hence the canonical class $K_X = 0$. In particular $c_1(T_X^{1,0}) = 0 \in Pic(X)$.

Theorem of Yau ensures that a **compact** Kähler manifold $(X, \omega^{1,1})$ with vanishing first Chern class admits a unique Ricci flat metric with the same cohomology class as $\omega^{1,1}$.

Theorem of Tian and Todorov ensures that for a **compact** Calabi-Yau manifold X , the local deformation theory is unobstructed. As a result, the moduli space \mathcal{M}_X of complex structures on X is a smooth orbifold of complex dimension $h^{1,d-1} = rk H^1(X, \Omega^{d-1})$.

Calabi-Yau manifolds of dimension $d = 3$ play especially important role in the String Theory (they compactify 10-dimensional superstring background to the 4-dimensional space-time). Mirror symmetry phenomenon was discovered in case of the quintic in \mathbf{CP}^4 . In that case $h^{1,0}(X) = 0$ and the moduli space has roughly 10^5 connected component. For a Calabi-Yau manifold its Hodge dimond is symmetric with respect to the diagonal. If we denote $h^{1,2} = l, h^{2,1} = m$ then for the quintic we have $l = 1, m = 101$.

There is an interesting class of non-compact CY manifolds called “local”. For $d = 3$ one can construct examples by considering the total spaces of anticanonical bundles over surfaces, e.g. \mathbf{CP}^2 . Another example is given by the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over \mathbf{P}^1 . Yet another example is provided by the CY manifold associated with an affine curve $P(x, y) = 0$. Adding new variables u, v one has a non-compact subvariety in \mathbf{C}^4 defined by the equation

$$uv + P(x, y) = 0.$$

The latter carries a nowhere vanishing holomorphic 3-form

$$\Omega^{3,0} = \frac{dx \wedge dy \wedge du \wedge dv}{uv + P(x, y)}.$$

More generally one has a big class of “toric CYs”. In the case $d = 3$ they are glued from several copies of \mathbf{C}^3 .

Question 2.2.2 1) Find a satisfactory “metric” definition of local CY manifolds (e.g. is there a compactification \overline{X} by the divisor with normal crossings such that the top degree form $\Omega^{d,0}$ has poles on $D = \overline{X} - X$?)

2) Find analogs of Yau and Tian-Todorov theorems in this case.

Local CYs are typically cones over varieties which have positive Ricci curvature. Structure of the latter is far from being clear.

2.3 Calabi-Yau manifolds and mirror symmetry

Mirror symmetry conjecture says (roughly speaking) that for a “cusp” in the moduli space \mathcal{M}_X one can find another CY manifold X^\vee (called “mirror dual to X ”) such that its Hodge dimond is obtained from the one for X by rotation by 90 degrees. Cusp is defined by a path $X_t, t \rightarrow 0$ such that the monodromy around the point $t = 0$ acting on the middle cohomology has a maximal Jordan block (i.e. of size d).

Mirror symmetry admits many reformulations. Let us mention one of them. Let $n_{0,d}$ be the genus zero and degree d Gromov-Witten invariant of X^\vee (i.e. properly defined number of rational curves of degree $d \in H_2(X^\vee, \mathbf{Z})$). Let us consider a generating function

$$F_0([\omega]) = \frac{1}{6} \int_{X^\vee} \omega^3 + \sum_{d \neq 0} n_{0,d} \exp\left(\int_d \omega\right).$$

Here ω is the Kähler form. Hence F_0 is a germ of analytic function in a domain in $H^2(X, \mathbf{C})$. We can extend it to a function \bar{F}_0 on $H^0(X, \mathbf{C}) \oplus H^2(X, \mathbf{C})$ by the formula

$$\bar{F}_0(t, [\omega]) = t^2 F_0([\omega]/t^2).$$

Clearly we obtain a homogeneous function of degree +2. The graph of $d\bar{F}_0$ defines a complex Lagrangian cone in even cohomology of X^\vee (symplectic structure is given by Poincaré duality, and Calabi-Yau property ensures isomorphism of H^2 and H^4).

On the other hand, let us consider the moduli space of “gauged” Calabi-Yau manifolds (i.e. pairs $(X, \Omega_X^{3,0})$). The period map locally embeds this moduli space as a complex Lagrangian cone $\mathcal{L}_X \subset H^3(X^\vee, \mathbf{C})$.

Theorem 2.3.1 *The Lagrangian cone defined via the graph of $d\bar{F}_0$ is isomorphic (modulo the action of $Sp(2m+2, \mathbf{C})$ on $H^3(X, \mathbf{C})$) to the germ of \mathcal{L}_X at infinity.*

Notice that under this “mirror isomorphism” the integral lattice $H^3(X, \mathbf{Z})$ is identified not with $H^{ev}(X^\vee, \mathbf{Z})$ but with $\hat{\Gamma}_{T_{X^\vee}} \wedge H^{ev}(X^\vee, \mathbf{Z})$, where

$$\hat{\Gamma}_{T_{X^\vee}} = 1 - \frac{\zeta(2)}{2} c_2(T_{X^\vee}) + \frac{\zeta(3)}{6} c_3(T_{X^\vee})$$

is the Γ -class of the tangent bundle.

There is a generalization of the above Theorem to the higher genus Gromov-Witten invariants (holomorphic anomaly equation).

2.4 Calabi-Yau categories

One can axiomatize the condition $K_X = 0$ in order to obtain a definition of an algebraic Calabi-Yau variety (over any ground field \mathbf{k}). Here is the idea how to formulate it categorically. The Serre duality turns into a non-degenerate pairing

$$\text{Ext}^i(E, F) \otimes \text{Ext}^{d-i}(F, E) \rightarrow \mathbf{k}.$$

Hence the derived category of coherent sheaves $D^b(X)$ is endowed with a non-degenerated pairing

$$(\bullet, \bullet) : \text{Hom}(E, F) \otimes \text{Hom}(F, E) \rightarrow \mathbf{k}[-d]$$

(latter is the shifted constant complex \mathbf{k}).

The category $D^b(X)$ is triangulated. If the ground field is \mathbf{C} the category is equivalent to the triangulated envelope (realized as the category of twisted complexes) of the dg-category of dg-modules over the dg-algebra $\Omega^{0,\bullet}(X)$, which are projective as graded modules (i.e. if we forget differentials). Working with this dg-model we observe that besides of CY pairing on Hom 's we have also a differential of degree +1:

$$m_1 : \text{Hom}(E_0, E_1) \rightarrow \text{Hom}(E_0, E_1)[1],$$

and a composition

$$m_2 : \text{Hom}(E_0, E_1) \otimes \text{Hom}(E_1, E_2) \rightarrow \text{Hom}(E_0, E_2).$$

These two structures are compatible with each other and with the pairing in the natural way.

As our notation suggests, one can further axiomatize the structure, replacing dg-categories by A_∞ -categories. In this way we arrive to the notion of *d-dimensional Calabi-Yau category*. Then one has a collection of higher compositions of degrees $2 - n$:

$$m_n : \otimes_{0 \leq i \leq n-1} \text{Hom}(E_i, E_{i+1}) \rightarrow \text{Hom}(E_0, E_n)[2 - n],$$

and the CY pairing

$$(\bullet, \bullet) : \text{Hom}(E, F) \otimes \text{Hom}(F, E) \rightarrow \mathbf{k}[-d].$$

The compatibility of m_n 's with the Calabi-Yau pairing says that the polynomial functional

$$(m_n(a_1, \dots, a_n), a_{n+1})$$

is cyclically symmetric. Let us assume that the dimension $d = 3$. In that case for every object E we can define its **potential** as the following formal series:

$$W_E(a) = \sum_{n \geq 1} \frac{(m_n(a, \dots, a), a)}{n + 1}.$$

Here $a \in \text{Hom}^1(E, E)$. We can informally think of a as of an element of the tangent space to the moduli of formal deformations of E . Then the potential can be intuitively thought of as a function which is defined on a space slightly bigger than the space of objects (“off-shell”). Objects belong to the critical locus of this function, which is locally regular along the moduli space of objects and formal in the transversal direction.

Important: our theory of motivic DT-invariants deals with CY categories of dimension 3 only.

Only in this case $W_E(a)$ is a formal function (all summands have degree zero). Calabi-Yau categories can be associated with not necessarily compact Calabi-Yau manifolds. E.g. local Calabi-Yau threefolds give rise to compact 3-dimensional Calabi-Yau categories.

2.5 Geometric example of DT-invariant: holomorphic Casson invariant

Assume that X is a compact complex Calabi-Yau 3-fold. Holomorphic Chern-Simons functional is defined by the formula:

$$CS_{\mathbf{C}}(A_0 + \alpha) = \int_X \text{Tr} \left(\frac{1}{2} \bar{\partial}_{A_0} \alpha \wedge \alpha + \frac{1}{3} \alpha \wedge \alpha \wedge \alpha \right) \wedge \Omega^{3,0}.$$

Here A_0 is a $(0, 1)$ -connection on a complex C^∞ vector bundle of fixed topological type, say, the trivial one, α is a vector-valued 1-form. In the categorical notation it can be written as $\frac{1}{2}(m_1(a), a) + \frac{1}{3}(m_2(a, a), a)$.

Critical points of HCS are exactly complex structures. Hence the properly defined “number of critical points” is the “number” of holomorphic vector bundles with fixed class in the K -theory. In order to define the count properly one has to impose the stability condition and count semistable vector bundles (otherwise the moduli space is not well-defined). In this way one obtains *holomorphic Casson invariant* of Richard Thomas (the term is explained by analogy with real Chern-Simons when one counts flat connections instead of those which have vanishing $(0, 2)$ part of the curvature).

There is a mirror dual story also considered by Thomas in the end of 90's. In that case the functional has the form

$$f_{\mathcal{C}}(A, L) = \int_{L_0}^L (F_A + \omega^{1,1})^2$$

for the mirror dual Calabi-Yau manifold X^\vee endowed with the Kähler form $\omega^{1,1}$. Critical points are pairs: Lagrangian submanifold L and a flat connection A on a line bundle over it. The stability restriction is not that obvious here. It turns out that mirror dual to semistable vector bundles are *special Lagrangian submanifolds* (SLAGs for short). Hence we have a mirror dual count of SLAGs with fixed class in $H^3(X^\vee, \mathbf{Z})$. Notice that a pair (L, A) defines an object in the Fukaya category of X^\vee . Notice also that HCS functional is additive on short exact sequences. Thus in both cases we deal in fact with derived categories. Therefore from categorical point of view we have the following problem.

Problem

Let \mathcal{C} be a 3CY category (i.e. triangulated A_∞ -category with compactible CY pairing of degree 3). Fix a class $\gamma \in K_0(\mathcal{C})$. Fix a stability condition σ on \mathcal{C} (this notion was introduced by Tom Bridgeland). Define the number $\Omega(\gamma) \in \mathbf{Z}$ as the virtual number of σ -semistable objects of \mathcal{C} which have class γ .

Then $\Omega(\gamma)$ should be called the *DT-invariant of the category \mathcal{C}* . One can also make the generating function $\sum_\gamma \Omega(\gamma)e_\gamma$, where e_γ are parameters.

We will see later that there is a better way to think about DT-invariants $\Omega(\gamma)$. Moreover we will also see that the generating function is not the best way to encode a collection of integers. In particular partition functions in physics should be thought of as only first approximation to a more complicated object. Our theory of DT-invariants indicates that it is useful to think that e_γ are not just formal parameters but also carry some algebraic structure (e.g. a structure of Lie algebra).

2.6 Algebraic example: representations of quivers

In the geometric examples above we tried to count critical points of a functional (if there is a moduli space then we can take some well-behaved geometric invariant, e.g. Euler characteristic).

Here is an algebraic example which in a sense underlies many geometric examples as well.

Let us fix a finite quiver Q , with the set I of vertices, and $a_{ij} \in \mathbf{Z}_{\geq 0}$ arrows from i to j for $i, j \in I$. For any dimension vector

$$\gamma = (\gamma^i)_{i \in I} \in \mathbf{Z}_{\geq 0}^I$$

consider the space of representations of Q in complex coordinate vector spaces of dimensions $(\gamma^i)_{i \in I}$

$$\mathbf{M}_\gamma = \mathbf{M}_\gamma^Q \simeq \prod_{i, j \in I} \mathbf{C}^{a_{ij} \gamma^i \gamma^j}$$

endowed with the action by conjugation of a complex algebraic group

$$\mathbf{G}_\gamma := \prod_{i \in I} GL(\gamma^i, \mathbf{C}) .$$

The quotient stack $\mathbf{M}_\gamma / \mathbf{G}_\gamma$ is the stack of representations of Q with dimension vector γ .

Let W be a cyclically invariant element of the path algebra $\mathbf{C}Q$ (we can think of it as of an element of the vector space $\mathbf{C}Q / [\mathbf{C}Q, \mathbf{C}Q]$). A pair (Q, W) is called **quiver with potential**. Then one can associate with such a pair a 3CY category in the way very much similar to the above story with holomorphic Chern-Simons. It goes such as follows.

a) Observe that $W_\gamma = Tr(W) : \mathbf{M}_\gamma \rightarrow \mathbf{C}$ is a regular function. It is \mathbf{G}_γ -invariant. If $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ is an exact sequence of representations of Q of dimensions $\gamma_1, \gamma_1 + \gamma_2, \gamma_2$ respectively then $W_{\gamma_1 + \gamma_2}(E) = W_{\gamma_1}(E_1) + W_{\gamma_2}(E_2)$.

b) Define the abelian category $Crit(W)$ which is the full subcategory of the category of representations of Q consisting of critical points $Crit(W_\gamma)$ of all W_γ . It is closed under extensions. Spaces of objects are \mathbf{G}_γ -invariant.

Then one can prove that there is a 3CY-category which have a t -structure with the heart isomorphic to the abelian category $Crit(W)$. Moreover, one can generalize this story further, considering instead of sets $Crit(W_\gamma)$ some closed \mathbf{G}_γ -invariant subsets \mathbf{M}_γ^{sp} (special) of $Crit(W_\gamma)$ which satisfy the condition that for any short exact sequence as above the representation E is special if and only if both E_1 and E_2 are special. We will do this later.

This gives us pure algebraic example of 3CY category.

Remark 2.6.1 *Many people think that a DT-invariant is the properly defined number of ideal sheaves on a CY 3-fold. Count of ideal sheaves was initiated in the work of Nekrasov, Okounkov, Reshetikhin, Vafa. Subsequently,*

Maulik-Nekrasov-Okounkov-Pandharipande related the generating function for the ideal sheaves to the one for Gromov-Witten invariants. From our point of view ideal sheaves (or more generally stable pairs of Pandharipande and Thomas) correspond to semistable sheaves on CY 3-folds with the dimension of support less or equal than 1. The stability condition is “very large”. But there are other “chambers” in the space of stability conditions, where DT-invariants have a description very different from the one given in terms of the ideal sheaves. DT-invariants in different chambers are related by the wall-crossing formulas. This will be discussed in the next lecture.

Remark 2.6.2 *The above examples give a hint on how to define DT-invariants for 3CY categories. Namely, let us think of the space of objects $Ob(\mathcal{C})$ as a space of critical points of the potential W . Assume for simplicity that the critical value is zero. Then we are interested in those invariants of the category \mathcal{C} which can be defined in terms of the Milnor fiber of W . For example we can take e.g. equivariant cohomology or the Euler characteristic of the fiber $W^{-1}(\varepsilon)$ which is close to the critical one.*

2.7 DT vs GW

According to the (almost proven) conjecture of Maulik-Nekrasov-Okounkov-Pandharipande (arXiv 0312059), in the compact case the DT-numbers $\Omega(\gamma)$ determine all Gromov-Witten invariants for the same Calabi-Yau threefold. Since the latter are much more difficult to compute, the problem of understanding and computing DT-invariants has a great importance for symplectic geometry as well. But this relationship raises many questions. For example, the so-called *holomorphic anomaly equation* plays no role in the theory of DT-invariants. Also, DT-invariants depend on twice as many parameters as GW-invariants do. Another problem comes from the fact that DT-invariants depend on a choice of stability condition. But very little is known about stability conditions on $D^b(X)$ when X is compact. In relation to this last problem, let us mention that as we will see below there is a way to define and study properties of DT-invariants by categorical and algebraic means, avoiding difficult foundational geometric questions. From this unifying point of view we will see the relationship of our approach to DT-invariants to representation theory of quivers, complex integrable systems, black holes, tropical geometry, cluster algebras and many other things. Those theories in turn motivate many questions of the theory of DT-invariants (e.g. the Ooguri-Strominger-

Vafa conjecture predicts that in the case of local Calabi-Yau threefolds one has asymptotic formulas of the type $\Omega(n\gamma) = e^{C|\gamma|^2 n^2} (1 + o(1))$, for some $C > 0$).

3 Wall-crossing formulas

3.1 A-model wall-crossing: example in the Fukaya category

Objects of the Fukaya category are pairs: (*Lagrangian submanifold, local system on it*). Let us ignore local systems and consider only Lagrangian submanifolds of X .

We denote by \mathcal{M}_X the moduli space of complex structures on a *CY* 3-fold X (what is called “moduli of Calabi-Yau manifolds”) and by \mathcal{L}_X the moduli space of “gauged” Calabi-Yau manifolds. This means that points of \mathcal{L}_X parametrize pairs $b = (\tau, \Omega_\tau^{3,0})$ consisting of a complex structure τ on X and the corresponding holomorphic volume form. Clearly there is a map $p : \mathcal{L}_X \rightarrow \mathcal{M}_X$ with the fiber \mathbf{C}^* . The period map $b \mapsto \Omega_\tau^{3,0}$ gives a local embedding of \mathcal{L}_X as a complex Lagrangian cone in $H^3(X, \mathbf{C})$.

With a generic point $b \in \mathcal{L}_X$ we associate a linear functional (called central charge or stability function):

$$Z_b : H_3(X, \mathbf{Z}) \rightarrow \mathbf{C}, Z_b(\gamma) = \int_\gamma \Omega_\tau^{3,0}.$$

Then the genericity implies that there are no \mathbf{Q} -independent γ 's which are mapped by Z to the same straight line.

Let us choose a Kähler form. Then we can speak about Lagrangian manifolds and special Lagrangian manifolds (SLAGs). In the latter case $Arg(L) := Arg(\Omega^{3,0})|_L$ is constant. By looking at the gradient flow of the functional $\int_L |dArg(\Omega^{3,0})|_L|^2$ on Lagrangian submanifolds of Maslov index zero, one can convince oneself that every such Lagrangian submanifold “flows” to a singular “Lagrangian cycle” with smooth components which are SLAGs. Therefore SLAGs should be thought of as stable “building blocks” for the Fukaya category.

The DT-invariant $\Omega_b(\gamma) \in \mathbf{Z}, \gamma \in H_3(X, \mathbf{Z})$ depends on the point b (i.e. it depends on a complex structure). It is locally constant (since it is integer), but it can jump on real codimension one “walls”. It is a “number of SLAGs

with homology class γ . Hence after moving along a loop in the moduli space \mathcal{M}_X of complex structures, we arrive to the same $\Omega_b(\gamma)$. But the monodromy acts on $H^3(X, \mathbf{Z})$ via the symplectic linear group $Sp(2m+2)$. Hence we arrive to a constraint on DT-invariants: $\Omega(g\gamma) = \Omega(\gamma), g \in Sp(2m+2, \mathbf{Z})$. But there are also local constraints which we are going to discuss now.

If L_1, L_2 are SLAGS with primitive homology classes γ_1, γ_2 then moving from the region where $Arg(L_1) > Arg(L_2)$ we arrive to the region where $Arg(L_2) > Arg(L_1)$. Equivalently we can speak about inequalities $Arg Z(\gamma_1) > Arg Z(\gamma_2)$ and $Arg Z(\gamma_2) > Arg Z(\gamma_1)$. The wall is the space of b 's where $Z_b(\gamma_1) \in \mathbf{R}_{>0} Z_b(\gamma_2)$ (hence the arguments coincide). When we cross the wall, a new SLAG L can arise: the Lagrangian connected sum of L_1 and L_2 (small handle develops between L_1 and L_2). Its homology class is $\gamma_1 + \gamma_2$. Then the change of the number of SLAGs is the wall-crossing formula:

$$\Delta\Omega(\gamma_1 + \gamma_2) = \langle \gamma_1, \gamma_2 \rangle \Omega(\gamma_1) \Omega(\gamma_2),$$

where $\langle \bullet, \bullet \rangle$ is the Poincaré pairing on $H_3(X, \mathbf{Z})$ (in the compact case we can replace homology by cohomology).

In mirror dual story the wall-crossing happens when we count holomorphic semistable bundles because the notion of stability depends on a choice of Kähler structure. There is an obvious similarity of this $6d$ story with $4d$ Donaldson theory. recall that Donaldson invariants can jump on the real codimension 1 wall in the space of metrics if $b_+ = 1$.

Remark 3.1.1 *In the case when $H^1(L) \neq 0$ the Lagrangian submanifold is not isolated. If its SLAG then the tangent space to the moduli of deformations of L is the space of harmonic 1-forms on L . One can ask how to define $\Omega_b(\gamma)$ as an integral over the virtual fundamental class (the latter does not exist for the moduli space of SLAGs). E.g. one can consider pairs $(L, (E, \nabla))$, where (E, ∇) is a complex local system on L , i.e. a bundle with flat connection (or representation $\pi_1(L) \rightarrow GL(N, \mathbf{C})$). Rescaling $\nabla \mapsto k\nabla$ and taking the limit $k \rightarrow \infty$ one compactify the moduli space of the above pairs by “Higgs bundles” over L . Here one can keep in mind the analogy with degeneration of the ε -connection $\varepsilon\partial_z + \varphi$ on a complex curve into the Higgs field φ . Our story is a real 3-dimensional analog of that. In particular the reality of L should imply reality of the spectrum of the limiting Higgs field. Similarly to the case of curves, our Higgs field on $E \rightarrow L$ defines (via its eigenvalues) a subvariety in T^*L . One can show that it is Lagrangian. If L is SLAG then it is probably defined by a multivalued harmonic form on L , which generalizes*

the above-mentioned McLean description of the moduli space of deformations of L . Thus we obtain a “spectral variety” of the Higgs bundle on L . In the case $d = 2$ we have a conventional spectral curve from the theory of Higgs bundles on curves. In the much more interesting case $d = 3$ which we discuss here, one can speculate that in the way we obtain a compactification of the moduli space of pairs $(L, (E, \nabla))$ as well as a virtual fundamental class of the compactification. One can hope that the wall-crossing formulas can be derived geometrically from the study of the dependence of this virtual fundamental class from the complex structure of X .

3.2 Same example but from the algebraic perspective

With two SLAGs L_1, L_2 with the intersection index $\langle \gamma_1, \gamma_2 \rangle = n$ we can associate the Kronecker quiver K_n with two vertices 1, 2 and n arrows from 1 to 2. Simplest case is $n = 1$. Let $Q = A_2$ be a quiver with two vertices 1, 2 and an arrows $1 \rightarrow 2$. A finite-dimensional representation of Q of dimension (γ^1, γ^2) is defined by a linear map $f : \mathbf{C}^{\gamma^1} \rightarrow \mathbf{C}^{\gamma^2}$. Analog of the central charge Z is now given by a homomorphism

$$Z : \mathbf{Z}^2 \rightarrow \mathbf{C}.$$

It is sometimes called *stability function*. We will assume that the image $Z(\mathbf{Z}_{\geq 0}^2 - \{0\})$ belongs to the upper half-plane. Then the stability function is the same as a pair of complex numbers (z_1, z_2) with imaginary parts belonging to the upper half-plane.. Semistable objects are defined similarly to the case of vector bundles on curves: semistable objects do not contain subobjects with bigger argument of the central charge. In this way we define $Arg(E) := Arg(Z(E))$. Equivalently, one can take imaginary part of Z , which gives an additive map $Im Z : \mathbf{Z}_{\geq 0}^2 \rightarrow \mathbf{R}_{\geq 0}$. Then the stability condition can be expressed in terms of a pair of real non-negative numbers (which we abusing the notation will denote by z_i) in terms of the slope function $\mu(\gamma^1, \gamma^2) = \frac{z_1\gamma_1 + z_2\gamma_2}{\gamma_1 + \gamma_2}$. Notice that although the inequality of slopes is equivalent to the inequality of the arguments, it is better to use the central charge as the main ingredient of the theory. Indeed, it is an additive function on the K_0 -group of the abelian category of quiver representations.

Let us choose $Z_0 = (z_1, z_2) = (0, 1)$. Then the slope is $\frac{\gamma_2}{\gamma_1 + \gamma_2}$. Irreducible representations S_1 and S_2 of dimensions $(0, 1)$ and $(1, 0)$ are semistable (in fact stable) and $Arg(S_1) > Arg(S_2)$. Any other representation with $\gamma_1 \geq 1$

contains $0 \rightarrow \mathbf{C}$ and hence cannot be semistable. Hence semistables are multiples $S_1^{\oplus n} := nS_1, S_2^{\oplus n} := nS_2, n \geq 1$. Let us choose a path Z_t in the space of stability functions from Z_0 to $Z_1 = (1, 0)$. Then for some t the arguments of S_1 and S_2 coincide. On the other hand, for the stability function Z_1 we have semistables nS_1, nS_2 and nS , where S is the representation $id : \mathbf{C} \rightarrow \mathbf{C}$ of dimension $(1, 1)$.

Thus we see that $\Omega_{Z_0}(\gamma)$ is equal to 1 for γ equal $(n, 0)$ or $(0, n)$ and is zero otherwise, while $\Omega_{Z_1}(\gamma)$ has an additional non-trivial value 1 for $\gamma = n(\gamma_1 + \gamma_2) = (n, n), n \geq 1$. One can say that after crossing the wall a new stable object is created: it is an extension (from the point of view of triangulated categories it is better to speak about exact triangle):

$$S_1 \rightarrow S \rightarrow S_2.$$

From this point of view it is clear why the SLAG L from the geometric story should be thought of as an extension $L_1 \rightarrow L \rightarrow L_2$.

This simple algebraic example gives a naive idea of how one can think about numbers $\Omega_Z(\gamma)$ and wall-crossing. Naively, one should form the moduli space \mathcal{M}_γ of Z -semistable objects having class γ . Here “class” can be the dimension vector or the Chern class or some other discrete parameter which is additive with respect to the K -theory class of an object. Then look at $\mathcal{M}_\gamma(\mathbf{F}_q)$, the number of points over a finite field (or take its finite extension). Hopefully, there is a limit of this number as $q \rightarrow 1$. This would be the motivic DT-invariant. We will see later how to modify this naive picture, and what we should take instead of the number of points over a finite field.

But first we are going to discuss a formalism of wall-crossing formulas, which does not depend on the origin of DT-invariants. Basically, it is a piece of linear algebra.

3.3 Stability data and wall-crossing formulas for graded Lie algebras

Let us fix a commutative unital \mathbf{Q} -algebra \mathbf{k} , a free abelian group Γ of finite rank, and a graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ over \mathbf{k} .

Definition 3.3.1 *Stability data on \mathfrak{g} is a pair $\sigma = (Z, a)$ such that:*

1) $Z : \Gamma \rightarrow \mathbf{R}^2 \simeq \mathbf{C}$ is a homomorphism of abelian groups called the central charge (or stability function);

2) $a = (a(\gamma))_{\gamma \in \Gamma \setminus \{0\}}$ is a collection of elements $a(\gamma) \in \mathfrak{g}_\gamma$, satisfying the following

Support Property:

There exists a non-degenerate quadratic form Q on $\Gamma_{\mathbf{R}}$ such that

1) $Q|_{\text{Ker } Z} < 0$;

2) $\text{Supp } a \subset \{\gamma \in \Gamma \setminus \{0\} | Q(\gamma) \geq 0\}$,

where we use the same notation Z for the natural extension of Z to $\Gamma_{\mathbf{R}}$.

Typically there is an involution $\eta : \Gamma \rightarrow \Gamma$ and one considers *symmetric* stability data, i.e. $a(\eta(\gamma)) = a(\gamma)$.

There is another data which are equivalent to the previous one, but sometime more convenient. Let \mathcal{S} be the set of strict sectors in \mathbf{R}^2 , possibly degenerate (i.e. rays). Recall that a strict sector is the one which is less than 180° and which has the vertex at the origin.

We consider the set of triples (Z, Q, A) such that:

a) $Z : \Gamma \rightarrow \mathbf{R}^2$ is a homomorphism of abelian groups;

b) Q is a non-degenerate quadratic form on $\Gamma_{\mathbf{R}}$ such that $Q|_{\text{Ker } Z} < 0$;

c) A is the map $V \mapsto A_V \in G_{V,Z,Q}$, where $V \in \mathcal{S}$ and $G_{V,Z,Q}$ is a pronilpotent group such that its Lie algebra

$$\text{Lie}(G_{V,Z,Q}) = \mathfrak{g}_{V,Z,Q} = \prod_{\gamma \in \Gamma \cap C(V,Z,Q)} \mathfrak{g}_\gamma,$$

where $C(V, Z, Q)$ is the convex cone generated by the set $S(V, Z, Q) = \{x \in \Gamma_{\mathbf{R}} \setminus \{0\} | Z(x) \in V, Q(x) \geq 0\}$.

We impose the following axiom on the set of triples (Z, Q, A) :

Factorization Property:

If $V = V_1 \sqcup V_2$ (in the clockwise order) then the element A_V is given by the product $A_V = A_{V_1} A_{V_2}$ where the equality takes place in $G_{V,Z,Q}$. There is a natural equivalence relation on the set of triples (since Q is not unique).

Theorem 3.3.2 *There is a one-to-one correspondence between the set of equivalence classes of triples (Z, Q, A) and the set of stability data.*

Indeed, the Factorization Property implies that A_V factorizes as a clockwise product of the elements A_l over all rays $l \subset V$ with the vertex at the origin. Each A_l is equal to $\exp(\sum_{Z(\gamma) \in l, Q(\gamma) \geq 0} a(\gamma))$. Similarly one proves the statement in the opposite direction.

On the set $Stab(\mathfrak{g})$ of stability data on \mathfrak{g} there is a Hausdorff topology such that the projection to the central charge is a local homeomorphism. Roughly speaking the continuity of a path in this topology is the continuity of its projection to central charges and the property that considered as a path in the space of collections of $(a(\gamma))_{\gamma \in \Gamma - \{0\}}$ it enjoys the property that every group element A_V does not change as long as no $Z(\gamma)$ with $a(\gamma) \neq 0$ enters V .

Definition 3.3.3 *A wall of first kind (notation \mathcal{W}_1) is a subset of the set of stability data (Z, Q, A) such that for the central charge $Z : \Gamma \rightarrow \mathbf{R}^2$ there exist \mathbf{Q} -independent points $\gamma_1, \gamma_2 \in \Gamma$ with the property $Arg(Z(\gamma_1)) = Arg(Z(\gamma_2))$.*

Let us consider a *generic* continuous path in the space of stability data, and let $Z_t, t \in [0, 1]$ be the corresponding path in the space of central charges. Let $t_0 \in [0, 1]$ corresponds to an intersection of the path with \mathcal{W}_1 . Then there is a $2d$ lattice mapped by Z_{t_0} into a straight line. Its intersection with the cone $Q \geq 0$ gives rise to the union of strict sectors $Cone_{t_0} \cup -Cone_{t_0}$. Consider a small loop around the origin in the plane of values of Z . The the product of the group elements A_l for all rays intersecting the loop is trivial. This leads to the two different factorization formulas for each element A_V . This observation is called the **wall-crossing formula**. Therefore the wall-crossing formula is essentially a corollary of the fact that for any strict sector V the group element A_V can be uniquely factorized as a certain product over all rays $l \subset V$:

$$\overrightarrow{\prod}_{l \subset V} A_l = \overleftarrow{\prod}_{l \subset V} A'_l,$$

where LHS and RHS correspond to the limits of $A_l(t)$ as $t \rightarrow t_0^+$ and $t \rightarrow t_0^-$ along the path. More explicitly, we have the following result, see 0811.2435:

Proposition 3.3.4 *(General wall-crossing formula) Let $\gamma = (m, n) \in \mathbf{Z}^2$, and $a(\gamma)^\pm(t_0) := a^\pm(m, n)$. Then the wall-crossing formula takes the form:*

$$\overrightarrow{\prod}_{(m,n)=1} \exp \left(\sum_{k \geq 1} a^-(km, kn) \right) = \overleftarrow{\prod}_{(m,n)=1} \exp \left(\sum_{k \geq 1} a^+(km, kn) \right),$$

where in the LHS we take the product over all coprime m, n in the increasing order, while in the RHS we take the product over all coprime m, n in the

decreasing order. Either of the products is equal to $\exp(\sum_{m^2+n^2 \neq 0} a^0(m, n))$, where the sum is taken over all positive m, n and $a^0(m, n) = a(\gamma)(t_0)$ for $\gamma = (m, n)$.

3.4 An example of the wall-crossing formula

This is an example which is related to Calabi-Yau categories.

Let Γ be a free abelian group endowed with a skew-symmetric integer-valued bilinear form $\langle \bullet, \bullet \rangle$. Consider a Lie algebra over \mathbf{Q} with the basis $(e_\gamma)_{\gamma \in \Gamma}$ such that

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}.$$

This Lie algebra is isomorphic (non-canonically) to the Lie algebra of Laurent polynomials on the algebraic torus $\mathbf{T} := \mathbf{T}_\Gamma = \text{Hom}(\Gamma, \mathbf{G}_m)$, endowed with the translation-invariant Poisson bracket associated with $\langle \bullet, \bullet \rangle$.

Let $Z : \Gamma \rightarrow \mathbf{C}$ be an additive map which is generic in the sense that there are no two \mathbf{Q} -independent elements of the lattice which are mapped by Z into the same real line. Otherwise Z belongs to the wall of first kind. Instead of choosing a quadratic form Q we can equivalently choose an arbitrary norm $\| \bullet \|$ on the real vector space $\Gamma_{\mathbf{R}} = \Gamma \otimes \mathbf{R}$. Finally, assume that we are given an **even** map $\Omega : \Gamma \setminus \{0\} \rightarrow \mathbf{Z}$ supported on the set B of such $\gamma \in \Gamma$ that $\| \gamma \| \leq C|Z(\gamma)|$ for some constant $C > 0$. Let $V \subset \mathbf{R}^2$ be a strict sector which has the vertex at the origin, and $C(V)$ be the convex hull of $Z^{-1}(V) \cap B$. We define an element $A_V \in G_V := \exp(\prod_{\gamma \in \Gamma \cap C(V)} \mathbf{Q} \cdot e_\gamma)$,

$$A_V := \overrightarrow{\prod}_{\gamma \in C(V) \cap \Gamma} \exp \left(-\Omega(\gamma) \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right),$$

where the product is taken into a clockwise order, and G_V is considered as a pronilpotent group.

A generic path $Z_t, 0 \leq t \leq 1$ in the space of the above additive maps Z intersects the wall of first kind at $t = t_0$ for which there is a lattice $\Gamma_0 \subset \Gamma$ of rank two such that $Z_{t_0}(\Gamma_0)$ belongs to a real line $\mathbf{R} \cdot e^{i\alpha} \subset \mathbf{C}$.

Denote by $k \in \mathbf{Z}$ the value of the form $\langle \bullet, \bullet \rangle$ on a fixed basis of $\Gamma_0 \simeq \mathbf{Z}^2$. We assume that $k \neq 0$, otherwise there will be no jump in values of Ω on Γ_0 . Let us consider the pronilpotent group generated by products of the following formal symplectomorphisms (automorphisms of $\mathbf{Q}[[x, y]]$) preserving

the symplectic form $-k^{-1}(xy)^{-1}dx \wedge dy$:

$$T_{a,b}^{(k)} : (x, y) \mapsto (x \cdot (1 - (-1)^{kab} x^a y^b)^{-kb}, y \cdot (1 - (-1)^{kab} x^a y^b)^{ka}), a, b \geq 0, a+b \geq 1.$$

Let $T_{a,b} := T_{a,b}^{(1)}$. Any exact symplectomorphism ϕ of $\mathbf{Q}[[x, y]]$ can be decomposed uniquely into a clockwise and an anti-clockwise product which gives a wall-crossing formula:

$$\phi = \overrightarrow{\prod}_{a,b} (T_{a,b}^{(k)})^{c_{a,b}} = \overleftarrow{\prod}_{a,b} (T_{a,b}^{(k)})^{d_{a,b}}$$

with certain exponents $c_{a,b}, d_{a,b} \in \mathbf{Q}$ which depend on the stability data. The passage from the clockwise order (when the slope $a/b \in [0, +\infty] \cap \mathbb{P}^1(\mathbf{Q})$ decreases) to the anti-clockwise order (the slope increases) gives the change of $\Omega_{|\Gamma_0}$ as we cross the wall of first kind. Later we will explain why the exponents in the wall-crossing formulas are integer numbers.

Proposition 3.4.1 *If one decomposes the product $T_{1,0} \cdot T_{0,k}, k > 0$ in the opposite order:*

$$T_{1,0} \cdot T_{0,k} = \prod_{a/b \text{ increases}} T_{a,kb}^{kd(a,b,k)},$$

then $d(a, b, k) \in \mathbf{Z}$ for all a, b, k .

Here are decompositions for $k = 1, 2$:

$$T_{1,0} \cdot T_{0,1} = T_{0,1} \cdot T_{1,1} \cdot T_{1,0},$$

$$T_{1,0} \cdot T_{0,2} = T_{0,2} \cdot T_{1,4} \cdot T_{2,6} \cdots T_{1,2}^{-2} \cdots T_{3,4} \cdot T_{2,2} \cdot T_{1,0}.$$

Slopes of the symplectomorphisms $T_{a,b}^{(k)}$ which appear in the wall-crossing formulas with non-trivial exponents (can be called the spectrum of the problem) are not arbitrary. The corresponding rays belong to a sector with the vertex at the origin and boundary lines $x/y = \lambda, x/y = \lambda^{-1}$, where λ, λ^{-1} are the roots of the quadratic equation $\lambda^2 - k\lambda + 1 = 0$, which is the characteristic polynomial of a certain matrix $g \in SL(2, \mathbf{Z})$. Then the spectrum is periodic with respect to the natural action of g on \mathbf{R}^2 .

3.5 Another example: stability on $\mathfrak{gl}(n)$ and $2d$ wall-crossing of Cecotti and Vafa

Previous example can leave an impression that non-trivial wall-crossing formulas are related to infinite-dimensional Lie algebras. Here we give a realistic finite-dimensional example.

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{Q})$ be the Lie algebra of the general linear group. We consider it as a Γ -graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$, where

$$\Gamma = \{(k_1, \dots, k_n) \mid k_i \in \mathbf{Z}, \sum_{1 \leq i \leq n} k_i = 0\}$$

is the root lattice. We endow \mathfrak{g} with the Cartan involution η . Algebra \mathfrak{g} has the standard basis $E_{ij} \in \mathfrak{g}_{\gamma_{ij}}$ consisting of matrices with the single non-zero entry at the place (i, j) equal to 1. Then $\eta(E_{ij}) = -E_{ji}$. According to our terminology this means that we consider symmetric (with respect to η) stability data on \mathfrak{g} .

We notice that

$$Hom(\Gamma, \mathbf{C}) \simeq \mathbf{C}^n / \mathbf{C} \cdot (1, \dots, 1) .$$

We define a subspace $Hom^\circ(\Gamma, \mathbf{C}) \subset Hom(\Gamma, \mathbf{C})$ consisting (up to a shift by the multiples of the vector $(1, \dots, 1)$) of vectors (z_1, \dots, z_n) such that $z_i \neq z_j$ if $i \neq j$. Similarly we define a subspace $Hom^{\circ\circ}(\Gamma, \mathbf{C}) \subset Hom(\Gamma, \mathbf{C})$ consisting (up to the same shift) of such (z_1, \dots, z_n) that there is no z_i, z_j, z_k belonging to the same real line as long as $i \neq j \neq k$. Obviously there is an inclusion $Hom^{\circ\circ}(\Gamma, \mathbf{C}) \subset Hom^\circ(\Gamma, \mathbf{C})$.

For $Z \in Hom(\Gamma, \mathbf{C})$ we have $Z(\gamma_{ij}) = z_i - z_j$. If $Z \in Hom^{\circ\circ}(\Gamma, \mathbf{C})$ then symmetric stability data with such Z is the same as a skew-symmetric matrix (a_{ij}) with rational entries determined from the equality $a(\gamma_{ij}) = a_{ij}E_{ij}$. Every continuous path in $Hom^\circ(\Gamma, \mathbf{C})$ admits a unique lifting to $Stab(\mathfrak{g})$ as long as we fix the lifting of the initial point. The matrix (a_{ij}) changes when we cross walls in $Hom^\circ(\Gamma, \mathbf{C}) \setminus Hom^{\circ\circ}(\Gamma, \mathbf{C})$. A generic wall-crossing corresponds to the case when the point z_j crosses a straight segment joining z_i and $z_k, i \neq j \neq k$. In this case the only change in the matrix (a_{ij}) is of the form:

$$a_{ik} \mapsto a_{ik} + a_{ij}a_{jk} .$$

This follows from the wall-crossing formula:

$$\begin{aligned} \exp(a_{ij}E_{ij}) \exp(a_{ik}E_{ik}) \exp(a_{jk}E_{jk}) &= \\ &= \exp(a_{jk}E_{jk}) \exp((a_{ik} + a_{ij}a_{jk})E_{ik}) \exp(a_{ij}E_{ij}) . \end{aligned}$$

This wall-crossing formulas appeared in the work of Cecotti and Vafa in the study of the change of the number of solitons in $N = 2$ two-dimensional supersymmetric QFT. In their paper the numbers a_{ij} were integers, and the wall-crossing preserved integrality. In our considerations, for any $Z \in \text{Hom}^{\circ}(\Gamma, \mathbf{C})$ the fundamental group $\pi_1(\text{Hom}^{\circ}(\Gamma, \mathbf{C}), Z)$ acts on the space of skew-symmetric matrices by polynomial transformations with integer coefficients. It can be identified with the well-known actions of the pure braid group on the space of upper-triangular matrices in the theory of triangulated categories endowed with exceptional collections. Furthermore, the matrices $\exp(a_{ij}E_{ij}) = 1 + a_{ij}E_{ij}$ can be interpreted as Stokes matrices of a certain connection in a neighborhood of $0 \in \mathbf{C}$, which has irregular singularities (tt^* -connection from the work of Cecotti and Vafa).

In a recent paper of Gaiotto-Moore-Neitzke a formula which combines the examples of this and the previous subsections was proposed. Although mathematical meaning of the combined formula is not very deep (it is related to a very simple graded Lie algebra which is the algebra of automorphisms of a rank n vector bundle on the torus associated with the lattice Γ) the geometric and categorical meaning of that formula is not quite clear. We discuss briefly in the last lecture a possible relation of that “ $2d/4d$ formula” to the holomorphic Chern-Simons functional.

4 Cohomological Hall algebra of a quiver without potential

4.1 Why quivers?

In our paper 0811.2435 the theory of motivic DT-invariants is developed for *ind-constructible 3CY categories*. Ind-constructibility is a technical property, which more or less means that objects form a countable union of algebraic varieties acted by algebraic groups. Then one can use motivic integration for constructing invariants. In this lecture course we are not going to use motivic integration. We will work in a slightly less generality of *quivers with potential*. Let us explain why this covers almost all (if not all) practical needs.

We explained in the first lecture that 3CY categories can have geometric or algebraic origin. But for the purposes of general theory algebraic approach is more powerful. Moreover, one can express algebra in terms of geometry using Bondal-Van den Bergh theorem. It says that if the derived category

of coherent sheaves has a generator then it is equivalent to the (triangulated envelope of the) category of dg-modules over a dg-algebra. For example, derived category of coherent sheaves on \mathbf{P}^1 is equivalent to the derived category of representations of the Kronecker quiver K_2 (two vertices 1, 2 and two arrows from 1 to 2). Quivers appear when we have explicit description of generators of the category. For example, in case of $D^b(\mathbf{P}^1)$ vertices of the quiver K_2 correspond to the sheaves $O, O(1)$. For a local Calabi-Yau one can pull-back an exceptional collection of vector bundles from the underlying Fano variety.

One can see quivers in the mirror dual story as well. For example, if the Fukaya category is generated by Lagrangian spheres (i.e. objects $E_i, i \in I$ such that $Ext^\bullet(E_i, E_i) \simeq H^\bullet(S^3)$) then one has a quiver with the set of vertices I and the number of arrows $i \rightarrow j$ is equal to $a_{ij} = -\dim Ext^1(E_i, E_j)$ (to be more precise one should assume that $Ext^m(E_i, E_j)$ is trivial unless $m = 1, 2$).

Furthermore, setting $E = \bigoplus_i E_i$ and restricting the potential W_E to $Ext^1(E, E)$ one defines a **potential of the quiver** (it is cyclically invariant non-commutative polynomial in arrows). We see that there is a big class of 3CY categories which can be described in terms of quivers with potential. Our general theory becomes more direct in this case, and many technical issues disappear. *For that reason, from now on we will consider mostly 3CY categories arising from quivers with potential.*

4.2 Semistable representations of quivers

Quiver is a finite graph Q . Let I be the set of vertices. We will work over a fixed ground field \mathbf{k} . The a representation of Q of dimension $\gamma = (\gamma^i)_{i \in I}$ is given by a collection of \mathbf{k} -vector spaces $V_i, i \in I$ and a linear map $f_a : V_i \rightarrow V_j$ for each arrow $a : i \rightarrow j$. Stability function is given by a group homomorphism $Z : \mathbf{Z}^I \rightarrow \mathbf{C}, Z(\gamma) = \sum_{i \in I} z_i \gamma^i$ such that $Im z_i > 0$. Given a stability function one defines the *argument* of a non-zero representation V of dimension γ by the formula $Arg(V) := Arg_Z(V) = Arg(Z(\gamma)) \in (0, \pi)$.

Definition 4.2.1 *In the above notation we say that V is Z -semistable if for any non-trivial proper subrepresentation V' one has $Arg(V') \leq Arg(V)$. We can V stable if the inequality is strict.*

We will denote the set of semistable representations of fixed dimension γ by \mathbf{M}_γ^{ss} . With the above definition one can prove the following result.

Proposition 4.2.2 *Any representation V admits a unique Harder-Narasimhan filtration (HN-filtration for short), i.e. a finite filtration*

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V$$

such that all quotients $gr_i(V) := V_i/V_{i-1}$ are Z -semistable and

$$Arg(gr_1(V)) > Arg(gr_2(V)) > \dots > Arg(gr_n(V)).$$

Furthermore, for any $\varphi \in (0, \pi)$ the category which consists of the trivial representation Z -semistables having the argument φ is *abelian*.

The above definitions were generalized by Bridgeland to the case of triangulated categories. He introduced the notion of **stability condition** (a.k.a. **stability structure**). In order to define a stability condition it suffices to choose a t -structure and a stability function (also known as **central charge**). In the case of quivers one has a triangulated category (derived category of the category of finite-dimensional representations), and the category of finite-dimensional representations is the heart of the obvious t -structure. The stability function Z is given by any homomorphism $Z : \mathbf{Z}^I \rightarrow \mathbf{C}$.

Numerical definition of the concept of stability is equivalent to the algebro-geometric one defined in terms of GIT (work of A. King). On the other hand, one knows that in algebraic geometry the notion of stability can be described in differential-geometric terms using hermitian metrics (Donaldson-Yau-Uhlenbeck theorem). It would be interesting to develop the corresponding formalism for categories.

Some hints toward to that could be found in the work of physicists, in particular, Nekrasov. From their work one can derive the idea that a hermitian metric on a sheaf of finite rank is “the same” as a Hilbert metric on the $*$ -representation of the algebra $\mathbf{C}[x_1, x_1^*, \dots, x_n, x_n^*]$ subject to the condition

$$\sum_i [x_i, x_i^*] = n \cdot Id, [x_i, x_j^*] = \delta_{ij} + R_{ij},$$

where R_{ij} is the trace-class operator. Then stability via GIT corresponds to solutions of the equation $\sum_i [x_i, x_i^*] = \sum_i \theta_i$.

All that can be made more precise in the case of free algebra (i.e. path algebra of quiver with one vertex and a number of loops). Let us fix a collection $\theta_i \in \mathbf{R}, i \in I$. Then (this is equivalent to GIT story) one can define semistable representations of dimension γ in terms of the slope function

$\frac{\sum_{i \in I} \theta_i \gamma^i}{\sum_{i \in I} \gamma^i}$, as in the above example of the quiver A_2 . Then for representations of slope zero, this notion of θ -stability is equivalent to the existence of the $*$ -representation of the algebra $C = \mathbf{C}[x_1, x_1^*, \dots, x_n, x_n^*]$ (n is the number of loops) such that

$$\sum_{i \in I} [x_i, x_i^*] = \sum_{i \in I} \theta_i p r_i,$$

where $p r_i$ is the idempotent corresponding to the vertex $i \in I$.

In the space of stability data $(\theta)_{i \in I}$ the condition $\sum_i \theta_i \gamma^i = 0$ singles out countably many hyperplanes. The above condition on commutators means that on our $*$ -algebra C there exists a non-trivial central trace, i.e. a linear functional $Tr : C \rightarrow \mathbf{C}$ such that $Tr(aa^*) > 0, Tr([a, b]) = 0$. We can relax the condition and look for such stability data θ and such representations of C (not necessarily finite-dimensional) for which such a trace does exist. Probably the set of such θ -s defines a closed compact in the projectivization of the vector space of all θ -s. It should contain the closure of such stability data for which there exists a semistable object of a given dimension. Informally traces can be thought of as a pair (θ, V) where $\dim V$ is not necessary integer.

4.3 Counting of stable objects: the case of finite fields

To illustrate the idea which will be later worked out in details, let us assume that the ground field is finite: $\mathbf{k} = \mathbb{F}_q$ where $q = p^r$ and p is prime. For any dimension vector $\gamma = (\gamma^i)_{i \in I}$ we define the *stack number*

$$s_\gamma = \sum_{[V], \dim(V) = \gamma} \frac{1}{|Aut(V)|},$$

where $|M|$ denotes the cardinality of the finite set M . It is easy to see that

$$s_\gamma = \frac{q^{\sum_{i,j \in I} a_{ij} \gamma^i \gamma^j}}{\prod_{i \in I} |GL(\gamma^i, \mathbb{F}_q)|}.$$

Here $|GL(m, \mathbb{F}_q)| = (q^m - 1)(q^m - q) \dots (q^m - q^{m-1})$, and a_{ij} is the number of arrows from the vertex i to the vertex j .

It is more convenient (for the reasons related e.g. to the wall-crossing formulas) to replace the stack number s_γ by an infinite series in non-commutative variables. In order to do that we introduce the **quantum torus** as the algebra generated by $e_\gamma, \gamma \in \mathbf{Z}_{\geq 0}^I$ subject to relations

$$e_{\gamma_1} e_{\gamma_2} = q^{E(\gamma_1, \gamma_2)} e_{\gamma_1 + \gamma_2},$$

where $E(\gamma_1, \gamma_2) = \sum_{i,j \in I} a_{ij} \gamma_2^i \gamma_1^j - \sum_{i \in I} \gamma_2^i \gamma_1^i$ is the Euler-Ringel bilinear form. It is naturally related to the stack numbers by the following observation:

$$q^{E(\gamma_1, \gamma_2)} = \sum_{[V]} \frac{1}{|Aut(V)|},$$

where the sum is taken over all isomorphism classes of short exact sequences $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$, where $\dim(V_i) = \gamma_i, i = 1, 2$.

Exercise 1 *Prove the above formula.*

Let us now introduce a *stack generating function*

$$A = \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} \left(\sum_{[V], \dim(V)=\gamma} \frac{1}{|Aut(V)|} \right) e_\gamma.$$

This generating function does not depend on a choice of stability function. In order to see this relation let us introduce (for a fixed $\varphi \in (0, \pi)$) the generating function

$$A_{\varphi, Z} = 1 + \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} \left(\sum_{[V], \dim(V)=\gamma, Arg Z(V)=\varphi, V \in \mathbf{M}_\gamma^{ss}} \frac{1}{|Aut(V)|} \right) e_\gamma.$$

We leave the following result as an exercise to the reader.

Proposition 4.3.1

$$A = \overrightarrow{\prod} A_{\varphi, Z} = \lim_{\varphi_1 < \dots < \varphi_n} A_{\varphi_n, Z} \dots A_{\varphi_1, Z}.$$

Here the product is taken in the clockwise order.

Example 4.3.2 *Dynkin quiver A_1 consist of one vertex and no arrows. Dimension vector is just a integer $n \in \mathbf{Z}_{\geq 0}$. Then a stability function is given by $Z(n) = nz, Im z > 0$. The quantum torus has generators $e_k, k \geq 0$ subject to the relations*

$$e_k e_l = q^{-kl} e_{k+l}.$$

Notice that if we set $x := e_1$ then $e_n = q^{\frac{n(n-1)}{2}} x^n$. Hence we can write for the stack series

$$A = \sum_{n \geq 0} \frac{e_n}{|GL(n, \mathbb{F}_q)|} = \sum_{n \geq 0} \frac{q^{\frac{n(n-1)}{2}}}{\prod_{0 \leq i \leq n} (q^n - q^i)} x^n.$$

The latter can be also written as an infinite product:

$$A = (-q^{-1}x; q^{-1})_{\infty},$$

where we use the Pochhammer symbol $(z; q)_{\infty} = (1-z)(1-qz)(1-q^2z)\dots$. One sees that A is an entire function in x .

Similarly, for the quiver with one vertex and one loop one has

$$A = \frac{1}{(x; q^{-1})_{\infty}}.$$

This is the case of affine Dynkin quiver. In a more complicated wild case of the quiver with one vertex and two loops one has

$$A = (-qx; q)_{\infty} (q^3x^2; q)_{\infty}^{-1} (-q^6x^3; q)_{\infty} (q^8x^4; q)_{\infty}^{-1} (q^{10}x^4; q)_{\infty}^{-1} \dots$$

Notice that $q > 1$, hence it is more natural (for convergence properties) to use q^{-1} in the Pochhammer symbols. One can do this, if considering the symbols as infinite series in x with coefficients in $\mathbf{Q}((q))$. Then, one has e.g. the identity

$$(x; q^{-1})_{\infty} (qx; q)_{\infty} = 1.$$

We see that in the above examples the series A factorizes as an infinite product of integer powers of Pochhammer symbols. This result holds in a much bigger generality, and in fact the integer exponents are the so-called refined DT-invariants (refined BPS states in physics). For a general quiver we have

$$A = \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} \frac{q^{\sum_{i,j} a_{ij} \gamma^i \gamma^j}}{\prod_{i \in I} |GL(\gamma^i, \mathbb{F}_q)|} e_{\gamma} = \overrightarrow{\prod} A_{\varphi, Z} = \overrightarrow{\prod}_{\gamma \neq 0} \prod_{|k| \leq \text{const}(\gamma)} ((-1)^{E(\gamma, \gamma)} q^k e_{\gamma}; q^{-1})_{\infty}^{c_Z(\gamma, k)}.$$

Here we assume that Z is a *generic* stability function, i.e. if γ_1 and γ_2 are \mathbf{Z} -independent elements of Z^I then $\text{Arg}(Z(\gamma_1)) \neq \text{Arg}(Z(\gamma_2))$. The exponents $c_Z(\gamma, k)$ are integers, which are called **refined** DT-invariants. **Numerical** DT-invariants are defined as $\Omega_Z(\gamma) = \sum_k c_Z(\gamma, k)$. The sum is well-defined because of the restriction $|k| \leq \text{const}(\gamma)$. Integrality of $c_Z(\gamma, k)$ implies integrality of $\Omega_Z(\gamma)$.

4.4 Counting of stable objects: arbitrary field case

Let \mathbf{k} be the ground field. Recall the setup. We have a finite quiver Q , with the set I of vertices, and $a_{ij} \in \mathbf{Z}_{\geq 0}$ arrows from i to j for $i, j \in I$. For any dimension vector

$$\gamma = (\gamma^i)_{i \in I} \in \mathbf{Z}_{\geq 0}^I$$

we have space of representations of Q in complex coordinate vector spaces of dimensions $(\gamma^i)_{i \in I}$

$$\mathbf{M}_\gamma = \mathbf{M}_\gamma^Q \simeq \prod_{i, j \in I} \mathbf{k}^{a_{ij} \gamma^i \gamma^j}$$

endowed with the action by conjugation of a complex algebraic group

$$\mathbf{G}_\gamma := \prod_{i \in I} GL(\gamma^i, \mathbf{k}).$$

Thus we have an Artin stack of finite type $\mathbf{M}_\gamma / \mathbf{G}_\gamma$ which is the moduli stack of representations of Q of dimension γ . In the case of finite field \mathbb{F}_q our invariants were rational numbers $\frac{|\mathbf{M}_\gamma(\mathbb{F}_q)|}{|\mathbf{G}_\gamma(\mathbb{F}_q)|}$ and $\frac{|\mathbf{M}_\gamma^{ss}(\mathbb{F}_q)|}{|\mathbf{G}_\gamma(\mathbb{F}_q)|}$ (the latter depends on a chosen stability function). What should we do in general?

Let us illustrate the idea in the case of varieties rather than stacks (i.e. there is no group action). If X is an algebraic variety over \mathbb{F}_q and $l \neq \text{char } \mathbb{F}_q$ then we can define compactly supported étale cohomology $H_c^\bullet(X, \mathbf{Q}_l)$. This a finite-dimensional \mathbf{Q}_l -vector space acted by the Frobenius automorphism $Fr_{\mathbb{F}_q} \in \text{Gal}(\overline{\mathbb{F}_q} / \mathbb{F}_q) \simeq \widehat{\mathbf{Z}}$. The Lefschetz fixed point formula (Weil conjecture) gives us

$$|X(\mathbb{F}_q)| = \sum_i (-1)^i \text{Tr}_{H_c^i}(Fr_{\mathbb{F}_q}),$$

where eigenvalues of the Frobenius λ_α are algebraic Weil numbers, i.e. for any complex embedding $i : \overline{\mathbf{Q}} \rightarrow \mathbf{C}$ one has $|i(\lambda_\alpha)| \in q^{\frac{1}{2}\mathbf{Z}}$.

For any finite-dimensional l -adic representation of the above Galois group such that eigenvalues of Fr are Weil numbers we define **Serre polynomial of E**

$$S(E) = \sum_{j \in \mathbf{Z}} \dim E_j t^j,$$

where E_j is the eigenspace of Fr of weight j . By definition it is a Laurent polynomial with integer coefficients. Now we define **Serre polynomial of X** by the formula

$$S(X) = \sum_{0 \leq i \leq 2 \dim X} (-1)^i S(H_c^i(X, \mathbf{Q}_l)).$$

For example $S(\mathbf{A}^n) = t^{2n}$, $S(\mathbf{P}^n) = 1 + t^2 + \dots + t^{2n}$. For a smooth projective curve C of genus g one has $S(C) = 1 - 2gt + t^2$.

For a closed subset $Y \subset X$ one has an exact sequence of groups

$$\dots \rightarrow H_c^i(X - Y) \rightarrow H_c^i(X) \rightarrow H_c^i(Y) \rightarrow \dots$$

Hence $S(X) = S(Y) + S(X - Y)$. Also $S(X_1 \times X_2) = S(X_1)S(X_2)$. Therefore $S(X)$ behaves similarly to $|X(\mathbb{F}_q)|$. They are both examples of *motivic invariants* of X (which can be also called “generalized Euler characteristic”).

Having all that in mind we get an idea how to deal with the case $\mathbf{k} = \mathbf{C}$. In that case we have cohomology groups $H^i(X, \mathbf{C})$ and $H_c^i(X, \mathbf{C})$. For any $i \geq 0$ they carry mixed Hodge structure (MHS). Recall that for a complex vector space E a MHS is defined in terms of the vector subspace $E_{\mathbf{Q}}$ over \mathbf{Q} s.t. $E = E_{\mathbf{Q}} \otimes \mathbf{C}$ and two filtrations: weight filtration $W_{\leq j} E_{\mathbf{Q}}$, $j \in \mathbf{Z}$ of $E_{\mathbf{Q}}$ and Hodge filtration $F^{\geq p} E$ of E . Main property is the decomposition of the associated graded vector space $gr_j^W = F^{\geq p} \oplus \overline{F}^{\geq (j-p)}$. Then one defined the Serre polynomial of E by the formula

$$S(E) = \sum_{j \in \mathbf{Z}} \dim(gr_j^W E) t^j.$$

This polynomial enjoys the same properties as the one for finite fields. It can be thought of as algebro-geometric version of Poincaré polynomial in topology.

In order to calculate Serre polynomial one needs to know weight filtration on the cohomology groups with compact support. One can reduce the problem to the whole cohomology using the duality isomorphism

$$H_c^i(X) \simeq (H^{2 \dim X - i}(X))^* \otimes \mathbb{L}^{\otimes \dim X},$$

where $\mathbb{L} := H^2(\mathbf{P}^1)$ is the Tate motive of weight +2. Then in order to compute $W_{\leq k}H^\bullet(X) \subset H^\bullet(X)$ one can choose a smooth compactification $X \subset \overline{X}$ s.t. $\overline{X} - X = D := \cup_\alpha D_\alpha$ is the normal crossing divisor. Let $D^{(j)}$ denotes the union of j -dimensional strata of D , and $X^{(j)} = \overline{X} - D^{(j)}$. Then one has a chain of embedding $X \subset X^{(j)} \subset \overline{X}$. One can show that $W_{\leq k}H^i(X)$, $k = j + i + 1$ is the image of $H^i(X^{(j)})$ under the last embedding. Alternatively one can use Deligne's theorem which says that $gr_j^W(H_c^i(X))$ is the cohomology of degree $i - j$ of the complex (arrows are induced by the natural restrictions)

$$\dots \rightarrow H^j(\overline{X}) \rightarrow \oplus_\alpha H^j(D_\alpha) \rightarrow \oplus_{\alpha, \beta} H^j(D_\alpha \cap D_\beta) \rightarrow \dots$$

In the case of n Artin stack represented by a variety acted by an affine group we can use equivariant cohomology $H_G^i(X)$. More precisely we define the cohomology of the stack X/G in the following way. Choose an embedding $G \rightarrow GL(N)$ for some $N \geq 0$. Replace X by the variety $X' = X \times GGL(N)$. Consider the "universal" space $X'_N = X' \times GL(N)EGL(N)$, where $EGL(N) \rightarrow BGL(N)$ is the universal bundle for the group $GL(N)$. Here $BGL(N) = Gr(N, \infty) = \lim_{M \rightarrow \infty} Gr(N, N + M)$. One can show that for each $i \geq 0$ the sequence of cohomology groups $H^i(X'_N)$ is well-defined, i.e. the corresponding sequence of finite-dimensional cohomology groups stabilizes with respect to $M \rightarrow \infty$. Moreover, it does not depend on the choice of the embedding of G to $GL(N)$. We define $H^i(X/G) := H_G^i(X)$ to be this group.

One can show that the sequence of weights on $H_G^i(X)$ goes to infinity as $i \rightarrow \infty$. Then one has the Serre series

$$\sum_{i \geq 0} (-1)^i S(H_G^i(X)) \in \mathbf{Z}[[t]] \cap \mathbf{Q}(t).$$

One can also define compactly supported equivariant (stack) cohomology $H_{c,G}^i(X)$. In the case when X is smooth the definition is easy:

$$H_{c,G}^i(X) := (H^{2(\dim X - \dim G) - i}(X))^* \otimes \mathbb{L}^{\otimes (\dim X - \dim G)}.$$

The general case is a bit involved. For the compactly supported Serre series

$$S(X/G) := \sum_{i \geq 0} (-1)^i S(H_{c,G}^i(X)) \in \mathbf{Z}((t^{-1})) \cap \mathbf{Q}(t).$$

For a class of algebraic groups which includes products of general linear groups the Serre series $S(X/G)$ is equal to $\frac{S(X)}{S(G)}$.

With the above definitions we define for the quiver Q the series

$$A = \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} S(\mathbf{M}_\gamma/\mathbf{G}_\gamma)e_\gamma,$$

where e_γ are the generators of the quantum torus. Similarly, for a chosen stability function Z and angle φ we define

$$A_{\varphi,Z} = 1 + \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I, \text{Arg } Z(\gamma)=\varphi} S(\mathbf{M}_\gamma^{\text{ss}}/\mathbf{G}_\gamma)e_\gamma.$$

Then

$$A = \overrightarrow{\prod} A_{\varphi,Z}.$$

We are going to discuss these results in detail in subsequent lectures.

4.5 COHA: definition

We introduce a $\mathbf{Z}_{\geq 0}^I$ -graded abelian group

$$\mathcal{H} := \bigoplus_\gamma \mathcal{H}_\gamma,$$

where each component is defined as an equivariant cohomology

$$\mathcal{H}_\gamma := H_{\mathbf{G}_\gamma}^\bullet(\mathbf{M}_\gamma) := H^\bullet(\mathbf{M}_\gamma^{\text{univ}}) = \bigoplus_{n \geq 0} H^n(\mathbf{M}_\gamma^{\text{univ}}).$$

About the notation: as usual we define equivariant cohomology via the universal space:

$$\mathbf{M}_\gamma^{\text{univ}} := (\mathbf{E}\mathbf{G}_\gamma \times \mathbf{M}_\gamma) / \mathbf{G}_\gamma.$$

In this case the classifying space for \mathbf{G}_γ is a product of infinite Grassmannians.

Our first observation is that *the graded vector space \mathcal{H} carries an associative product* which makes it into an algebra, which we call **Cohomological Hall algebra (COHA for short)**. Definition of the product resembles the one for the conventional Hall algebra (which explains the name). In 0811.2435 we introduced the notion of *motivic Hall algebra* which indeed generalizes to the case of triangulated categories usual Hall algebras (it also

generalizes to the case of field of characteristic zero a definition of derived Hall algebra of Toën). COHA is different from motivic Hall algebra. It is non-trivial even for quiver with one vertex and no loops (see below). Conceptually, one can say that the motivic Hall algebra lives in the world of constructible sheaves, while COHA lives in the world of coherent sheaves. *Therefore our Cohomological Hall algebra is a new object in mathematics.*

Let us explain the definition of the product. Fix any $\gamma_1, \gamma_2 \in \mathbf{Z}_{\geq 0}^I$ and denote $\gamma := \gamma_1 + \gamma_2$. The product is defined with the help of the space $\mathbf{M}_{\gamma_1, \gamma_2}$, which is the space of representations of Q in coordinate spaces of dimensions $(\gamma_1^i + \gamma_2^i)_{i \in I}$ such that the standard coordinate subspaces of dimensions $(\gamma_1^i)_{i \in I}$ form a subrepresentation. The group $\mathbf{G}_{\gamma_1, \gamma_2} \subset \mathbf{G}_\gamma$ consisting of transformations preserving subspaces $(\mathbf{C}^{\gamma_1^i} \subset \mathbf{C}^{\gamma^i})_{i \in I}$ (i.e. the group of block upper-triangular matrices), acts on $\mathbf{M}_{\gamma_1, \gamma_2}$.

Let us consider a morphism

$$\mathbf{m}_{\gamma_1, \gamma_2} : \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \rightarrow \mathcal{H}_\gamma = \mathcal{H}_{\gamma_1 + \gamma_2},$$

which is the composition of the multiplication morphism (which becomes Künneth isomorphism after the extension of coefficients for cohomology from \mathbf{Z} to \mathbf{Q})

$$\otimes : H_{\mathbf{G}_{\gamma_1}}^\bullet(\mathbf{M}_{\gamma_1}) \otimes H_{\mathbf{G}_{\gamma_2}}^\bullet(\mathbf{M}_{\gamma_2}) \rightarrow H_{\mathbf{G}_{\gamma_1} \times \mathbf{G}_{\gamma_2}}^\bullet(\mathbf{M}_{\gamma_1} \times \mathbf{M}_{\gamma_2}),$$

and of the following sequence of 3 morphisms:

$$H_{\mathbf{G}_{\gamma_1} \times \mathbf{G}_{\gamma_2}}^\bullet(\mathbf{M}_{\gamma_1} \times \mathbf{M}_{\gamma_2}) \xrightarrow{\cong} H_{\mathbf{G}_{\gamma_1, \gamma_2}}^\bullet(\mathbf{M}_{\gamma_1, \gamma_2}) \rightarrow H_{\mathbf{G}_{\gamma_1, \gamma_2}}^{\bullet+2c_1}(\mathbf{M}_\gamma) \rightarrow H_{\mathbf{G}_\gamma}^{\bullet+2c_1+2c_2}(\mathbf{M}_\gamma),$$

where

1. the first arrow is an isomorphism induced by natural projections of spaces and groups, inducing homotopy equivalences

$$\mathbf{M}_{\gamma_1} \times \mathbf{M}_{\gamma_2} \xleftarrow{\cong} \mathbf{M}_{\gamma_1, \gamma_2}, \quad \mathbf{G}_{\gamma_1} \times \mathbf{G}_{\gamma_2} \xleftarrow{\cong} \mathbf{G}_{\gamma_1, \gamma_2},$$

2. the second arrow is the pushforward map associated with the closed $\mathbf{G}_{\gamma_1, \gamma_2}$ -equivariant embedding $\mathbf{M}_{\gamma_1, \gamma_2} \hookrightarrow \mathbf{M}_\gamma$ of complex manifolds,
3. the third arrow is the pushforward map associated with the fundamental class of compact complex manifold $\mathbf{G}_\gamma / \mathbf{G}_{\gamma_1, \gamma_2}$, which is the product of Grassmannians $\prod_{i \in I} Gr(\gamma_1^i, \mathbf{C}^{\gamma^i})$.

Shifts in the cohomological degrees are given by

$$c_1 = \dim_{\mathbf{C}} \mathbf{M}_\gamma - \dim_{\mathbf{C}} \mathbf{M}_{\gamma_1, \gamma_2}, \quad c_2 = -\dim_{\mathbf{C}} \mathbf{G}_\gamma / \mathbf{G}_{\gamma_1, \gamma_2}.$$

We endow \mathcal{H} with a product $\mathfrak{m} : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$, $\mathfrak{m} := \sum_{\gamma_1, \gamma_2} \mathfrak{m}_{\gamma_1, \gamma_2}$.

Theorem 4.5.1 *The product \mathfrak{m} on \mathcal{H} is associative.*

Now we can give a formal definition.

Definition 4.5.2 *Associative unital $\mathbf{Z}_{\geq 0}^I$ -graded algebra \mathcal{H} with the product \mathfrak{m} is called the **Cohomological Hall algebra** associated with the quiver Q .*

The multiplication does not preserve the cohomological grading. The shift is given by

$$2(c_1 + c_2) = -2\chi_Q(\gamma_1, \gamma_2),$$

where

$$\chi_Q(\gamma_1, \gamma_2) := - \sum_{i, j \in I} a_{ij} \gamma_1^j \gamma_2^i + \sum_{i \in I} \gamma_1^i \gamma_2^i$$

is the Euler (a.k.a. Ringel) form on the K_0 group of the category of finite-dimensional representations of Q :

$$\chi_Q(\gamma_1, \gamma_2) = \dim \operatorname{Hom}(E_1, E_2) - \dim \operatorname{Ext}^1(E_1, E_2) = \chi(\operatorname{Ext}^\bullet(E_1, E_2)),$$

where E_1, E_2 are arbitrary representations of Q , of dimension vectors γ_1, γ_2 .

Remark 4.5.3 *COHA is a mathematical incarnation of BPS algebra envisioned by Harvey, Moore, Losev, Nekrasov and Shatashvili in the middle of 90's. Until our work there was no mathematical definition of BPS algebra, although various authors expected that it should be a "Nakajima type" construction. Probably the new idea is to consider multi-particle states instead of one-particle states as in previous attempts.*

4.6 Explicit formula for the product

For any γ the abelian group \mathcal{H}_γ is the cohomology of the classifying space $B\mathbf{G}_\gamma$, as the manifold \mathbf{M}_γ is contractible. It is well-known that $H^\bullet(B\mathbf{G}_\gamma)$ can be canonically identified with the algebra of symmetric polynomials with

integer coefficients in n variables of degree $+2$ for any $n \geq 0$, via the embedding

$$H^\bullet(\mathrm{BGL}(n, \mathbf{C})) \hookrightarrow H^\bullet(B(\mathbf{C}^\times)^n) \simeq \mathbf{Z}[x_1, \dots, x_n],$$

induced by the diagonal embedding $(\mathbf{C}^\times)^n \hookrightarrow \mathrm{GL}(n, \mathbf{C})$. Therefore, \mathcal{H}_γ is realized as the abelian group of polynomials in variables $(x_{i,\alpha})_{i \in I, \alpha \in \{1, \dots, \gamma^i\}}$ symmetric under the group $\prod_{i \in I} \mathrm{Sym}_{\gamma^i}$ of permutations preserving index i and permuting index α . The torus equivariant localization gives an explicit formula.

Theorem 4.6.1 *The product $f_1 \cdot f_2$ of elements $f_i \in \mathcal{H}_{\gamma_i}$, $i = 1, 2$ is given by the symmetric function $g((x_{i,\alpha})_{i \in I, \alpha \in \{1, \dots, \gamma^i\}})$, where $\gamma := \gamma_1 + \gamma_2$, obtained from the following function in variables $(x'_{i,\alpha})_{i \in I, \alpha \in \{1, \dots, \gamma_1^i\}}$ and $(x''_{i,\alpha})_{i \in I, \alpha \in \{1, \dots, \gamma_2^i\}}$:*

$$f_1((x'_{i,\alpha})) f_2((x''_{i,\alpha})) \frac{\prod_{i,j \in I} \prod_{\alpha_1=1}^{\gamma_1^i} \prod_{\alpha_2=1}^{\gamma_2^j} (x''_{j,\alpha_2} - x'_{i,\alpha_1})^{a_{ij}}}{\prod_{i \in I} \prod_{\alpha_1=1}^{\gamma_1^i} \prod_{\alpha_2=1}^{\gamma_2^i} (x''_{i,\alpha_2} - x'_{i,\alpha_1)}},$$

by taking the sum over all shuffles for any given $i \in I$ of the variables $x'_{i,\alpha}, x''_{i,\alpha}$ (the sum is over $\prod_{i \in I} \binom{\gamma^i}{\gamma_1^i}$ shuffles).

Remark 4.6.2 *This is a special case of Odesskii-Feigin algebra (“shuffle algebra”). The shuffle algebra with such rational kernel was not considered previously.*

4.7 Example: quivers with one vertex

This example as well as example in the next subsection can be found in our 1006.2706.

Let $Q = Q_d$ be now a quiver with just one vertex and $d \geq 0$ loops. Then the product formula from the previous section specializes to

$$(f_1 \cdot f_2)(x_1, \dots, x_{n+m}) := \sum_{i_1, \dots, j_m} f_1(x_{i_1}, \dots, x_{i_n}) f_2(x_{j_1}, \dots, x_{j_m}) \left(\prod_{k=1}^n \prod_{l=1}^m (x_{j_l} - x_{i_k}) \right)^{d-1}$$

for symmetric polynomials, where f_1 has n variables, and f_2 has m variables. The sum is taken over all $\{i_1 < \dots < i_n, j_1 < \dots < j_m, \{i_1, \dots, i_n, j_1, \dots, j_m\} =$

$\{1, \dots, n + m\}$. The product $f_1 \cdot f_2$ is a symmetric polynomial in $n + m$ variables.

We introduce a double grading on algebra \mathcal{H} , by declaring that a homogeneous symmetric polynomial of degree k in n variables has bigrading $(n, 2k + (1 - d)n^2)$. Equivalently, one can shift the cohomological grading in $H^\bullet(\text{BGL}(n, \mathbf{C}))$ by $[(d - 1)n^2]$.

It follows directly from the product formula that the bigraded algebra is commutative for odd d , and supercommutative for even d . The parity in this algebra is given by the parity of the shifted cohomological degree. It is easy to see that for $d = 0$ the algebra \mathcal{H} is an exterior algebra (i.e. Grassmann algebra) generated by odd elements $\psi_1, \psi_3, \psi_5, \dots$ of bidegrees $(1, 1), (1, 3), (1, 5), \dots$. Generators $(\psi_{2i+1})_{i \geq 0}$ correspond to the additive generators $(x^i)_{i \geq 0}$ of

$$H^\bullet(\mathbf{C}P^\infty) = H^\bullet(\text{BGL}(1, \mathbf{C})) \simeq \mathbf{Z}[x] \simeq \mathbf{Z}[x_1].$$

A monomial in the exterior algebra

$$\psi_{2i_1+1} \cdots \psi_{2i_n+1} \in \mathcal{H}_{n, \sum_{k=1}^n (2i_k+1)}, \quad 0 \leq i_1 < \cdots < i_n$$

corresponds to the Schur symmetric function $s_\lambda(x_1, \dots, x_n)$, where

$$\lambda = (i_n + (1 - n), i_{n-1} + (2 - n), \dots, i_1)$$

is a partition of length $\leq n$.

Similarly, for $d = 1$ algebra \mathcal{H} is isomorphic (after tensoring by \mathbf{Q}) to the algebra of symmetric functions in infinitely many variables, and it is a polynomial algebra generated by even elements $\phi_0, \phi_2, \phi_4, \dots$ of bidegrees $(1, 0), (1, 2), (1, 4), \dots$. Again, the generators $(\phi_{2i})_{i \geq 0}$ correspond to the additive generators $(x^i)_{i \geq 0}$ of $H^\bullet(\mathbf{C}P^\infty) \simeq \mathbf{Z}[x]$.

Notice that the underlying additive group of the algebra \mathcal{H} is equal to $\bigoplus_{n \geq 0} H^\bullet(\text{BGL}(n, \mathbf{C}))$ and hence does not depend on d . The isomorphism (after tensoring by \mathbf{Q}) between the underlying additive groups of the free polynomial algebra ($d = 1$) and of the free exterior algebra ($d = 0$), is in fact a part of the well-known **boson-fermion correspondence**. But the Fock space structure is still missing.

Now we come to the definition of motivic DT-series and motivic DT-invariants is this toy-model case. It will become clear later why we call them

motivic. In the case of the quiver Q_d the motivic DT-series is the Hilbert-Poincaré series $P_d = P_d(z, q^{1/2})$ of bigraded algebra \mathcal{H} twisted by the sign $(-1)^{\text{parity}}$ is the generalized q -exponential function:

$$\sum_{n \geq 0, m \in \mathbf{Z}} (-1)^m \dim(\mathcal{H}_{n,m}) z^n q^{m/2} = \sum_{n \geq 0} \frac{(-q^{1/2})^{(1-d)n^2}}{(1-q) \dots (1-q^n)} z^n \in \mathbf{Z}((q^{1/2}))[[z]] .$$

In cases $d = 0$ and $d = 1$ this series decomposes in an infinite product:

$$P_0 = (q^{1/2}z; q)_\infty = \prod_{i \geq 0} (1 - q^{i+1/2}z), \quad P_1 = \frac{1}{(z; q)_\infty} = \prod_{i \geq 0} \frac{1}{1 - q^i z},$$

where we use the standard notation for the q -Pochhammer symbol:

$$(x; q)_\infty := (1-x)(1-qx)(1-q^2x) \dots .$$

We will explain later the approach to factorization formulas for P_d . In this particular example it looks like this:

Theorem 4.7.1 *For any $d \geq 0$ there exist integers $\delta^{(d)}(n, m)$ for all $n \geq 1$ and $m \in (d-1)n + 2\mathbf{Z} = (1-d)n^2 + 2\mathbf{Z}$, such that for a given number n we have $\delta^{(d)}(n, m) \neq 0$ only for **finitely many** values of m , and*

$$P_d = \prod_{n \geq 1} \prod_{m \in \mathbf{Z}} (q^{m/2} z^n; q)_\infty^{\delta^{(d)}(n,m)} .$$

Exponents are motivic DT-invariants (known as refined BPS invariants in physics).

The above Theorem implies the following decomposition

$$\frac{P_d(z, q^{1/2})}{P_d(qz, q^{1/2})} = \prod_{n \geq 1} \prod_{m \in \mathbf{Z}} \prod_{i=0}^{n-1} (1 - q^{m/2+i} z^n)^{\delta^{(d)}(n,m)} .$$

Therefore, the limit

$$P_d^{\text{cl}}(z) := \lim_{q^{1/2} \rightarrow 1} \frac{P_d(z, q^{1/2})}{P_d(qz, q^{1/2})} \in 1 + z\mathbf{Z}[[z]]$$

exists and has the form

$$P_d^{cl} = \prod_{n \geq 1} (1 - z^n)^{nc^{(d)}(n)}, \quad c^{(d)}(n) \in \mathbf{Z},$$

where $c^{(d)}(n) = \sum_m \delta^{(d)}(n, m)$ is the numerical DT-invariant.

4.8 Non-symmetric example: quiver A_2

The quiver A_2 has two vertices $\{1, 2\}$ and one arrow $1 \leftarrow 2$. The Cohomological Hall algebra \mathcal{H}

contains two subalgebras $\mathcal{H}_L, \mathcal{H}_R$ corresponding to representations supported at the vertices 1 and 2 respectively.

Clearly each subalgebra $\mathcal{H}_L, \mathcal{H}_R$ is isomorphic to the Cohomological Hall algebra for the quiver $A_1 = Q_0$. Hence it is

an infinitely generated exterior algebra (see Section 2.5). Let us denote the generators by $\xi_i, i = 0, 1, \dots$ for the vertex 1 and by $\eta_i, i = 0, 1, \dots$ for the vertex 2. Each generator ξ_i or η_i corresponds to an additive generator of the group $H^{2i}(BGL(1, \mathbb{C})) \simeq \mathbb{Z} \cdot x^i$. Then one can check that $\xi_i, \eta_j, i, j \geq 0$ satisfy the relations

$$\xi_i \xi_j + \xi_j \xi_i = \eta_i \eta_j + \eta_j \eta_i = 0, \quad \eta_i \xi_j = \xi_{j+1} \eta_i - \xi_j \eta_{i+1}.$$

Let us introduce the elements $\nu_i^1 = \xi_0 \eta_i, i \geq 0$ and $\nu_i^2 = \xi_i \eta_0, i \geq 0$. It is easy to see that $\nu_i^1 \nu_j^1 + \nu_j^1 \nu_i^1 = 0$, and similarly the generators ν_i^2 anticommute. Thus we have two infinite Grassmann subalgebras in \mathcal{H} corresponding to these two choices: $\mathcal{H}^{(1)} \simeq \bigwedge (\nu_i^1)_{i \geq 0}$ and

$\mathcal{H}^{(2)} \simeq \bigwedge (\nu_i^2)_{i \geq 0}$. One can directly check the following result.

Proposition 4.8.1 *The multiplication (from the left to the right) induces isomorphisms of graded abelian groups*

$$\mathcal{H}_L \otimes \mathcal{H}_R \xrightarrow{\sim} \mathcal{H}, \quad \mathcal{H}_R \otimes \mathcal{H}^{(i)} \otimes \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}, \quad i = 1, 2.$$

Passing to generating series we obtain the standard identity

$$(q^{1/2} \widehat{\mathbf{e}}_1; q)_\infty \cdot (q^{1/2} \widehat{\mathbf{e}}_2; q)_\infty = (q^{1/2} \widehat{\mathbf{e}}_2; q)_\infty \cdot (q^{1/2} \widehat{\mathbf{e}}_{12}; q)_\infty \cdot (q^{1/2} \widehat{\mathbf{e}}_1; q)_\infty,$$

where non-commuting variables $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_{12}$ satisfy relations of the Heisenberg group (with $-q^{1/2}$ corresponding to the central element):

$$\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_2 = q^{-1} \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_1 = -q^{-1/2} \widehat{\mathbf{e}}_{12}.$$

This is the 5-term identity for the quantum dilogarithm.

5 Generalization of quivers: smooth algebras

So far we have discussed finite-dimensional modules over the path algebra $\mathbf{k}Q$ of the quiver Q . A natural generalization is given by the class of **smooth** associative algebras (they were also called formally smooth by Cuntz and Quillen). Recall the definition.

Definition 5.0.2 *\mathbf{k} -algebra R is called smooth if the bimodule of Kähler differentials $\Omega^1(R) := \text{Ker}(m : R \otimes R \rightarrow R)$ is projective.*

For example if $R = \mathbf{k}\langle x_1, \dots, x_n \rangle$ is a free algebra (equivalently, the path algebra of the quiver Q_n with one vertex and n loops) then $\Omega^1(R) = \bigoplus_{1 \leq i \leq n} R \otimes dx_i \otimes R$, where $dx_i := 1 \otimes x_i - x_i \otimes 1$.

The abelian category of R -modules has cohomological dimension 1. Then, informally speaking a “non-commutative scheme” $\text{Spec}(R)$ is a “non-commutative curve”.

An associative algebra can be written as a quotient of the free algebra: $R \simeq \mathbf{k}\langle x_1, \dots, x_n \rangle / (r_1 = 0, r_2 = 0, \dots, r_m = 0 \dots)$. Let us form the “algebra of differential forms”

$$\Omega(R) \simeq \mathbf{k}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle / (r_1 = 0, \dots, r_m = 0 \dots, \sum_i y_i \partial / \partial x_i (r_j) = 0), j = 1, 2, \dots$$

The algebra $\Omega(R)$ is graded by the number of y_i :

$$\Omega(R) = R \oplus \Omega^1(R) \oplus \Omega^2(R) \oplus \dots$$

Using this algebra one can reformulate smoothness in a more practical way.

Theorem 5.0.3 *Algebra R is smooth if and only if there exists a derivation $D : \Omega(R) \rightarrow \Omega(R)$ such that $D(x_i) = y_i, D(y_i) \in \Omega^2(R) = \Omega^1(R) \otimes_R \Omega^1(R)$.*

Notice that one can always find a *super derivation* D^{super} of the algebra $\Omega(R)$ which satisfies the properties $D^{\text{super}}(x_i) = y_i, D^{\text{super}}(y_i) = 0$. By definition it satisfies *super Leibniz rule* $D^{\text{super}}(ab) = D^{\text{super}}(a)b + (-1)^{\text{deg } a} a D^{\text{super}}(b)$.

Here are few examples of smooth algebras:

1. Path algebra $\mathbf{k}Q$ of a quiver Q .
2. Matrix algebra $Mat(n, \mathbf{k})$ for any n .
3. Algebra $\mathcal{O}(C)$ of functions on a smooth affine curve C .
4. Non-commutative quadrics, i.e. algebras $\mathbf{k}\langle x_i, y_i \rangle / (\sum_i x_i y_i = 1), 1 \leq i \leq n$.
5. Algebra R^{op} in the case when R is formally smooth.
6. Free product $R_1 * R_2$ of two formally smooth algebras R_1, R_2 .
7. Algebra $R\langle f_1, \dots, f_k \rangle$ in case if R is formally smooth and $f_i, 1 \leq i \leq k$ are elements of R . More generally one can add inverses to morphisms of finitely generated projective R -modules.

The relationship of formal smooth in the non-commutative sense with smoothness in algebraic geometry becomes more transparent in the observation that for a finitely-generated formally smooth R and any $N \geq 1$ the space $Rep_N(R) = Hom(R, Mat(n, \mathbf{k}))$ is a smooth affine scheme. This fact can be used e.g. in order to define a differential-geometric structure on the “non-commutative scheme” $Spec(R)$ by simply saying that all $Rep_N(R)$ have this structure.

Let $W \in R/[R, R]$ and $\widetilde{W} \in R$ be any representative of W . Then we have a collection of functions on $Rep_N(R)$ defined by the formula $W_N(\rho) = Tr(\rho(\widetilde{W}))$, where $\rho : R \rightarrow Mat(N, \mathbf{k})$. More generally one has a map $Sym^\bullet(R/[R, R]) \rightarrow \mathcal{O}(Rep_N(R))$. Furthermore, vector fields correspond to derivations of R , elements of the space $\Omega^\bullet(R)/[\Omega^\bullet(R), \Omega^\bullet(R)]$ (endowed with a super commutator) correspond to differential forms, a choice of derivation on the latter space corresponds to the choice of connection on the tangent bundle which has zero torsion.

The above results about COHA and DT-series admit a generalization to smooth algebras. We will discuss all that in a more general framework, when the data include a potential.

6 COHA of a quiver with potential

6.1 Cohomology theories for a space with a function

For a quiver with potential we have a collection of spaces M_γ endowed with regular functions $W_\gamma = Tr(W)$. The gauge group G_γ acts on M_γ preserving the function. Therefore if we choose an appropriate cohomology theory

$\mathbf{H}^\bullet(X, f)$ serving an algebraic variety endowed with a regular function, then we can define \mathcal{H}_γ as $\mathbf{H}_{\mathbf{G}_\gamma}^\bullet(\mathbf{M}_\gamma, W_\gamma) := \mathbf{H}^\bullet(\mathbf{M}_\gamma^{univ}, W_\gamma)$. In the case $f = 0$ the cohomology theory should coincide with one of the known theories, i.e. with Betti, de Rham or étale cohomology.

In fact, it would be even nicer to have a tensor “category of exponential motives” which generalizes, for example, Grothendieck’s category of pure motives to the case of pairs $[(X, f)]$ instead of just $[X]$. Then a cohomology theory should give us a cohomology functor (also called *realization*) from that category to graded vector spaces.

This program was realized in our paper 1006.2706. Definition of the tensor category of exponential motives is quite involved. Since computations use concrete realizations, let us briefly review the cohomology theories which we will use.

6.1.1 Betti realization

Definition 6.1.1 *For a complex algebraic variety X and a function $f \in \mathcal{O}(X)$ regarded as a regular map $f : X \rightarrow \mathbf{C}$, we define the **rapid decay cohomology** $H^\bullet(X, f)$ as the limit of the cohomology of the pair $H^\bullet(X, f^{-1}(S_t))$ for real $t \rightarrow -\infty$, where*

$$S_t := \{z \in \mathbf{C} \mid \operatorname{Re} z < t\} .$$

The cohomology stabilizes at some $t_0 \in \mathbf{R}$, $t_0 \ll 0$ (also in the definition one can replace $f^{-1}(S_t)$ by $f^{-1}(t)$). The cohomology $H^\bullet(X, f)$ behaves similarly to the usual cohomology. In particular, for a map $\pi : Y \rightarrow X$ compatible with functions $f_Y \in \mathcal{O}(Y)$, $f_X \in \mathcal{O}(X)$ in the sense that $f_Y = \pi^* f_X$, we have the pullback $\pi^* : H^\bullet(X, f_X) \rightarrow H^\bullet(Y, f_Y)$. If π is proper and both X and Y are smooth, then we have the pushforward morphism $\pi_* : H^\bullet(Y, f_Y) \rightarrow H^{\bullet+2(\dim_{\mathbf{C}}(X)-\dim_{\mathbf{C}}(Y))}(X, f_X)$. Similarly to the usual cohomology, there is a multiplication (Künneth) morphism

$$\otimes : H^\bullet(X, f_X) \otimes H^\bullet(Y, f_Y) \rightarrow H^\bullet(X \times Y, f_X \boxplus f_Y) ,$$

where the Thom-Sebastiani sum \boxplus is given by

$$f_X \boxplus f_Y := \operatorname{pr}_{X \times Y \rightarrow X}^* f_X + \operatorname{pr}_{X \times Y \rightarrow Y}^* f_Y .$$

We define the *Cohomological Hall algebra of the pair (Q, W)* (in Betti realization) as

$$\mathcal{H} = \bigoplus_\gamma \mathcal{H}_\gamma , \quad \mathcal{H}_\gamma := H_{\mathbf{G}_\gamma}^\bullet(\mathbf{M}_\gamma, W_\gamma) .$$

Definition of the product is similar to the one for the quiver without potential.

6.1.2 De Rham and étale realizations

For smooth X the de Rham realization is given by the de Rham cohomology of X with coefficients in the holonomic \mathcal{D}_X -module $\exp(u^{-1} \cdot f) \cdot \mathcal{O}_X$. In other words it is a finite-dimensional \mathbf{Z} -graded vector space over \mathbf{k} which is the hypercohomology in Zariski topology

$$H_{DR,u}^\bullet(X, f) := \mathbb{H}^\bullet(X_{Zar}, (\Omega_X^\bullet, d + u^{-1} df \wedge \cdot)), \quad H_{DR}^\bullet(X, f) := H_{DR,1}^\bullet(X, f).$$

The abstract comparison isomorphism between complexified Betti and de Rham realization gives us:

$$H_{DR}^\bullet(X, f) \simeq \mathbf{C} \otimes H^\bullet(X, f) .$$

In the affine case it is given by the integration of complex-analytic closed forms on $X(\mathbf{C})$ of the type $\exp(f) \alpha$, where α is an algebraic form on X such that

$$d\alpha + df \wedge \alpha = 0 \iff d(\exp(f) \alpha) = 0 ,$$

over closed real semi-algebraic chains with the “boundary at infinity” in the direction $Re(f) \rightarrow -\infty$. The integral is absolutely convergent because the form $\exp(f) \alpha$ decays rapidly at infinity. This explains the term “rapid decay cohomology”.

There is also étale realization of $H^\bullet(X, f)$ given by the l -adic cohomology of the direct image perverse sheaf $f_* \mathbf{Q}_{l,X}$. Then using Deligne-Fourier transform one can write

$$\begin{aligned} & \sum_{0 \leq i \leq 2 \dim X} (-1)^i Tr_{H_c^i(X, f)} Fr_k = \\ & = \sum_{x \in X(\mathbb{F}_q)} e^{\frac{2\pi i}{p} Tr_{\mathbb{F}_q | \mathbb{F}_p}(f(x))} \in \mathbf{Q}(\mu_p), \end{aligned}$$

here $\mathbf{k} = \mathbb{F}_q$, $q = p^r$ and μ_p is the primitive p -th root of 1.

We are not going to discuss this story. Let us just say that there are comparison theorems between Betti, de Rham and étale realizations. But there are other realizations for which the comparison isomorphism does not take place, so-called critical cohomology. We will briefly discuss them later.

6.2 Example

This example can be also found in our 1006.2706.

Let us describe the algebra \mathcal{H} in the case of the quiver $Q = Q_1$ with one vertex and one loop l , and potential $W = \sum_{i=0}^N c_i l^i$, $c_N \neq 0$, given by a polynomial of degree $N \in \mathbf{Z}_{\geq 0}$ in one variable. Dimension vector γ for such Q is given by an integer $n \geq 0$. For simplicity, we consider Betti realization.

In the case $N = 0$, the question reduces to Q without potential, which we considered before. It follows from the explicit formula that the algebra \mathcal{H} is the polynomial algebra of infinitely many variables.

In the case $N = 1$ the cohomology of pair vanishes for matrices of size greater than 0, hence $\mathcal{H} = \mathcal{H}_0 = \mathbf{Z}$.

In the case $N = 2$ we may assume without loss of generality that $W = -l^2$. It is easy to see that the cohomology of

$$(\text{Mat}(n \times n, \mathbf{C}), x \mapsto -\text{Tr}(x^2))$$

can be identified under the restriction map with the cohomology

$$(\text{Herm}(n), x \mapsto -\text{Tr}(x^2)),$$

where $\text{Herm}(n)$ is the space of Hermitean matrices. The latter cohomology group is the same as $H^\bullet(D^{n^2}, \partial D^{n^2})$, where D^{n^2} is the standard closed unit ball in $\mathbf{R}^{n^2} \simeq \text{Herm}(n)$. Hence $H^\bullet(\text{Mat}(n \times n, \mathbf{C}), x \mapsto -\text{Tr}(x^2))$ is isomorphic to \mathbf{Z} concentrated in the cohomological degree n^2 . Moreover, one can use the unitary group $U(n)$ instead of the homotopy equivalent group $\text{GL}(n, \mathbf{C})$ in the definition of equivariant cohomology. Group $U(n)$ acts on the pair $(D^{n^2}, \partial D^{n^2})$, and Thom isomorphism gives a canonical isomorphism of cohomology groups

$$\mathcal{H}_n \simeq H^\bullet(\text{BGL}(n, \mathbf{C}))[-n^2] \simeq H_{U(n)}^\bullet(D^{n^2}, \partial D^{n^2}).$$

Let us endow \mathcal{H} with the natural bigrading coming from cohomological and weight gradings (corresponding exponential motives are pure). One can show that \mathcal{H} coincides as a bigraded abelian group with the algebra associated with the quiver Q_0 with one vertex and zero arrows. Furthermore, comparing Grassmannians which appear in the definition of the multiplication for the quiver Q_0 with those which arise for Q_1 with potential $W = -l^2$, one can check that the multiplications coincide as well. Hence the algebra $\mathcal{H} =$

$\mathcal{H}^{(Q,W)}$ is the exterior algebra with infinitely many generators (again, this follows from the explicit formula).

In the case of degree $N \geq 3$ one can show that the bigraded algebra \mathcal{H} is isomorphic to the $(N - 1)$ -st tensor power of the exterior algebra corresponding to the case $N = 2$.

This example (and some other considerations) shows that COHA should be intimately related to matrix integrals. More generally, one can think of matrix integrals as of “periods”. In algebraic geometry a period is a pairing of a de Rham form with Betti cycle. Cohomological Hall algebra contains all such matrix periods, *without specifying cycles*. From the algebro-geometric point of view, periods correspond to the pairing between two different realization of the same object of the appropriate category of motives. Since we integrate not quite algebraic forms, but forms which are products of algebraic ones with exponents of regular functions, we should speak about “exponential motives”. Such a notion was introduced by Kontsevich and Zagier, but was not studied much. From this perspective Cohomological Hall algebra should be upgraded to an associative algebra in the tensor category of exponential motives. We will discuss such a generalization later.

6.3 Twisted graded algebras in tensor categories

There is some abstract nonsense which is useful for various generalizations of COHA which we will discuss. This is a formalism of algebras in tensor categories graded by the Heisenberg group. It goes like this.

Let Γ be an abelian group and $B : \Gamma \otimes \Gamma \rightarrow \mathbf{Z}$ be a bilinear form (in our case $\Gamma = \mathbf{Z}^I$ is the group of dimension vectors and $B = \chi_Q$). We associate with (Γ, B) the discrete Heisenberg group $\text{Heis}_{\Gamma, B}$ which is the set $\Gamma \times \mathbf{Z}$ endowed with the multiplication

$$(\gamma_1, k_1) \cdot (\gamma_2, k_2) := (\gamma_1 + \gamma_2, k_1 + k_2 - 2B(\gamma_1, \gamma_2)) .$$

Let (\mathcal{T}, \otimes) be a symmetric monoidal category endowed with an **even** invertible object $\mathbb{T}_{\mathcal{T}}$, i.e. such an object that the commutativity morphism $\mathbb{T}_{\mathcal{T}} \otimes \mathbb{T}_{\mathcal{T}} \rightarrow \mathbb{T}_{\mathcal{T}} \otimes \mathbb{T}_{\mathcal{T}}$ is the identity morphism. Let us fix (Γ, B) as above. We define a **twisted graded monoid** as a collection of objects $(\mathcal{H}_{\gamma})_{\gamma \in \Gamma}$ together with a collection of morphisms

$$m_{\gamma_1, \gamma_2} : \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \rightarrow \mathbb{T}_{\mathcal{T}}^{\otimes B(\gamma_1, \gamma_2)} \otimes \mathcal{H}_{\gamma_1 + \gamma_2} ,$$

and a unit morphism $1_{\mathcal{C}} \rightarrow \mathcal{H}_0$, satisfying an obvious extension of the usual associativity and the unity axioms. In the additive case we will speak about **twisted graded algebra**.

For example, in case of a quiver without potential the category (\mathcal{T}, \otimes) is the tensor category of \mathbf{Z} -graded abelian groups (with the usual Koszul sign rule for the commutativity morphism), and $\mathbb{T}_{\mathcal{T}} = \mathbf{Z}[-2]$ is the group \mathbf{Z} placed in degree $+2$, i.e. the cohomology of the pair $H^\bullet(\mathbf{C}P^1, pt)$. Object \mathcal{H}_γ is just $H^\bullet(M_\gamma^{univ}) = H^\bullet(BG_\gamma)$.

Let us assume additionally that the form B is symmetric, and we are given an invertible object $\mathbb{T}_{\mathcal{T}}^{\otimes 1/2} \in \mathcal{T}$ such that $(\mathbb{T}_{\mathcal{T}}^{\otimes 1/2})^{\otimes 2} \simeq \mathbb{T}_{\mathcal{T}}$. Then for any twisted graded monoid $\mathcal{H} = (\mathcal{H}_\gamma)_{\gamma \in \Gamma}$ we define the modified graded monoid by

$$\mathcal{H}_\gamma^{mod} := \mathcal{H}_\gamma \otimes \left(\mathbb{T}_{\mathcal{T}}^{\otimes 1/2} \right)^{\otimes B(\gamma, \gamma)},$$

which is an ordinary (untwisted) Γ -graded monoid in \mathcal{T} . In particular, for the category of \mathbf{Z} -graded abelian groups we take $\mathbb{T}_{\mathcal{T}}^{\otimes 1/2}$ to be \mathbf{Z} in degree $+1$. Notice that in this case the object $\mathbb{T}_{\mathcal{T}}^{\otimes 1/2}$ is not even.

6.4 Cohomology theories

We need a bit more of abstract nonsense.

Definition 6.4.1 *Let \mathbf{k} be a field, and K be another field, $\text{char}(K) = 0$. Assume that we are given a K -linear Tannakian category \mathcal{C} . A **pre-cohomology theory** over \mathbf{k} with values in \mathcal{C} is a contravariant tensor functor \mathbf{H}^\bullet from the category of schemes of finite type over \mathbf{k} (version: affine schemes of finite type endowed with a function) to the tensor category $\mathcal{C}^{\mathbf{Z}\text{-gr}}$ of \mathbf{Z} -graded objects in \mathcal{C} endowed with Koszul rule of signs.*

Here is the list of standard examples of (pre-)cohomology theories, $\mathcal{C} = K\text{-mod}$:

- (case $\text{char} \mathbf{k} = 0$, $\mathbf{k} \subset \mathbf{C}$): rational Betti cohomology, $K = \mathbf{Q}$,
- (case $\text{char} \mathbf{k} = 0$): de Rham cohomology, $K = \mathbf{k}$,
- (case $\text{char} \mathbf{k} \neq l$ for prime l): étale cohomology, $K = \mathbf{Q}_l$.

In the case $\mathbf{k} \subset \mathbf{C}$ and Betti cohomology one can enhance \mathcal{C} from $\mathbf{Q}\text{-mod}$ to the Tannakian category of polarizable mixed Hodge structures. Similarly, in the étale case we can take \mathcal{C} to be the category of continuous l -adic representations of the absolute Galois group of \mathbf{k} .

In fact we are dealing with **cohomology theories**. Such a theory is given by a tensor functor from \mathcal{C} to vector spaces over some extension of the field K , and such that it is obtained from one of the standard theories by the change of scalars.

We denote $\mathbf{H}^2(\mathbb{P}_{\mathbf{k}}^1)$ by $K(-1)$ understood as an element of \mathcal{C} , and set

$$\mathbb{T} = \mathbb{T}_{\mathbf{H}^\bullet} := K(-1)[-2] \in \mathcal{C}^{\mathbf{Z}^{-gr}}.$$

This should be called **Tate motive**.

6.5 Category of exponential mixed Hodge structures

In the paper 1006.2706 we defined a tensor category $EMHS$ of **exponential mixed Hodge structures** (their objects can be also called *exponential motives*). De Rham and Betti cohomology discussed last time are just examples of cohomology theories \mathbf{H}^\bullet (in the theory of motives cohomology functors are called realizations). Realization functor takes value in an appropriate Tannakian category \mathcal{C} . In the framework of $EMHS$ we can define COHA, Serre polynomials (or rather series) of objects, etc. For completeness we reproduce some details below.

Definition 6.5.1 *Cohomological Hall algebra of (Q, W) (in realization \mathbf{H}) is an associative twisted graded algebra in \mathcal{C} defined by the formula*

$$\mathcal{H} := \bigoplus_{\gamma} \mathcal{H}_{\gamma}, \quad \mathcal{H}_{\gamma} := \mathbf{H}^\bullet(\mathbf{M}_{\gamma}/\mathbf{G}_{\gamma}, W_{\gamma}) := \mathbf{H}^\bullet(\mathbf{M}_{\gamma}^{univ}, W_{\gamma}^{univ}) \quad \forall \gamma \in \mathbf{Z}_{\geq 0}^I$$

in the obvious notation.

All examples which we considered before were special cases of this construction.

Definition of the category of $EMHS$ is involved, and we skip it here. Instead we give below a brief summary of the construction and properties.

1) The notion of Hodge structure can be reformulated in terms of D -modules on the line. This is natural from the point of view of de Rham version

of cohomology, which uses the parameter u . The latter can be thought of as a coordinate on the line. After the Fourier transform the resulting holonomic D -module is in fact a bundle with connection on the punctured line which has regular singularities at zero and infinity. In case of exponential mixed Hodge structures we allow irregular singularities of exponential type at infinity. This means that solutions can grow exponentially.

2) After using Riemann-Hilbert correspondence we get a perverse sheaf. We require that it has trivial cohomology.

3) It is important for us that objects of $EMHS$ carry weight filtration, similarly to the case of usual Hodge theory. In order to achieve that we define $EMHS$ as a full subcategory of the category of *mixed Hodge modules on the line*, introduced by M. Saito in 90's. There is a natural retraction functor from the category of mixed Hodge modules to the category $EMHS$. Then using the notion of weight filtration for mixed Hodge modules we prove that it descends to $EMHS$. In particular, we can speak about **pure objects of weight n** in the category $EMHS$. If (X, f) is a scheme endowed with a function, then the corresponding object of $EMHS$ will be denoted by $[(X, f)]$. Graded components of COHA are objects of $EMHS$.

We define **Serre polynomial** of an exponential mixed Hodge structure E as

$$S(E) = \sum_i rk\ gr_i^{exp}(E)q^{i/2} \in \mathbf{Z}[q^{\pm 1/2}],$$

where gr_{\bullet}^{exp} denotes the weight filtration on $EMHS$.

By additivity we extend S to a functional on the K_0 -group of the bounded derived category of $EMHS$. One can prove that the map S is a ring homomorphism.

6.6 A generalization to smooth algebras

As we already discussed in the case of trivial potential, (formally) smooth algebras provide a very natural generalization of our theory. Recall briefly the definition.

Definition 6.6.1 *An associative unital algebra R over a field \mathbf{k} is called **smooth** if it is finitely generated and formally smooth in the sense of D. Quillen and J. Cuntz, i.e. if the bimodule $\Omega_R^1 := Ker(R \otimes_{\mathbf{k}} R \xrightarrow{mult} R)$ is projective. Here $mult : R \otimes_{\mathbf{k}} R \rightarrow R$ is the product.*

Basic examples of smooth algebras are matrix algebras, path algebras of finite quivers, and algebras of functions on smooth affine curves. Hence it is natural to axiomatize a class of R -modules which generalizes finite-dimensional representations of quivers.

For a finite set I , we call an unital associative algebra R/\mathbf{k} **I -bigraded** if R is decomposed (as a vector space) into the direct sum

$R = \bigoplus_{i,j \in I} R_{ij}$ in such a way that $R_{ij} \cdot R_{jk} \subset R_{ik}$. Equivalently, R is I -bigraded if we are given a morphism of unital algebras $\mathbf{k}^I \rightarrow R$.

Let now R be an I -bigraded *smooth* algebra. It follows that any finite-dimensional representation E of R decomposes into a direct sum of finite-dimensional vector spaces $E_i, i \in I$.

For any dimension vector $\gamma = (\gamma^i)_{i \in I} \in \mathbf{Z}_{\geq 0}^I$ the scheme $\mathbf{M}_\gamma = \mathbf{M}_\gamma^R$ of representations of R in coordinate spaces $E_i = \mathbf{k}^{\gamma^i}, i \in I$ is a smooth affine scheme. Any choice of a finite set of I -bigraded generators of R gives a closed embedding of \mathbf{M}_γ into the affine space \mathbf{M}_γ^Q for some quiver Q with the set of vertices equal to I .

Let us make the following **Assumption**:

We are given a bilinear form $\chi_R : \mathbf{Z}^I \otimes \mathbf{Z}^I \rightarrow \mathbf{Z}$ such that for any two dimension vectors $\gamma_1, \gamma_2 \in \mathbf{Z}_{\geq 0}^I$ and for any two representations $E_i \in \mathbf{M}_{\gamma_i}(\bar{\mathbf{k}})$ we have the equality

$$\dim \text{Hom}(E_1, E_2) - \dim \text{Ext}^1(E_1, E_2) = \chi_R(\gamma_1, \gamma_2).$$

Here $\bar{\mathbf{k}}$ is an algebraic closure of \mathbf{k} , and E_1, E_2 are considered as representations of algebra $R \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ over $\bar{\mathbf{k}}$.

The assumption implies that the smooth scheme \mathbf{M}_γ is *equidimensional* for any given γ and

$$\dim \mathbf{M}_\gamma = -\chi_R(\gamma, \gamma) + \sum_i (\gamma^i)^2.$$

Let R be a smooth I -bigraded algebra over field \mathbf{k} endowed with a bilinear form χ_R on \mathbf{Z}^I satisfying the Assumption.

Let us assume that we are given an element

$$W \in R/[R, R]$$

represented by some element $\widetilde{W} \in R$, $W = \widetilde{W} \pmod{[R, R]}$. The element W (or its lifting \widetilde{W}) is called a *potential*.

Then for any $\gamma \in \mathbf{Z}_{\geq 0}^I$ we obtain a function W_γ on the affine variety \mathbf{M}_γ , invariant under the action of \mathbf{G}_γ . The value of W_γ at any representation is given by the trace of the image of \widetilde{W} . For any short exact sequence

$$0 \rightarrow E_1 \hookrightarrow E \twoheadrightarrow E_2 \rightarrow 0$$

of representations of R we have $W_{\gamma_1+\gamma_2}(E) = W_{\gamma_1}(E_1) + W_{\gamma_2}(E_2)$, where $\gamma_i, i = 1, 2$ are dimension vectors of $E_i, i = 1, 2$. Then we can repeat the considerations we did for quivers in the case of I -bigraded smooth algebra R . This gives us the definition of **Cohomological Hall algebra of the pair** (R, W) . *Moreover, all results of the lectures stated for COHA of quivers admit a straightforward generalization to this case.*

7 Motivic COHA and motivic DT-series

7.1 Motivic DT-series and quantum tori

The cohomology theory \mathbf{H}^\bullet used in the definition of Cohomological Hall algebra takes values in a Tannakian category \mathcal{C} . Hence cohomology groups carry an action of a pro-affine algebraic group (group of automorphisms of \mathbf{H}^\bullet) which we call the **motivic Galois group** $Gal_{\mathbf{H}^\bullet}^{mot}$ for the theory \mathbf{H}^\bullet .

We assume that there is a notion of weight filtration in \mathcal{C} . For example, in the case of rapid decay (tensoring by \mathbf{Q}) or de Rham cohomology we consider $\mathbf{H}^\bullet(X)$ for any variety X as a vector space graded by cohomological degree and endowed with the weight filtration. In terms of $Gal_{\mathbf{H}^\bullet}^{mot}$ this means that we have an embedding $w : \mathbb{G}_m \hookrightarrow Gal_{\mathbf{H}^\bullet}^{mot}$ (defined up to conjugation) such that the Lie algebra of $Gal_{\mathbf{H}^\bullet}^{mot}$ has non-positive weights with respect to the adjoint \mathbb{G}_m -action. For any representation E of $Gal_{\mathbf{H}^\bullet}^{mot}$ the weight filtration is defined by

$$W_i E := \bigoplus_{j \leq i} E_j, \quad i, j \in \mathbf{Z},$$

where $E_j \subset E$ is the eigenspace of $w(\mathbb{G}_m)$ with weight j .

Let us assume that for the Tannakian category \mathcal{C} (target of the cohomology functor) we have the notion of weight filtration. We will say that COHA \mathcal{H} is *pure* if for any γ the graded space $\mathbf{H}^\bullet(\mathbf{M}_\gamma^{univ}, W_\gamma^{univ})$ is pure, i.e. its n -th component is of weight n for any $n \in \mathbf{Z}$. In order to prove that \mathcal{H} is pure it is sufficient to check that $\mathbf{H}^\bullet(\mathbf{M}_\gamma, W_\gamma)$ is pure. Indeed, in this case the spectral

sequence

$$\mathbf{H}^\bullet(\mathbf{M}_\gamma, W_\gamma) \otimes \mathbf{H}^\bullet(B\mathbf{G}_\gamma) \implies \mathbf{H}^\bullet(\mathbf{M}_\gamma^{univ}, W_\gamma^{univ})$$

collapses because $\mathbf{H}^\bullet(B\mathbf{G}_\gamma)$ is pure, and hence for every weight n we have a complex supported only in degree n .

Examples of pure COHA include all quivers with zero potentials as well as the above example of Q_1 with one vertex, one loop and an arbitrary potential. For the smooth algebra $R = \mathbf{C}[t, t^{-1}]$ (quiver Q_1 with an invertible loop), and $W = 0$, the corresponding COHA is *not* pure.

Remark 7.1.1 *a) In the case when R is the path algebra of a quiver Q , the space of representations \mathbf{M}_γ is the space of collections of matrices. Integrals of $\exp(W_\gamma/u)$ over appropriate non-compact cycles in \mathbf{M}_γ (usually over the locus of Hermitean or unitary matrices) are exactly objects of study in the theory of matrix models in mathematical physics. Those integrals are encoded in the comparison isomorphism between Betti and de Rham realizations of $\mathbf{H}_{EHMS}^\bullet(\mathbf{M}_\gamma/\mathbf{G}_\gamma)$ and (as we already pointed out) can be interpreted as periods of the corresponding “exponential motives”.*

b) The non-zero constant u parametrizing the comparison isomorphism corresponds to the string coupling constant g_s . Notice that the parameter u is the same for all dimension vectors γ . Moreover, we do not have a distinguished integration cycle or a volume element. As a result we do not consider the “large N ” (in our notation “large $|\gamma|$ ”) behavior of matrix integrals. We think that developing such a theory is an important problem. It should find applications not only in the theory of matrix integrals but also in the Chern-Simons theory with complex gauge group.

Let us consider the K_0 -ring \mathcal{M} of the tensor category \mathcal{C} (i.e. of the category of finite-dimensional representations of $Gal_{\mathbf{H}^\bullet}^{mot}$). It contains an invertible element \mathbb{L} corresponding to the Tate motive $\mathbf{H}^2(\mathbb{P}^1)$ of weight $+2$. We complete \mathcal{M} by adding infinite sums of pure motives with weights approaching to $+\infty$. Notice that this completion (which we denote by $\widehat{\mathcal{M}}$) differs from the completion used in the theory of motivic integration, where weights are allowed to go to $-\infty$.

With a pair (Q, W) (quiver with potential) we associate the following series with coefficients in $\widehat{\mathcal{M}}$:

$$A = A^{(Q, W)} := \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} [\mathcal{H}_\gamma] \mathbf{e}_\gamma,$$

where variables \mathbf{e}_γ are additive generators of the associative unital algebra \mathcal{R}_+ over \mathcal{M} isomorphic to the subalgebra of the motivic quantum torus \mathcal{R} . The relations are given by the formulas

$$\mathbf{e}_{\gamma_1} \cdot \mathbf{e}_{\gamma_2} = \mathbb{L}^{-\chi_Q(\gamma_1, \gamma_2)} \mathbf{e}_{\gamma_1 + \gamma_2} \quad \forall \gamma_1, \gamma_2 \in \mathbf{Z}_{\geq 0}^I, \quad \mathbf{e}_0 = 1,$$

and coefficients of the series A are given by

$$[\mathcal{H}_\gamma] := \sum_{k \geq 0} (-1)^k [\mathbf{H}_{\mathbf{G}_\gamma}^k(\mathbf{M}_\gamma, W_\gamma)] \in \widehat{\mathcal{M}}.$$

The series A belongs to the completion $\widehat{\mathcal{R}}_+$ consisting of infinite series in \mathbf{e}_γ . It has the form

$$A = 1 + \text{higher order terms}$$

and is therefore invertible.

Definition 7.1.2 *We call A the **motivic Donaldson-Thomas series** of the pair (Q, W) .*

In the same way one can use *cohomology with compact support* (which makes sense for objects of *EMHS*) and equivalently define

$$A = \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} S(\mathbf{H}_{c, \mathbf{G}_\gamma}^\bullet(\mathbf{M}_\gamma, W_\gamma)) \mathbf{e}_\gamma,$$

where S denotes the Serre polynomial.

7.2 An application: finite order birational maps

Skew-symmetrization of the Euler form gives a symplectic form on the dimension lattice Γ of representations of a quiver. Then we can consider the corresponding quantum torus generated by $\widehat{e}_\gamma, \gamma \in \Gamma$ subject to the relations

$$\widehat{e}_{\gamma_1} \widehat{e}_{\gamma_2} = (-q^{1/2})^{\langle \gamma_1, \gamma_2 \rangle} \widehat{e}_{\gamma_1 + \gamma_2}.$$

We restrict ourselves to the subalgebra graded by the “positive cone” Γ_+ consisting of non-negative integer combinations of the dimension vectors. Corresponding motivic DT-series has the form

$$A = \sum_{\gamma \in \Gamma_+} \frac{S(H_c^\bullet(\mathbf{M}_\gamma / \mathbf{G}_\gamma))}{(-q^{1/2})^{\dim \mathbf{M}_\gamma - \dim \mathbf{G}_\gamma}} \widehat{e}_\gamma.$$

The following result will be proved in the subsequent lectures.

Theorem 7.2.1 *We have:*

$$A = \overrightarrow{\prod}_{\gamma} \prod_{|k| < c(\gamma)} (q^{k/2} \widehat{e}_{\gamma}; q^{-1})_{\infty}^{\Omega_Z(\gamma, k)},$$

where Z is the central charge of some stability condition and $\Omega_Z(\gamma, k)$ are integer numbers (called DT-invariants). The arrow indicates that the product is taken in the order of increasing arguments $\text{Arg}(Z(\gamma))$.

The element A gives rise to the adjoint action $v \mapsto AvA^{-1}$ of the above quantum torus (better, to its “doubled” version introduced by Fock and Goncharov). From the above theorem one deduces that there is a quasi-classical limit as $q \rightarrow 1$ of this automorphism. It acts on the Poisson algebra of functions which is the quasi-classical limit of the quantum torus. In particular, one can write its action on the coordinates $x_i, i \in I$ which correspond to the standard basis of Γ . Assuming that the quasi-classical limit of the automorphism is rational, we can compose it with the antipodal involution $x_i \mapsto \frac{1}{x_i}, i \in I$. The resulting map has the form

$$x_i \mapsto \tilde{x}_i := \left(x_i \prod_j g_j^{a_{ij} - a_{ji}} \right)^{-1}, \quad y_i \mapsto \tilde{y}_i := (g_i y_i)^{-1},$$

where $(g_i)_{i \in I}$ is the unique solution of the system of equations

$$g_i = 1 + x_i \prod_j g_j^{a_{ij}} \in \mathbf{Z}[[x_j]_{j \in I}],$$

and a_{ij} is the number of arrows between vertices i and j .

Theorem 7.2.2 *For an acyclic quiver which is mutation equivalent to a Dynkin quiver the above automorphism has finite order. The order is equal to either $h + 2$ or $(h + 2)/2$ depending on the Dynkin diagram, where h is the Coxeter number*

7.3 Potentials linear in a group of variables

In computations of motivic DT-series there is a special case of quivers with potential when one can reduce the number of variables. The idea can be explained in the following geometric situation. Let $\pi : X \rightarrow Y$ be a morphism of smooth complex algebraic varieties with fibers which are affine spaces. Suppose that an affine algebraic group G acts on X, Y preserving the affine structure on fibers. Let $f \in \mathcal{O}(X)^G$ be the invariant function which is affine on fibers (i.e. it is polynomial of degree less or equal than 1). Let $Z \subset Y$ be a closed subset consisting of points y such that f is constant on $\pi^{-1}(y)$. Then we have the induced by f the function on Z : $f_Z \in \mathcal{O}(Z)^G$.

Proposition 7.3.1 *Under the above assumptions, there is a natural isomorphism*

$$\mathbf{H}^\bullet(X/G, f) \simeq \mathbf{H}_c^\bullet(Z/G, -f_Z)^\vee \otimes \mathbb{T}^{\otimes \dim Y/G},$$

where the cohomology of a stack is defined via the universal bundle construction as in Section 1.

Corollary 7.3.2 *In the obvious notation we have the following equality of Serre polynomials:*

$$S(X/G, f) = S(Z/G, f_Z) \cdot q^{\dim X - \dim Y}$$

The proof is just an application of the duality between cohomology and cohomology with compact support and the long exact sequence of the pair $(X, X - \pi^{-1}(Z))$.

Suppose that we have two quivers Q and \widehat{Q} with the same set of vertices I and such that arrows of Q form a subset in the set of arrows of \widehat{Q} .

Let $R = \mathbf{C}Q^{op}$ and $\widehat{R} = \mathbf{C}\widehat{Q}^{op}$. Suppose that $\widehat{W} \in \widehat{R}$ be a potential (i.e. cyclically invariant non-commutative polynomial) which is linear with respect to the arrows of \widehat{Q} which are not arrows of Q . Let R_0 be the quotient algebra of \widehat{R} with respect to the Jacobi ideal generated by the cyclic derivatives of \widehat{W} . Then we have the situation described in the above Proposition with $X = \mathbf{M}_{\gamma, \widehat{Q}}, Y = \mathbf{M}_{\gamma, Q}$, the morphism π which forgets extra arrows of \widehat{Q} . The invariant closed subset $Z \subset Y$ can be identified with the set of representations of R_0 of some dimension γ_0 which is easy to compute. Then Z is in general singular. Then the above Corollary gives for the DT-series the answer of the form

$$\sum_{\gamma} S(\mathbf{M}_{\gamma, R_0}/\mathbf{G}_{\gamma}) (-q^{1/2})^{\text{vir dim } \mathbf{M}_{\gamma} - \text{vir dim } \mathbf{G}_{\gamma}} \widehat{e}_{\gamma}.$$

Here the virtual dimension of M_γ is equal to the difference between the number of vertices and number of arrows.

Here are two examples.

Example 7.3.3 Let Q_3 be the quiver with one vertex and three loops x, y, z and $W = xyz - zyx$. Then $A := A^{(Q_3, W)}$ is given by the following formula:

$$A = \prod_{n, m \geq 1} (1 - \mathbb{L}^{m-2} \widehat{\mathbf{e}}_1^n)^{-1}.$$

Let us consider the classical limit of the quantum torus as $\mathbb{L} \rightarrow 1$. One can show that there is a limit of the conjugation $A \widehat{\mathbf{e}}_1 A^{-1}$ which has the form $A^{(1), cl} \mathbf{e}_1^{cl}$, where \mathbf{e}_1^{cl} is the classical limit of $\widehat{\mathbf{e}}_1$. The above formula implies that $A^{(1), cl} = \prod_{n \geq 1} (1 - (\mathbf{e}_1^{cl})^n)^{-n}$ is the MacMahon function.

Example 7.3.4 here we take the quiver Q which has three vertices $I = \{1, 2, 3\}$, six arrows $\alpha_{12}, \alpha_{23}, \alpha_{31}, \beta_{12}, \beta_{23}, \beta_{31}$ and potential

$$W = \alpha_{31} \cdot \alpha_{23} \cdot \alpha_{12} + \beta_{31} \cdot \beta_{23} \cdot \beta_{12}.$$

This quiver is invariant under mutations.

Using the fact that W is linear in α_{31}, β_{31} , we reduce the question to the calculation of $[\mathbf{H}_c^\bullet(S_\gamma)]$, where $S_\gamma, \gamma \in \mathbf{Z}_{\geq 0}^3$ is the stack of representations of Q of dimension vector γ with removed arrows α_{31}, β_{31} , and relations $\alpha_{23} \cdot \alpha_{12} = \beta_{23} \cdot \beta_{12} = 0$. It has the same cohomology as a similar stack for the quiver Q' with five vertices $I = 1, 1', 2, 3, 3'$, six arrows $\alpha_{12}, \alpha_{23}, \beta_{1'2}, \beta_{23}, \beta_{23'}, \delta_{11'}, \delta_{33'}$ with relations $\alpha_{23} \cdot \alpha_{12} = 0, \beta_{23'} \cdot \beta_{1'2} = 0$ and conditions that $\delta_{11'}, \delta_{33'}$ are invertible. If one removes from Q' the arrows $\delta_{11'}, \delta_{33'}$ (which corresponds to the division by a simple factor $[\mathbf{H}_c^\bullet(GL(\gamma^1) \times GL(\gamma^2))]$) then one obtains a tame problem of linear algebra with 13 indecomposable objects and with dimension vectors

$$\left\{ (\gamma^1, \gamma^{1'}, \gamma^2, \gamma^3, \gamma^{3'}) \mid \gamma^i \in \{0, 1\}, \gamma^1 \gamma^2 \gamma^3 = \gamma^{1'} \gamma^2 \gamma^{3'} = 0 \right\}.$$

The result is a complicated sum over 13 indices of q -hypergeometric type.

8 Stability conditions and motivic DT-invariants

So far we have discussed the theory which did not depend on a stability condition. In particular we defined *motivic DT-series*, but has not defined *motivic*

DT-invariants. We will do that below. In order to define DT-invariants we will use the concept of stability condition as well as a new theory of admissible series. It will allow us to define the DT-invariants as exponents in a certain factorization formula for the DT-series (factors will depend on a chosen stability condition).

8.1 Stability and Harder-Narasimhan filtration

Let Q be a quiver with the set of vertices I .

Definition 8.1.1 A central charge Z (a.k.a. **stability function**) is an additive map

$Z : \mathbf{Z}^I \rightarrow \mathbf{C}$ such that the image of any standard base vector lies in the upper-half plane $\mathbb{H}_+ := \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$.

Central charge Z is called **generic** if there are no two \mathbf{Q} -independent elements of $\mathbf{Z}_{\geq 0}^I$ which are mapped by Z to the same straight line.

Complement to the set of generic central charges is the countable union of real locally closed algebraic varieties of finite type called *walls*. The walls are responsible for the wall-crossing phenomenon of our DT-invariants (which we are going to introduce later).

For a given finite-dimensional representation of Q we define

$$\text{Arg}(E) := \text{Arg}(Z(\text{cl}(E))) \in (0, \pi) ,$$

where $\gamma = \text{cl}(E) \in \mathbf{Z}_{\geq 0}^I$ is the dimension vector of the object E . We will also use the shorthand notation $Z(E) := Z(\text{cl}(E))$.

Definition 8.1.2 A non-zero object E is called **semistable** (for the central charge Z) if there is no non-zero subobject $F \subset E$ such that $\text{Arg}(F) > \text{Arg}(E)$.

It is easy to see that the set of semistable objects is the set \mathbf{C} -points of a Zariski open G_γ -invariant subset $\mathbf{M}_\gamma^{ss} \subset \mathbf{M}_\gamma(\mathbf{C})$ defined over \mathbf{C} . In particular it is smooth.

Any non-zero finite-dimensional representation E admits a canonical **Harder-Narasimhan filtration** (HN-filtration in short), i.e. an increasing filtration $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ with $n \geq 1$ such that all the quotients $F_i := E_i/E_{i-1}$, $i = 1, \dots, n$ are semistable and

$$\text{Arg}(F_1) > \cdots > \text{Arg}(F_n) .$$

Let W be a potential for Q . Recall that the tensor category $EMHS$ contains an invertible (with respect to the tensor product) object \mathbb{T} corresponding to $\mathbf{H}^2(\mathbb{P}^1)$ (Tate motive). Using the natural filtration of the space \mathbf{M}_γ associated with HN-filtration of its objects one can construct a spectral sequence converging to \mathcal{H}_γ with the first term isomorphic to

$$\bigoplus_{n \geq 0} \bigoplus_{\gamma_1, \dots, \gamma_n \in \mathbf{Z}_{\geq 0}^I, -0, \text{Arg} \gamma_1 > \dots > \text{Arg} \gamma_n} \mathbf{H}_{\prod_{i \in I} \mathbf{G}_{\gamma_i}}^\bullet (\mathbf{M}_{\gamma_1}^{ss} \times \cdots \times \mathbf{M}_{\gamma_n}^{ss}, W_\gamma) \otimes \mathbb{T}^{\otimes \sum_{i < j} (-\chi_Q(\gamma_i, \gamma_j))} .$$

Here $W_\gamma = \text{Tr}(W)$ and χ_Q is the Euler form.

Let $V \subset \mathbb{H}_+$ be a sector, i.e. $V = V + V = \mathbf{R}_{>0} \cdot V$ and $0 \notin V$. Denote by $\mathbf{M}_{V, \gamma} \subset \mathbf{M}_\gamma$ the set of all representations whose HN -factors have classes in $Z^{-1}(V)$. It is easy to see that for any γ the set $\mathbf{M}_{V, \gamma}$ is Zariski open and \mathbf{G}_γ -invariant. We define the **Cohomological Hall vector space** by

$$\mathcal{H}_V := \bigoplus_\gamma \mathcal{H}_{V, \gamma} = \bigoplus_\gamma \mathbf{H}_{\mathbf{G}_\gamma}^\bullet (\mathbf{M}_{V, \gamma}, W_\gamma) .$$

Similarly to the above, one has a spectral sequence converging to \mathcal{H}_V where we use only $Z(\gamma) \in V$.

In general the space \mathcal{H}_V does not carry a product. Nevertheless, for V being a ray $l = \exp(i\phi) \cdot \mathbf{R}_{>0}$ as well as for V equal to the whole upper-half plane, the product is well-defined.

Let $Z : \mathbf{Z}^I \rightarrow \mathbf{C}$ be a central charge. Then for any sector $V \subset \mathbb{H}_+$ we define the **motivic DT-series associated with V** similarly to what we did in Section 5.4:

$$A_V := \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} [\mathcal{H}_{V, \gamma}] \mathbf{e}_\gamma .$$

The above-mentioned spectral sequence implies the following **Factorization Formula** (a.k.a wall-crossing formula):

$$A = A_{\mathbb{H}_+} = \prod_l^{\curvearrowright} A_l ,$$

where the product is taken in the clockwise order over all rays $l \subset \mathbb{H}_+$ containing non-zero points in $Z(\mathbf{Z}_{\geq 0}^I)$. Each factor A_l corresponds to semistable objects with the central charge in l :

$$A_l = 1 + \sum_{\gamma \in Z^{-1}(l)} \sum_{k \geq 0} (-1)^k [\mathbf{H}_{G_\gamma}^k(\mathbf{M}_\gamma^{ss}, W_\gamma)] \mathbf{e}_\gamma \in \widehat{\mathcal{R}}_+.$$

Also, for any pair of disjoint sectors $V_1, V_2 \subset \mathbb{H}_+$ whose union is also a sector, and such that V_1 lies on the left of V_2 , we have

$$A_{V_1 \cup V_2} = A_{V_1} A_{V_2}.$$

Finally, we mention that in other cases there should be a generalization of all above results to the case of formally smooth algebras (in the sense of Cuntz and Quillen) endowed with potential. The path algebra of Q is an example of formally smooth algebra.

9 Admissible series and motivic DT-invariants

9.1 Admissible series

Definition 9.1.1 *A series*

$$F \in \mathbf{Z}((q^{1/2}))[[x = (x_i)_{i \in I}]],$$

where $q^{1/2}, (x_i)_{i \in I}$ variables is called **admissible** if it has a form

$$F = \prod_{\gamma \in \mathbf{Z}_{\geq 0}^I} \prod_{n \in \mathbf{Z}} (q^{n/2} x^\gamma; q)_\infty^{c(\gamma, n)} \in 1 + x \cdot \mathbf{Z}((q^{1/2}))[[x]],$$

where $c(\gamma, n) \in \mathbf{Z}$ for all n, γ , and for any given γ we have $c(\gamma, n) = 0$ for $|n| \gg 0$. Here $(x; q)_\infty = \prod_{n \geq 1} (1 - xq^n)$ and $x^\gamma = \prod_{1 \leq i \leq n} x_i^{\gamma_i}$.

In the case of one variable admissibility is equivalent to the property

$$F = \exp \left(- \sum_{n, m \geq 1} \frac{f_n(q^{m/2})}{m(1 - q^m)} x^{nm} \right),$$

where $f_n = f_n(t)$ belongs to $\mathbf{Z}[t^{\pm 1}]$ for all $n \geq 1$. The equivalence of two descriptions follows from the identity

$$\log (q^{i/2} x^n; q)_\infty = - \sum_{m \geq 1} \frac{(q^{m/2})^i}{m(1 - q^m)} x^{nm}.$$

It is easy to see that any series which belongs to the multiplicative group

$$1 + x \cdot \mathbf{Z}[q^{\pm 1/2}] [[x]]$$

is admissible.

Admissibility of a series $F = F(x; q^{1/2}) \in \mathbf{Z}((q^{\pm 1/2}))[[x]]$ implies certain divisibility properties. Namely, let us define a new series by the formula

$$G(x; q^{1/2}) := \frac{F(x; q^{1/2})}{F(qx; q^{1/2})} \in \mathbf{Z}[q^{\pm 1/2}] [[x]].$$

Then the evaluation at $q^{1/2} = 1$ of the series G is of the form

$$G(x; 1) := \lim_{q^{1/2} \rightarrow 1} G(x; q^{1/2}) = \prod_{n \geq 1} (1 - x^n)^{nc(n)} \in 1 + x\mathbf{Z}[[x]], \quad c(n) \in \mathbf{Z}.$$

Therefore $G(x; 1) = \prod_{n \geq 1} (1 - x^n)^{b(n)}$, where the exponents $b(n)$ are integers divisible by n .

Obviously, admissible series form a group under multiplication. One can also prove the following.

Theorem 9.1.2 *For a given symmetric integer matrix $\mathcal{B} = (b_{ij})_{i,j \in I}$, a series*

$$F = \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} a_{\gamma} x^{\gamma} \in B((q^{1/2}))[[x_i]_{i \in I}], \quad x^{\gamma} := \prod_I x_i^{\gamma_i}, \quad a_{\gamma} \in B((q^{1/2}))$$

is admissible if and only if

$$\tilde{F} := \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} (-q^{1/2})^{\sum_{ij} b_{ij} \gamma_i \gamma_j} a_{\gamma} x^{\gamma}$$

is admissible.

Existing proof of Theorem is complicated. It is based on the notion of factorization algebra in the category of coherent sheaves. A similar notion, but in the category of D -modules was used by Beilinson and Drinfeld in their work on chiral algebras. Yet another version of factorization algebras (in the framework of C^{∞} algebras) was introduced by Costello in his work on quantum field theory. In our case we consider the tower of certain *coherent*

sheaves on the tower of spaces \mathbf{M}_γ . The relationship of the work of Beilinson-Drinfeld and Costello to the operator product expansion in QFT is clear. One can speculate that there is a similar formalism of OPE in the framework of $N = 2$ theories associated with Calabi-Yau superstring compactifications which underlies our factorization algebras.

Using the above Theorem one can define the notion of *quantum admissible series*. Namely, let us choose an ordered basis in \mathbf{Z}^I and consider a series in the quantum torus with the fixed order of generators. The above Theorem says that the property of the “classical limit” of the series to be admissible does not depend on the order (change of the order leads to the multiplication of coefficients by a q -power of a quadratic form). Thus we say that by definition the series is quantum admissible if the corresponding classical series is admissible.

Proposition 9.1.3 *Let us choose a generic central charge $Z : \mathbf{Z}_{\geq 0}^I \rightarrow \mathbb{H}_+$. Then the set of quantum admissible series coincides with the set of products*

$$\prod_{l=\mathbf{R}_{>0}Z(\gamma_0)}^{\widehat{}} F_l(\widehat{\mathbf{e}}_{\gamma_0}),$$

where the product in the clockwise order is taken over all rays generated by primitive vectors $\gamma_0 \in \mathbf{Z}_{\geq 0}^I$, and $F_l(t)$ is an admissible series in one variable.

The proof is obtained by induction such as follows. Pick an ordered basis $\gamma_1, \gamma_1, \dots, \gamma_m$ of the dimension lattice such that γ_1 is primitive and $\text{Arg}(Z(\gamma_1))$ is the largest among all $\text{Arg}(Z(\gamma_i))$. Then use the induction by the number m and the fact that the product (from the left) of a quantum admissible series by a quantum admissible series corresponding to the ray $\mathbf{R}_{>0} \cdot Z(\gamma_1)$ is again quantum admissible (and also the fact that the inverse to a quantum admissible series in one variable is admissible).

Proposition 9.1.4 *For quantum variables $yx = qxy$ the collection of elements*

$$\prod_{(a,b) \in \mathbf{Z}_{\geq 0}^2 - \{0\}}^{\widehat{}} \prod_{|k| \leq \text{const}(a,b)} ((-1)^{ab} q^{k/2} x^a y^b; q)_\infty^{c(a,b;k)}$$

with $c(a, b; k) \in \mathbf{Z}$, is closed with respect to the product.

The corresponding group can be called *the quantum tropical vertex group* since for a quantum admissible series F the automorphism $Ad(F)$ in the limit $q^{1/2} \rightarrow 1$ gives rise to a formal symplectomorphism of the symplectic torus considered in our 2004 paper as well as in the recent paper by Gross-Pandharipande-Siebert (where it was called the tropical vertex group). Product structure on the tropical vertex group controls various enumerative problems. For example the number of rational curves in \mathbf{CP}^2 which pass through given points on two given divisors and tangent to a third divisor with prescribed order is given by the coefficient of the product of two elements of the group.

9.2 Admissibility of DT-series and motivic DT-invariants

For a fixed (Q, W) let us choose a central charge $Z : I \rightarrow \mathbb{H}_+$. We also fix a cohomology theory \mathbf{H}^\bullet with values in the category $\mathcal{C}^{\mathbb{Z}-gr}$ of \mathbb{Z} -graded objects associated with a Tannakian category \mathcal{C} . Also let us choose a degree +1 tensor square root $\mathbb{T}^{\otimes 1/2}$ of $\mathbb{T} \in Ob(\mathcal{C}^{\mathbb{Z}-gr})$. As before we assume that there is a notion of weight filtration for objects of \mathcal{C} .

We denote by $B_{\mathcal{C}}$ the K_0 -ring of the subcategory of \mathcal{C} consisting of objects of weight 0. Hence we have:

$$K_0(\mathcal{C}) = B_{\mathcal{C}}[q^{\pm 1/2}], \quad q^{1/2} := \mathbb{L}^{1/2} = [\mathbb{T}^{\otimes 1/2}[1]].$$

Theorem 9.2.1 *For any sector $V \subset \mathbb{H}_+$ the generating series A_V is quantum admissible.*

Corollary 9.2.2 *Let us assume that for some ray $l = \exp(i\phi)\mathbf{R}_{>0} \subset \mathbb{H}_+$ the restriction of the form χ_Q to the sublattice $\Gamma_l := Z^{-1}(\exp(i\phi)\mathbf{R}) \subset \mathbf{Z}^I$ is symmetric. Then there exist elements $\Omega^{mot}(\gamma) \in K_0(\mathcal{C})$ for which the following formula holds*

$$\sum_{\gamma \in Z^{-1}(l)} [\mathcal{H}_{l,\gamma}^{mod}] x^\gamma = \text{Sym} \left(\sum_{\gamma \in Z^{-1}(l)} \Omega^{mot}(\gamma) \cdot [\mathbf{H}^\bullet(\mathbb{P}^\infty)] \cdot x^\gamma \right),$$

where $\mathcal{H}_{l,\gamma}^{mod} = \mathcal{H}_{l,\gamma} \otimes (\mathbb{T}^{\otimes 1/2})^{\chi_Q(\gamma,\gamma)}$.

Definition 9.2.3 *Under the above assumptions the element $\Omega^{\text{mot}}(\gamma)$ is called **motivic Donaldson-Thomas invariant** of the pair (Q, W) , stability function Z and dimension vector γ .*

We can apply homomorphism $K_0(\mathcal{C}) \rightarrow \mathbf{Z}[q^{\pm 1/2}]$ and get the so-called *refined* (or quantum) Donaldson-Thomas invariants, or we can apply Euler characteristic $K_0(\mathcal{C}) \rightarrow \mathbf{Z}$ (i.e. evaluate at $q^{1/2} = 1$) and obtain numerical Donaldson-Thomas invariants. Admissibility property implies integrality of numerical DT-invariants. Also we can replace (Q, W) by (R, W) where R is a smooth algebra.

Let us briefly discuss our approach to the proof in the case when the sector V is the whole upper-half plane.

We want to prove that the series

$$A = \sum_{\gamma} (-q^{1/2})^{\dim M_{\gamma} - \dim G_{\gamma}} S(\mathbf{H}_{\mathbf{c}}^{\bullet}(M_{\gamma}/G_{\gamma}), W_{\gamma}) \widehat{e}_{\gamma}$$

is quantum admissible. Notice that each coefficient in the sum contains the power of $q^{1/2}$ which is an integer quadratic form. Then we apply Theorem 9.1.4. Hence in order to prove the admissibility of the series A it suffices to prove that the following series in commuting variables is admissible:

$$\sum_{\gamma} S(\mathbf{H}_{\mathbf{c}}^{\bullet}(M_{\gamma}/G_{\gamma}), W_{\gamma}) x^{\gamma}.$$

Furthermore, it suffices to prove that the following series is admissible

$$B = \sum_{\gamma} S(\mathbf{H}_{\mathbf{c}}^{\bullet}(M_{\gamma}/Sym_{\gamma} \times T_{\gamma}), W_{\gamma}) x^{\gamma}.$$

Here T_{γ} is the maximal torus in G_{γ} , and Sym_{γ} is the product of the permutation groups corresponding to the coordinates of γ (i.e. it is the Weyl group of G_{γ}). The reduction to the normalizer of the maximal torus of G_{γ} is based on the isomorphism $\mathbf{H}^{\bullet}(G_{\gamma}/(Sym_{\gamma} \times T_{\gamma})) \simeq \mathbf{H}^{\bullet}(pt)$. The latter group is a module over

$$\mathbf{H}^{\bullet}(BG_{\gamma}) = (\mathbf{H}^{\bullet}(BT_{\gamma}))^{Sym_{\gamma}}.$$

The above reduction is illustrated by the isomorphisms $\mathbf{H}_{GL(n, \mathbf{C})}^{\bullet}(X, \mathbf{Q}) \simeq \mathbf{H}_{Sym_n \times (\mathbf{C}^*)^n}^{\bullet}(X, \mathbf{Q}) \simeq \mathbf{H}_{(\mathbf{C}^*)^n}^{\bullet}(X, \mathbf{Q})^{Sym_n}$.

Definition 9.2.4 *A non-trivial representation of a smooth algebra R in coordinate spaces is called T -indecomposable if it can not be decomposed into the direct sum of two non-trivial subrepresentations in coordinate subspaces.*

Let

$$\mathbf{M}_\gamma^{T-ind} \subset \mathbf{M}_\gamma$$

be the set of T -indecomposable representations of R in coordinate spaces of dimension vector γ . It is a $(Sym_\gamma \times \mathbb{T}_\gamma)$ -invariant open subset of \mathbf{M}_γ .

In order to prove that B is admissible we observe that there is a constructible equivalence of ind-Artin stacks

$$\sqcup_{\gamma \in \mathbf{Z}_{\geq 0}^I} \mathbf{M}_\gamma / (Sym_\gamma \times \mathbb{T}_\gamma) \simeq Sym \left(\sqcup_{\gamma \in \mathbf{Z}_{\geq 0}^I - \{0\}} \mathbf{M}_\gamma^{T-ind} / (Sym_\gamma \times \mathbb{T}_\gamma) \right),$$

where for any ind-Artin stack X we define $Sym(X) := \sqcup_{k \geq 0} X^k / Sym_k$.

Since constructible equivalences preserve Serre series, this implies the identity

$$\sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} [\mathbf{H}_{c, \mathbb{T}_\gamma}^\bullet(\mathbf{M}_\gamma, -W_\gamma)^{Sym_\gamma}] x^\gamma = Sym \left(\sum_{\gamma \in \mathbf{Z}_{\geq 0}^I - \{0\}} [\mathbf{H}_{c, \mathbb{T}_\gamma}^\bullet(\mathbf{M}_\gamma^{T-ind}, -W_\gamma)^{Sym_\gamma}] x^\gamma \right).$$

Notice that $\mathbf{H}_c^\bullet(Sym(Y)) \simeq Sym(\mathbf{H}_c^\bullet(Y))$, where $Sym(V) := \bigoplus_{n \geq 0} Sym^n(V)$. Hence

$$B = S(Sym(\bigoplus_\gamma \mathbf{H}_c^\bullet(\mathbf{M}_\gamma^{T-ind}) x^\gamma)).$$

Here x^γ should be understood as a 1-dimensional representation of the group \mathbb{G}_m .

The space $\mathbf{H}_c^\bullet(\mathbf{M}_\gamma^{T-ind})$ is “small” in the following sense. The stabilizer of a point in $\mathbf{M}_\gamma^{T-ind}$ has “continuous” part and “finite” part. The continuous part is isomorphic to the diagonally embedded multiplicative group \mathbb{G}_m^{diag} . Hence $\mathbf{H}_c^\bullet(\mathbf{M}_\gamma^{T-ind}) \simeq \mathbf{H}_c^\bullet(\mathbf{M}_\gamma / \mathbb{T}_\gamma)^{Sym_\gamma}$.

Passing to the dual spaces we obtain

$$\sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} [\mathcal{H}_\gamma] \mathbb{L}^{-dim \mathbf{M}_\gamma} x^\gamma = Sym \left(\sum_{\gamma \in \mathbf{Z}_{\geq 0}^I - \{0\}} [\mathbf{H}_{\mathbb{T}_\gamma}^\bullet(\mathbf{M}_\gamma^{T-ind}, W_\gamma)^{Sym_\gamma}] \mathbb{L}^{-dim \mathbf{M}_\gamma} x^\gamma \right).$$

In order to finish the computation we introduce the group PT_γ which is the quotient of T_γ by the diagonal subgroup \mathbb{G}_m^{diag} . Since the latter acts trivially on $\mathbf{M}_\gamma^{T-ind}$, the group PT_γ acts freely on this space. Therefore we have

$$\mathbf{H}_{\mathrm{T}_\gamma}^\bullet(\mathbf{M}_\gamma^{T-ind}, W_\gamma)^{Sym_\gamma} \simeq \mathbf{H}^\bullet(\mathbf{M}_\gamma^{T-ind}/\mathrm{PT}_\gamma, W_\gamma)^{Sym_\gamma} \otimes \mathbf{H}^\bullet(B\mathbb{G}_m^{diag}),$$

where in RHS we take ordinary cohomology of the quotient variety endowed with a function. This implies admissibility of the series B and hence quantum admissibility of the series A .

Let finally make few remarks about the proof of main theorem. which guarantees that quantum admissibility is preserved if we multiply the coefficients by powers of $q^{1/2}$ which are integer quadratic forms. We do not have a conceptual proof in the case when $b_{ij} < 0$. Besides of that, the idea is to look at the cohomology $(\mathbf{H}_c^\bullet(\mathbf{M}_\gamma/\mathrm{T}_\gamma))^* \simeq \mathbf{H}_{\mathrm{T}_\gamma}^\bullet(\mathbf{M}_\gamma)$ as the coherent sheaf on the configuration space of points on the line. The collection of these coherent sheaves form a factorization system.

10 Motivic DT-invariants for 3CY categories

10.1 Stability conditions

The notion of stability condition in triangulated category introduced by Bridgeland combines two ideas: the idea of Harder-Narasimhan filtration in abelian category and the idea of a t-structure in triangulated category. A new ingredient which is motivated by physics is the notion of central charge. More precisely, it goes like this. Let \mathcal{C} be a triangulated category (say, over some ground field of characteristic zero). Suppose we are given a homomorphism of abelian groups $K_0(\mathcal{C}) \rightarrow \Gamma \simeq \mathbf{Z}^n$ (generalized Chern character, or “class map”). The a stability condition is given by the following data:

- an additive map $Z : \Gamma \rightarrow \mathbf{C}$, called the central charge,
- a collection \mathcal{C}^{ss} of (isomorphism classes of) non-zero objects in \mathcal{C} called the semistable ones, such that $Z(E) \neq 0$ for any $E \in \mathcal{C}^{ss}$, where we write $Z(E)$ for $Z(cl(E))$,
- a choice $LogZ(E) \in \mathbf{C}$ of the logarithm of $Z(E)$ defined for any $E \in \mathcal{C}^{ss}$.

The latter means that for any semistable object E we can define its argument $Arg(E) := Arg(Z(cl(E))) \in \mathbf{R}/2\pi\mathbf{Z}$.

- for all $E \in \mathcal{C}^{ss}$ and for all $n \in \mathbf{Z}$ we have $E[n] \in \mathcal{C}^{ss}$ and

$$ArgZ(E[n]) = ArgZ(E) + \pi n ,$$

- for all $E_1, E_2 \in \mathcal{C}^{ss}$ with $Arg(E_1) > Arg(E_2)$ we have

$$Ext_{\mathcal{C}}^{\leq 0}(E_1, E_2) = 0 ,$$

- for any object $E \in Ob(\mathcal{C})$ there exist $n \geq 0$ and a chain of morphisms $0 = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n = E$ (an analog of filtration) such that the corresponding “quotients” $F_i := Cone(E_{i-1} \rightarrow E_i)$, $i = 1, \dots, n$ are semistable and $Arg(F_1) > Arg(F_2) > \dots > Arg(F_n)$,
- (Support Property) Pick a norm $\| \cdot \|$ on $\Gamma \otimes \mathbf{R}$, then there exists $C > 0$ such that for all $E \in \mathcal{C}^{ss}$ one has $\| E \| \leq C|Z(E)|$.

This can be compared with the notion of t-structure. There we have a collection of full subcategories $\mathcal{C}^i \subset \mathcal{C}, i \in \mathbf{Z}$ such that $\mathcal{C}^i[1] = \mathcal{C}^{i-1}$ and

- $Hom(\mathcal{C}^i, \mathcal{C}^j) = 0, i < j$;
- For any object F of \mathcal{C} there exists a chain of morphisms

$$0 = \dots = F_{-(n+1)} = F_{-n} \rightarrow \dots \rightarrow F_{-1} \rightarrow F_0 \rightarrow \dots \rightarrow F_n = F_{n+1} = \dots = F$$

and $Cone(F_{i-1} \rightarrow F_i) \in \mathcal{C}^i$.

Notation: $\tau_{\leq i}(F) = F_i$.

For example in the derived category of modules over an algebra we have

$$\tau_{\leq i}(M^\bullet) = \dots \rightarrow M^{i-2} \rightarrow M^{i-1} \rightarrow Ker(M^i \rightarrow M^{i+1}) \rightarrow 0.$$

At the level of cohomology it is indeed a truncation:

$$H^{i-1}(M^\bullet) \rightarrow H^i(M^\bullet) \rightarrow 0.$$

A stability condition defines t-structure which consists of non-zero semistable objects E such that $Z(\text{cl}(E)) = x + iy, y > 0, x \notin \mathbf{R}_{<0}$. Rotation $Z \mapsto Ze^{i\phi}$ gives a new t-structure. It follows from the axioms that the space of stability conditions $\text{Stab}(\mathcal{C})$ is a complex manifold.

Let now X be a compact Calabi-Yau 3-fold. Then we have the Fukaya category $\mathcal{F}(X)$. Near a cusp in the moduli space of complexified Kähler structures mirror symmetry predicts an equivalence $\mathcal{F}(X) \simeq D^b(X^\vee)$ for the “mirror dual” Calabi-Yau 3-fold X^\vee . There are corresponding “Chern characters”

$$\begin{aligned} \text{cl} : K_0(D^b(X^\vee)) &\rightarrow H^{ev}(X^\vee, \mathbf{Q}), \\ \text{cl} : K_0(\mathcal{F}(X)) &\rightarrow H_3(X, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}). \end{aligned}$$

In the case of Fukaya category we have the central charge $Z : H_3(X, \mathbf{Z}) \rightarrow \mathbf{C}$ given by the formula $\gamma \mapsto \int_\gamma \Omega_X^{3,0}$ (we disregard local systems in our considerations). Furthermore the class of semistable objects is defined as a class of extensions of special Lagrangian submanifolds (SLAGs). On the other hand, there are no examples of stability conditions on $D^b(X^\vee)$.

Let $h^{1,2}(X) = b$ (i.e. $\text{rk } H^3(X, \mathbf{Z}) = 2b + 2$). Physicists speak about Π -stability (predecessor of the Bridgeland theory). The space of “physical” stability conditions is a complex Lagrangian cone $\mathcal{L}_X \subset H^3(X, \mathbf{C})$ which parametrizes pairs (*complex structure, holomorphic volume form*). Hence the dimension of the physical space of stability conditions is $b + 1$, which is twice smaller than the one for Bridgeland theory.

Question 10.1.1 *How to derive physical stability conditions from the Bridgeland ones?*

One possibility to obtain an answer consists in a new concept which is somewhere in between the notion of Bridgeland stability and the notion of t-structure.

Definition 10.1.2 *an \mathbf{R} -t-structure on \mathcal{C} is given by the following data:*

- 1) *a t-structure on \mathcal{C} ;*
- 2) *homomorphism $\text{Im } Z : \Gamma \rightarrow \mathbf{R}$ such that for any object E which belongs to the heart \mathcal{C}^0 of the t-structure we have: $\text{Im}(\text{cl}(E)) \geq 0$.*

There are also some axioms which are natural generalizations of those in the Bridgeland theory. In the above geometric example we see that taking the imaginary part of Z we arrive to an \mathbf{R} -t-structure which depends on

$2b + 2$ *real* parameters, i.e. it is of the “right” size. Notice that we can keep $Im Z$ fixed but change $Re Z$, thus keeping the t-structure unchanged. In general the space of \mathbf{R} -t-structures is a Hausdorff topological space which is locally homeomorphic to $Hom(\Gamma, \mathbf{R})$. We expect that \mathbf{R} -t-structures is a more fundamental object than stability structures.

10.2 Critical COHA and 3CY categories

Let us start with an example. Fix a quiver Q and a potential W , Recall the notion of cyclic derivative of W . E.g. if $W = xyz - zyx$ (up to cyclic permutations) then the cyclic derivative $\partial W/\partial x = yz - zy$ (for each monomial we choose a cyclic representative with the symbol x on the very left and then delete it from a monomial). Ginzburg suggested a construction of a dg-algebra \widehat{R} quasi-isomorphic to the algebra R which is the quotient of the path algebra $\mathbf{C}Q$ by the two-sided ideal ∂W generated by all cyclic derivatives (R is a non-commutative version of the Jacobi algebra). The dg-algebra \widehat{R} is concentrated in degrees $0, -1, -2$. In degree 0 it is generated by arrows of Q , in degree -1 it is generated by opposite arrows, and in degree -2 it is generated by all pr_i^* . Here pr_i^* is the dual to the projector pr_i associated with the vertex i of Q . The differential is trivial on the generators of degree zero. If $\alpha^* : j \rightarrow i$ then $d(\alpha) = \partial W/\partial \alpha$. Finally, $d(pr_i^*) = pr_i(\sum_{\alpha} [\alpha^*, \alpha])pr_i$.

Lemma 10.2.1 $d^2 = 0$.

Proof. Follows from the identity $\sum_{\alpha} [\partial W/\partial \alpha, \alpha] = 0$. ■

Then we have a triangulated 3CY category which is a triangulated envelope of the dg-category of finite-dimensional dg-modules over \widehat{R} . The heart consists of the union of critical sets $Crit(Tr(W_{\gamma}))$, where W_{γ} is the regular function defined by W in the representation of dimension γ .

Potentials which arise from 3CY categories are in general infinite series, not polynomials. For that reason one cannot use rapid decay or de Rham cohomology. Instead we use the cohomology with coefficients in the sheaf of vanishing cycles. This theory is more complicated than the theory of rapid decay cohomology. Recall that the definition of the sheaf of vanishing cycles goes back to Deligne. In particular, to an analytic function $f : X \rightarrow \mathbf{C}$ on a complex analytic manifold, one can assign the object $\phi_f(\mathbf{Q}_X) \in D_c^b(X_0)$ in the bounded derived category of constructible sheaves, where $X_0 = f^{-1}(0) \subset X$.

The fiber over $x_0 \in X_0$ is defined as

$$\varinjlim_{r \rightarrow 0} H^\bullet(B(x_0, r), B(x_0, r) \cap f^{-1}(\mathbf{R}_{<0}), \mathbf{Q}).$$

Notice that the inductive limit does not have non-trivial higher derived functors. This is a natural generalization of the cohomology of Milnor fiber of f . Indeed, the inductive limit is supported on $\text{Crit}(f) \cap X_0$. For isolated singularity it is concentrated in the middle cohomology and its dimension is equal to the Milnor number μ . Then one arrives to the natural question. Suppose that $X^s \subset X_0$ is closed subvariety. Then we have a well-defined $\phi_f(\mathbf{Q}_Y)|_{X^s} \in D_c^b(X^s)$. Answer to the following question is not given in the existing literature.

Question 10.2.2 *How to define the above restriction for a formal series f (i.e. a function in the formal neighborhood of $X^s \subset X_0 \cap \text{Crit}(f)$)?*

Another open question is about comparison of the above critical cohomology with the corresponding de Rham version. More precisely, let $f : X \rightarrow \mathbf{C}$ be proper. Let us consider the \mathbf{Q} -vector space

$$(i) \quad \bigoplus_{z_i} H_c^i(f^{-1}(z_i), \phi_{f-z_i}(\mathbf{Q}_X)),$$

where the sum is taken over all critical values z_i of the function f . Let us consider:

- (ii) $H^i(X, (\Omega_X[[u]], ud + df \wedge \bullet))$ as a free finite rank module over $\mathbf{C}[[u]]$.
- (iii) $H^i(X, (\Omega_X, df \wedge \bullet))$ as a \mathbf{C} -vector space.

Then the vector space (i) should have the same rank as (ii) and (iii).

The cohomology groups $H_c^i(X^s, \phi_{f-z_i}(\mathbf{Q}_X))$ is an example of the special type of *EMHS*. In order to study it one can use results of M. Saito on mixed Hodge modules. If X^s is algebraic so is the intersection with $f^{-1}(z_i)$. Hence we get a weight filtration on the critical cohomology groups. Applying Serre polynomials one arrives to the definition of COHA in critical case. Without discussing many details of this construction let us summarize main steps of the above idea.

Let R be a smooth I -bigraded algebra over a field \mathbf{k} , endowed with a bilinear form χ_R on \mathbf{Z}^I compatible with the Euler form, and a potential $W \in R/[R, R]$. Also, suppose that we are given additional data, consisting of a collection of \mathbf{G}_γ -invariant closed subsets $M_\gamma^{sp} \subset M_\gamma$ for all $\gamma \in \mathbf{Z}_{\geq 0}^I$ (superscript *sp* means “special”) satisfying the following conditions:

- for any γ we have $\mathbf{M}_\gamma^{sp} \subset \text{Crit}(W_\gamma)$, i.e. 1-form dW_γ vanishes at \mathbf{M}_γ^{sp} ,
- for any short exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ of representations of $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R$ with dimension vectors $\gamma_1, \gamma := \gamma_1 + \gamma_2, \gamma_2$ correspondingly, such that all E_1, E_2, E are critical points of the potential, the representation E belongs to \mathbf{M}_γ^{sp} if and only if both representations E_1, E_2 belong to $\mathbf{M}_{\gamma_1}^{sp}, \mathbf{M}_{\gamma_2}^{sp}$ respectively.

The last condition implies that the collection of representations $\mathbf{M}_\gamma^{sp}(\bar{\mathbf{k}})$ for all $\gamma \in \mathbf{Z}_{\geq 0}^I$ form an abelian category, which is a Serre subcategory of the abelian category $\text{Crit}(W)(\bar{\mathbf{k}}) := \sqcup_\gamma \text{Crit}(W_\gamma)(\bar{\mathbf{k}})$ (which is itself a full subcategory of $\bar{\mathbf{k}} \otimes_{\mathbf{k}} R - \text{mod}$).

For example, one can always take

$$\mathbf{M}_\gamma^{sp} := \text{Crit}(W_\gamma) \quad \forall \gamma \in \mathbf{Z}_{\geq 0}^I.$$

We can construct more examples such as follows. Pick an *arbitrary* subset $N \subset R$ and define \mathbf{M}_γ^{sp} as the set of representations belonging to $\text{Crit}(W_\gamma)$ for which all elements $n \in N$ act as *nilpotent* operators.

Assume that $\mathbf{k} = \mathbf{C}$. We will define the **critical COHA** as

$$\mathcal{H} = \bigoplus_{\gamma \in \mathbf{Z}_{\geq 0}^I} \mathcal{H}_\gamma,$$

where

$$\mathcal{H}_\gamma := \bigoplus_{z \in \mathbf{C}} \left(H_{G_\gamma, c}^\bullet(\mathbf{M}_\gamma^{sp} \cap W_\gamma^{(-1)}(z), \phi_{W_\gamma - z} \mathbf{Q}_{\mathbf{M}_\gamma}) \right)^\vee \otimes \mathbb{T}^{\otimes \dim M_\gamma / G_\gamma}.$$

Here we use equivariant cohomology with compact support with coefficients in the sheaf of vanishing cycles.

One can define a structure of EMHS of special type (called *monodromic mixed Hodge structure*) with Betti realization \mathcal{H}_γ , and a twisted associative product. For the critical COHA in Tannakian category EMHS there are analogs of results which we have formulated for rapid decay or de Rham cohomology. In particular we have motivic DT-series and DT-invariants (see below). But the *proofs in the critical case are much more involved*. This is due to the fact that several important results are either false or not proven yet. In particular there is no comparison theorem between critical and rapid decay cohomology, no Thom isomorphism for critical cohomology, etc.

There is a problem in application of the above theory to 3CY categories. Namely, there is no *canonically defined* object $\phi_W(\mathbf{Q}_X) \in D_c^b(\text{Ob}(\mathcal{C}))$, because the potential itself is not canonically defined. Indeed, we do not know on “how many” variables the potential W depends on. In particular, we can add to W a quadratic form in all but finitely many variables without changing the sheaf of vanishing cycles. The ambiguity is encoded in a “local system of rank one” on the space of objects which has monodromy ± 1 . This is what we called *orientation data*. Categorically, the fiber of the line bundle is $D_E = \sqrt{\text{Ext}^\bullet(E, E)}$, $E \in \text{Ob}(\mathcal{C})$. Notice that in the case of 3CY categories $\det(\text{Ext}^\bullet(E, E))$ is an *even* line. We want the property

$$D_{E \oplus F} \otimes D_E^{-1} \otimes D_F^{-1} \simeq \det(\text{Ext}^\bullet(E, E)).$$

The Calabi-Yau property ensures that the RHS is symmetric. For 3CY categories coming from smooth algebras such orientation data exist. They also exist at the quasi-classical level (for $D^b(X)$ it was shown by Joyce and Song).

We have seen above that the potential W does not have to be a globally defined function (e.g. a polynomial). Furthermore it can be partially a formal series. This is natural from the point of view of A_∞ -categories, since the products there are given by infinite series.

Let us consider one example in which an explicit form of this series is important. Let X be a compact real oriented 3-dimensional manifold. There is a 3CY category naturally associated with X . The heart of its t -structure consists of finite-dimensional representations $\pi_1(X, x_0) \rightarrow GL(n, \mathbf{C})$, $n \geq 0$. Equivalently, it is the category of local systems of complex vector spaces. If $X = S^3$ the corresponding 3CY category is related to the quiver with one vertex (since there is only one representation in each dimension). From the infinite-dimensional perspective our t -structure can be described as a set of critical points of the Chern-Simons functional

$$CS(A) = \int_X \text{Tr} \left(\frac{AdA}{2} + \frac{A^3}{3} \right).$$

Let us consider a cell decomposition of X and denote by $X^{(1)}$ its 1-skeleton. Then we have a surjective homomorphism $\pi_1(X^{(1)}) \rightarrow \pi_1(X)$. Hence for each $n \geq 0$ the closed subset of n -dimensional representations $\text{Rep}(\pi_1(X^{(1)})) \subset \text{Rep}(\pi_1(X))$ gives rise to a smooth Artin stack over \mathbf{Q} .

Question 10.2.3 *Find nice functions near the intersection of the critical locus of CS with this subset and write down the Chern-Simons functional as a formal series in terms of these functions.*

10.3 Line operators, framed objects and positivity

Let us discuss positivity property of classical and quantized symplectomorphisms. This property should be compared with the one discussed recently by Gaiotto-Moore-Neizke recently in the form of “positivity conjectures” for BPS and framed BPS states. From that point of view the positivity is closely related to the fact that line operators studied by physicists admit an expression as a linear combinations of characters of finite-dimensional representations of $SL_2(\mathbf{C})$. We expect that there are intrinsically defined compact algebraic varieties such that the above characters are in fact characters of the natural SL_2 -representations which appear in the Hodge theory.

Let \mathcal{C} be a triangulated A_∞ -category over \mathbf{C} . We also fix $\tau \in \text{Stab}(\mathcal{C})$, $F : \mathcal{C} \rightarrow D^b(\mathbf{C})$ a functor to the triangulated category of bounded complexes. For a fixed slope θ we denote by \mathcal{C}_θ^{ss} the abelian category of τ -semistable objects having the slope θ . We will impose the following assumption: F maps \mathcal{C}_θ^{ss} to the complexes concentrated in non-negative degrees.

Definition 10.3.1 *Framed object is a pair (E, f) where $E \in \text{Ob}(\mathcal{C}_\theta^{ss})$ and $f \in H^0(F(E))$.*

There is an obvious notion of isomorphic framed objects.

Definition 10.3.2 *We call framed object stable if there is no exact triangle $E' \rightarrow E \rightarrow E''$ such that both $E', E'' \in \text{Ob}(\mathcal{C}_\theta^{ss})$ and such that there is $f' \in H^0(F(E'))$ which is mapped to $f \in H^0(F(E))$.*

Proposition 10.3.3 *If (E, f) is a stable framed object then $\text{Aut}(E, f) = \{1\}$.*

Proof. Let $\alpha \in \text{Hom}^0(E, E)$ preserves f . It suffices to prove that $\alpha = id_E$. Let $\beta = \alpha - id_E$. Then $\beta(f) = 0$, and we want to prove that $\beta = 0$. Recall that the category \mathcal{C}_θ^{ss} is abelian, hence there exist $\text{Ker } \beta, \text{Im } \beta, \text{Coker } \beta$. Consider an exact short sequence in \mathcal{C}_θ^{ss} :

$$0 \rightarrow \text{Ker } \beta \rightarrow E \rightarrow \text{Im } \beta \rightarrow 0.$$

If $\text{Ker } \beta \neq 0$ then we have a non-trivial subobject in E having the same slope. Also $\text{Im } \beta$ does not coincide with E .

Let us remark that the functor H^0F transforms monomorphisms in \mathbf{C}_θ^{ss} into monomorphisms in the category $\text{Vect}_{\mathbf{C}}$ of complex vector spaces. Indeed, if we have an exact triangle $E' \rightarrow E \rightarrow E''$ in \mathcal{C} then applying the functor H^0F we obtain an exact long sequence

$$\dots \rightarrow H^{-1}F(E'') \rightarrow H^0F(E) \rightarrow H^0F(E') \rightarrow \dots$$

Assume that all terms of the exact triangle belong to the category \mathbf{C}_θ^{ss} . Then by assumption we know that $H^{-1}F(E'') = 0$. This shows that monomorphisms are transformed into monomorphisms. Hence H^0F is left exact on \mathbf{C}_θ^{ss} . Let $\pi : E \rightarrow \text{Im } \beta$ be the projection induced by the isomorphism $E/\text{Ker } \beta \simeq \text{Im } \beta$, and $j : \text{Im } \beta \rightarrow E$ be the natural embedding. Then $\beta = j \circ \pi$. Hence if $\beta(f) = 0$ then $j(f) = 0$. Applying the functor H^0F to the short exact sequence above, we obtain a long exact sequence of vector spaces

$$0 \rightarrow H^0F(\text{Ker } \beta) \rightarrow H^0F(E) \rightarrow H^0F(\text{Im } \beta) \rightarrow \dots$$

Since $\beta(f) = 0$, we see that the element f belongs to the image of the map $H^0F(\text{Ker } \beta) \rightarrow H^0F(E)$. ■

Let us see how compactness implies positivity. We fix a triangulated category, stability condition and a functor F satisfying the above conditions.

Conjecture 10.3.4 *Suppose that \mathcal{C} is ind-constructible and F is a constructible functor. Then the space of stable framed objects \mathcal{M}^{sfr} is compact and Hausdorff.*

Conjecture 10.3.5 *Suppose that in addition \mathcal{C} is a 3CY category. Then there is a formal manifold $\widehat{\mathcal{M}}^{sfr}$ and a formal function $W \in \mathcal{O}(\widehat{\mathcal{M}}^{sfr})$ such that:*

- a) \mathcal{M}^{sfr} is the set of critical points of W .
- b) For every $i \geq 0$ the cohomology group $H^i(\mathcal{M}^{sfr}, \phi_W)$ with the coefficients in the sheaf of vanishing cycles $\phi_W(\mathbf{Z}_{\mathcal{M}^{sfr}})$ carries a pure Hodge structure of weight 0 as well as the Lefschetz decomposition.

Assuming the Conjecture we arrive to the following:

Corollary 10.3.6 *The series $A^{sfr} := \sum_{\gamma \in \mathbf{Z}_{\geq 0}^I} [H^\bullet(\mathcal{M}^{sfr}, \phi_W)] \widehat{e}_\gamma$ enjoys the wall-crossing formulas (i.e. a mutation gives rise to a conjugation of A^{sfr})*

by the quantum dilogarithm). Applying Serre polynomial we obtain the series with coefficients which are characters of finite-dimensional SL_2 -representations.

In order to apply the above approach to quivers one needs sufficient conditions for the self-duality of the sheaf of vanishing cycles ϕ_W . One is given in the following conjecture.

Conjecture 10.3.7 *If (Q, W) is a quiver with generic potential then the for every dimensional vector $\gamma \in \mathbf{Z}_{\geq 0}^I$ the formal subscheme $\text{Crit}(W)$ of the scheme $\mathbf{M}_\gamma^{\text{nilp}}$ of nilpotent representations of Q of dimension γ is in fact a scheme of finite type.*

It seems plausible that in the framework of the last Conjecture the sheaf $\phi_W(\mathbf{Z}_{\mathbf{M}_\gamma})$ is Verdier self-dual. The latter implies purity of the mixed Hodge structure described in the above Corollary. In particular the WCFs mean that a mutation leads to a change of A^{sfr} by a conjugation of the quantum dilogarithm.

Self-duality of the sheaf of vanishing cycles is related to the following general fact.

Proposition 10.3.8 *Let X be a smooth complex algebraic variety, $f = (f_i)_{i \in I}$ is a finite collection of regular functions (thus $f : X \rightarrow \mathbf{C}^I$). Let us fix generic collection $(\lambda_i)_{i \in I}$ of complex numbers and define $g = \sum_{i \in I} \lambda_i f_i$. Then $\text{Crit}(g) \cap \widehat{f^{-1}(0)}$ is a union of connected components of the algebraic variety $\text{Crit}(g)$. Here $\widehat{f^{-1}(0)}$ denotes the formal neighborhood of the algebraic variety $f^{-1}(0)$.*

Notice that $\mathcal{O}(\mathbf{M}_\gamma)$ is an algebra which is finitely generated by traces of cyclically invariant paths in the quiver. We choose a set of generators $(f_i)_{i \in I}$. Then a generic potential W can be thought of as a function g from the above Proposition. It follows that the intersection of $\text{Crit}(g)$ with the formal neighborhood $\widehat{f^{-1}(0)}$ is open and closed at the same time, hence it is a union of connected components. Since those are connected components of $\text{Crit}(g)$, each of them is closed in \mathbf{M}_γ , and the restriction of ϕ_W to it is self-dual.

Assume the property that $\text{Crit}(W_\gamma) = \text{Tr}(W)|_{\mathbf{M}_\gamma}$ is a scheme. Let us fix the dimension γ and choose a stability condition $Z : \Gamma = \mathbf{Z}^I \rightarrow \mathbf{C}$ such that the standard basis $e_i = (0, \dots, 1, \dots)$ is mapped by Z to the upper-half plane.

Let us choose a generic ray l in the upper-half plane, so that $l = \mathbf{R}_{>0}Z(\gamma_0)$, where γ_0 is a primitive vector. Since the stability and the ray are generic, $Z^{-1}(l) \cap \Gamma = \mathbf{Z}_{>0}\gamma_0$. For each vertex $i \in I$ and $n \geq 1$ we define the moduli space $\mathcal{M}_{i,n}$ of stable framed objects (E, v) , where $Z(E) \in l$, $[E] = n\gamma_0$ and $v \in E_i$ is a non-zero vector (here $E = (E_i)_{i \in I}$). Stability of the framed object means that E is Z -semistable and moreover there is no submodule $E' \subset E$ having the same slope and such that $v \in E'_i$. It follows from results of King and Reineke that $\mathcal{M}_{i,n}$ is a smooth projective variety. It contains a closed subvariety $\mathcal{M}_{i,n}^{nilp}$ consisting of stable framed objects with nilpotent E . Then the potential W defines a function $W_{n\gamma_0}$ in the formal neighborhood of $\mathcal{M}_{i,n}^{nilp}$ which satisfies the same property as W_γ . Combining this with the above discussion we arrive to the following two conjectures.

Conjecture 10.3.9 *The critical cohomology $H_{crit}^\bullet(\mathcal{M}_{i,n}, W_{n\gamma_0})$ is a pure object of the category EMHS of exponential mixed Hodge structures.*

Conjecture 10.3.10 *The critical cohomology enjoys Lefschetz decomposition. In particular, its Serre polynomial is a sum of characters of finite-dimensional representations of SL_2 .*

Since even (resp odd) cohomology H_{crit}^i has even (resp odd) weight, then, assuming the above conjectures, we arrive to the following result.

Corollary 10.3.11 *Specialization of $S(H_{crit}^\bullet(\mathcal{M}_{i,n}, W_{n\gamma_0}))$ at $q^{1/2} = -1$ is a non-negative number equal to the dimension of the cohomology.*

Recall that for every ray l we have the corresponding factor A_l in the factorization of motivic DT-series $A^{Q,W}$. Let $A_l^{sfr}(\widehat{e}_{\gamma_0}) = 1 + \sum_{n \geq 1} [H_{crit}^\bullet(\mathcal{M}_{i,n}, W_{n\gamma_0})] \widehat{e}_{n\gamma_0}$. Then similarly to 0811.2435, Sect. 7 we obtain the following result.

Proposition 10.3.12 *The following formula holds*

$$A_l(\widehat{e}_{\gamma_0} q^{\widehat{\gamma}_0^i}) = A_l(\widehat{e}_{\gamma_0}) A_l^{sfr}(\widehat{e}_{\gamma_0}).$$

This Proposition immediatley implies

Corollary 10.3.13 *The Absence of Poles Conjecture from 0811.2435 holds.*

Finally we mention that the property of compactness can be approached from a categorical point of view, if we accept the following definition.

Definition 10.3.14 *We say that a triangulated A_∞ -category \mathcal{C} is proper if any triangulated A_∞ -functor $G : \mathbf{C} \rightarrow \text{Perf}(D^0)$ to the category of perfect modules over a punctured disc D^0 admits an extension to a functor $G : \mathbf{C} \rightarrow \text{Perf}(D)$ to a disc without the puncture.*

We expect that all the above conjectures hold in this setting.

11 Motivic DT-invariants, mutations and cluster transformations

11.1 Mutations of quivers with potentials

Mutations of quivers with potential keep the set of vertices unchanged, but change sets of arrows and the non-commutative polynomial (or series) representing the potential. In the case of trivial potential mutations generalize reflection functors of Gelfand-Ponomarev. Since mutations can create cycles in acyclic quivers, one can get a non-trivial potential starting with a quiver with the trivial potential. For that reason it is more convenient to present formula for the mutated potential as a sum of three summands (see below).

Let (Q, W) be a quiver with potential. We fix a vertex $i_0 \in I$ of Q which is loop-free (i.e. $a_{i_0 i_0} = 0$ for the incidence matrix $(a_{ij})_{i,j \in I}$). We will write the potential W as a finite \mathbf{k} -linear combination of cycles (in other words, cyclic paths) σ in Q

$$W = \sum_{\sigma} c_{\sigma} \sigma,$$

where \mathbf{k} is the ground field.

For any vertex $i \in I$ we have the corresponding cycle (i) of length 0 (the image of the projector corresponding to i). We define the (right) **mutation** (Q', W') of (Q, W) at the vertex i_0 in the following way:

- 1) The set of vertices of Q' is the same set I .
- 2) The new set of arrows and new matrix $(a'_{ij})_{i,j \in I}$ are defined such as follows:

$a'_{i_0 i_0} = 0$;
 $a'_{i_0 j} = a_{j i_0}$, $a'_{j i_0} = a_{i_0 j}$ for any $j \neq i_0$ (in terms of arrows: we reverse each arrow α which has head or tail at i_0 , i.e. replace α by a new arrow α^*);
 $a'_{j_1 j_2} = a_{j_1 j_2} + a_{j_1 i_0} a_{i_0 j_2}$ for $j_1, j_2 \neq i_0$ (in terms of arrows: for every pair of arrows $i_0 \xrightarrow{\beta} j_2$, $j_1 \xrightarrow{\alpha} i_0$, we create a new arrow $j_1 \xrightarrow{[\beta\alpha]} j_2$).

3) The mutated potential W' is defined as a sum of 3 terms:

$$W' = W_1 + W_2 + W_3,$$

where

$$W_1 = \sum_{j_1 \xrightarrow{\alpha} i_0 \xrightarrow{\beta} j_2} \beta^* \cdot [\beta\alpha] \cdot \alpha^*,$$

$$W_2 = \sum_{\sigma \neq (i_0)} c_\sigma \sigma^{mod},$$

$$W_3 = c_{(i_0)} \left(-(i_0) + \sum_{j \xrightarrow{\alpha} i_0} (j) \right) = c_{(i_0)} \left(-(i_0) + \sum_{j \in I} a_{ji_0}(j) \right).$$

Let us explain the notation in the formulas for $W_i, i = 1, 2, 3$.

The summand W_1 consists of cubic terms generated by cycles from i_0 to i_0 of the form $\beta^* \cdot [\alpha\beta] \cdot \alpha^*$, and it can be thought of as a ‘‘Lagrange multiplier’’.

The summand W_2 is obtained from W by modifying each cycle σ (except (i_0)) to a cycle σ^{mod} such as follows:

for each occurrence of i_0 in the cycle σ (there might be several of them) we replace the consecutive two-arrow product $(i_0 \xrightarrow{\beta} j') \cdot (j \xrightarrow{\alpha} i_0)$ of σ by an arrow $j \xrightarrow{[\beta\alpha]} j'$. New cycle is denoted by σ^{mod} , and it is taken with the same coefficient c_σ . In particular, if σ does not contain i_0 then $\sigma^{mod} = \sigma$.

This modification procedure is not applicable only to the cycle (i_0) of zero length.

The last summand W_3 can be thought of as a modification of the term in W corresponding to the exceptional cycle (i_0) . There is also a version of W_3 for the *left* mutation in which the same sum is taken over all arrows with the tail (and not the head) at i_0 .

11.2 Motivic DT-series and mutations

Let us choose a central charge $Z : \mathbf{Z}_{\geq 0}^I \rightarrow \overline{\mathbb{H}}_+$, where \mathbb{H}_+ is the (open) upper-half plane. It is completely determined by its values on the standard basis vectors, so we identify Z with a collection of complex numbers $(z_i)_{i \in I}$ belonging to \mathbb{H}_+ . We impose the condition that

$$\text{Arg}(z_{i_0}) > \text{Arg}(z_i), \quad \forall i \neq i_0 .$$

Denote by $\mathcal{H}^{(Q,W)}$ the Cohomological Hall algebra corresponding to (Q, W) and by $\mathcal{H}_V^{(Q,W)}$ the Cohomological Hall vector space corresponding to a subsector $V \subset \mathbb{H}_+$, with the apex at the origin. For $V = l$ (a ray in \mathbb{H}_+) this vector space is in fact a Cohomological Hall algebra, as we explained before. Let us introduce a ray $l_0 = \mathbf{R}_{>0}z_{i_0}$, and a sector $V_{0,+} = \{z \in \mathbb{H}_+ | \text{Arg}(z) < \text{Arg}(z_{i_0})\}$. The Factorization Formula implies

$$A^{(Q,W)} = A_{l_0}^{(Q,W)} A_{V_{0,+}}^{(Q,W)} .$$

We observe that $\mathcal{H}_{l_0}^{(Q,W)}$ corresponds to the category of representations of Q supported at the vertex i_0 . Since it is the same as the category of representations of the quiver A_1 with trivial potential, we can use the results of Section 2 and see that $\mathcal{H}_{l_0}^{(Q,W)}$ is a free exterior algebra, and

$$A_{l_0}^{(Q,W)} = (-\mathbf{e}_{i_0}; \mathbb{L})_\infty = (\mathbb{L}^{1/2}\widehat{\mathbf{e}}_{i_0}; \mathbb{L})_\infty .$$

Similarly, let us choose for the quiver Q' a central charge Z' such that for the corresponding collection of complex numbers $(z'_i)_{i \in I} \in \mathbb{H}_+$ the inequalities

$$\text{Arg}(z'_i) > \text{Arg}(z'_{i_0}), \quad \forall i \neq i_0$$

hold. Consider the ray $l'_0 := \mathbf{R}_{>0}z'_{i_0}$ and sector $V'_{0,-} := \{z \in \mathbb{H}_+ | \text{Arg}(z) > \text{Arg}(z'_{i_0})\}$.

Then we use again the Factorization Formula

$$A^{(Q',W')} = A_{V'_{0,-}}^{(Q',W')} A_{l'_0}^{(Q',W')}$$

and the equality

$$A_{l'_0}^{(Q',W')} = (\mathbb{L}^{1/2}\widehat{\mathbf{e}}'_{i_0}; \mathbb{L})_\infty .$$

Comparison of the generating functions for (Q, W) and (Q', W') is based on the following result.

Proposition 11.2.1 *Let $\gamma, \gamma' \in \mathbf{Z}^I$ are related as*

$$(\gamma')^i = \gamma^i \text{ for } i \neq i_0, \quad (\gamma')^{i_0} = \sum_j a_{i_0j} \gamma^j - \gamma^{i_0} .$$

Then there is an isomorphism of graded cohomology spaces

$$\mathcal{H}_{V_{0,+},\gamma}^{(Q,W),\bullet} \otimes \mathbb{T}^{\sum_j a_j i_0 \gamma^j (\gamma')^{i_0}} \simeq \mathcal{H}_{V'_{0,-},\gamma'}^{(Q',W'),\bullet}.$$

Corollary 11.2.2 *After the identification of quantum tori*

$$\widehat{\mathbf{e}}_\gamma \leftrightarrow \widehat{\mathbf{e}}'_{\gamma'}, \quad \mathbf{e}_\gamma \leftrightarrow \mathbb{L}^{\sum_j a_j i_0 \gamma^j (\gamma')^{i_0}} \mathbf{e}'_{\gamma'},$$

where γ and γ' are related as in the Proposition, the series $A_{V_{0,+}}^{(Q,W)}$ and $A_{V'_{0,-}}^{(Q',W')}$ coincide with each other.

11.3 Categorical meaning

The above result can be also explained categorically by lifting the mutations to the category of representations of Q . This generalization of Bernstein-Ponomarev reflection functors was introduced by people working on cluster algebras. Interpretation in terms of 3CY categories was suggested in Section 8 of our paper 0811.2435. There we explained how the motivic DT-series leads naturally to the (quantum) cluster transformations. More precisely, we prove that the equivalence classes of pairs (Q, W) (with formal series potential W) are in one-to-one correspondence with 3CY categories (as always in A_∞ sense) endowed with a collection of generators $E_i, i \in I$ (called cluster generators) such that $Ext^m(E_i, E_i)$ can be non-trivial for $0 \leq m \leq 3$ and $Ext^m(E_i, E_j)$ is non-trivial for $m = 1$ or $m = 2$ only. Special case: E_i are spherical generators (i.e. $Ext^\bullet(E_i, E_i) \simeq H^\bullet(S^3)$). Notice that we assume that there are no oriented 2-cycles in the quiver. Here I is the set of vertices of Q . The equivalence on the quivers side is a continuous change of non-commutative variables, while on the categorical side we consider A_∞ -equivalence which preserves the properties of cluster generators. Suppose that the potential starts with terms of degree at least 3. Then (assuming that everything is OK with vanishing cycles for formal series) we define a series $A_Q = 1 + \dots$ with values in the quantum torus. For the quiver with one vertex $A_Q = (q^{1/2} \widehat{e}_1; q^{-1})_\infty$. Suppose that the set of vertices $I = I_1 \sqcup I_2$ such that arrows can go only from $i_1 \in I_1$ to $i_2 \in I_2$, not in the opposite direction. Then we have the corresponding subquivers Q_1 and Q_2 . One can check that

$$A_Q = A_{Q_1} A_{Q_2}.$$

Mutation at the vertex i_0 can be interpreted categorically in terms of the cluster generators as a change of t -structure. Let us split $I = I_- \sqcup \{i_0\} \sqcup I_+$ so that one can have arrows from I_- to i_0 and from i_0 to I_+ , not in the opposite direction. Then we define new collection of cluster generators by the formulas

$$\begin{aligned} E'_i &= E_i, i \in I_-, \\ E'_{i_0} &= E_{i_0}[1], \\ E'_i &= \text{Cone}(E_{i_0} \otimes \text{Ext}^\bullet(E_{i_0}, E_i) \rightarrow E_i), i \in I_+. \end{aligned}$$

At the level of $K_0 \simeq \mathbf{Z}^I$ this categorical mutation become a reflection: if we denote $[E_i] = \gamma_i, [E'_i] = \gamma'_i$ then

$$\begin{aligned} \gamma'_i &= \gamma_i, i \in I_-, \\ \gamma'_{i_0} &= -\gamma_{i_0}, \\ \gamma'_i &= \gamma_i - \langle \gamma_{i_0}, \gamma_i \rangle \gamma_{i_0}, i \in I_+, \end{aligned}$$

where we use *skew-symmetric* Euler form on the K -theory. At the level of incidence matrix of the quiver, the mutation become cluster transformation of matrices. Then one can prove the following result.

Theorem 11.3.1 *Mutation applied to a quiver arising from cluster collection with generic potential is again a quiver corresponding to cluster collection and having generic potential.*

In this case we can apply mutations indefinitely. Furthermore there is natural mutation invariant orientation data (proved by Ben Davison, 1006.5475).

Remark 11.3.2 *Action of the mutation on W is a sort of non-commutative Legendre transform (which is more transparent if we think about variables as matrices and apply commutative Legendre transform with respect to the non-degenerate quadratic form W_2 in the series $W = W_2 + \dots$*

Comparison of the series A_Q gives us

$$(q^{1/2} \widehat{e}_{\gamma_{i_0}}; q^{-1})_\infty A_Q = A_{Q'} (q^{1/2} \widehat{e}_{-\gamma_{i_0}}; q^{-1})_\infty^{-1}$$

in the obvious notation. This should be understood as an equality of formal series in the variables \widehat{e}_γ . Furthermore, assume that Ad_{A_Q} is “quantum birational map” of the quantum torus. Then we can define $P_Q = \tau \circ Ad_{A_Q}$, where $\tau : \widehat{e}_\gamma \mapsto \widehat{e}_{-\gamma}$ is an involution.

Corollary 11.3.3 *The conjugacy class of P_Q in the group of automorphisms of the quantum torus does change under mutations:*

$$P_Q = CP_Q C^{-1},$$

where C is a quantum cluster transformation.

This result corresponds to the formulas from the previous subsection.

12 Collapsing Calabi-Yau manifolds, SYZ torus fibrations and Mirror Symmetry

12.1 Collapsing Calabi-Yau manifolds

Here we recall the approach to Mirror Symmetry suggested in our paper 0011041 and further developed in 0406564.

Suppose we have a holomorphic family $(X_t)_{0 < |t| \leq 1}$ of compact Calabi-Yau manifolds of fixed complex dimension n which has maximal degeneration as $t \rightarrow 0$. In other words it approaches the cusp in the moduli space \mathcal{M}_X of complex structures with maximal unipotent degeneration (the monodromy about $t = 0$ has maximal unipotent block of size n). Assume that the class of the Kähler form $[\omega_t^{1,1}] \in H^2(X_t, \mathbf{R})$ does not depend on t (for example, it belongs to $H^2(X_t, \mathbf{Z})$). Then we have a family of Calabi-Yau metrics g_t . One can argue that $\text{diam}(X_t, g_t) = C(\log \frac{1}{|t|})^{-1/2}(1 + o(1))$, $t \rightarrow 0$. We can rescale the metric without changing the Ricci curvature, so that the diameter of each X_t in the new metric g_t^{new} is equal to 1. We conjectured in 0011041 that there is a limit of the family of Riemannian manifolds (X_t, g_t^{new}) as $t \rightarrow 0$ which is the metric space (B, g_B) with the following properties:

- 1) $\dim_{\mathbf{R}} B = n = \frac{1}{2} \dim_{\mathbf{R}} X_t$.
- 2) g_B is a Riemannian metric with $\text{Ricci}(g_B) \geq 0$ outside of a singular subset B^{sing} which has codimension at least 2.
- 3) The Riemannian manifold $B^{\text{sm}} := B - B^{\text{sing}}$ carries an integral affine structure (\mathbf{Z} -affine structure for short).
- 4) There is a convex function h such that in affine coordinates one can (locally) write $g_B = \sum_{i,j} \frac{\partial^2 h}{\partial x_i \partial x_j} dx_i dx_j$.
- 5) $\det(g_B) = \text{const}$.

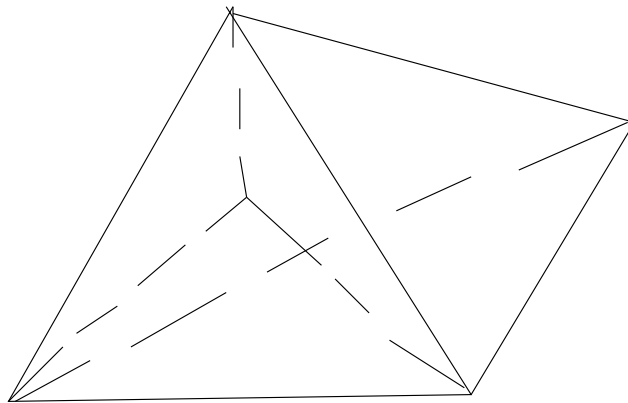
The non-negativeness of the Ricci curvature in 2) follows from Gromov's theory of collapse of Riemannian manifolds (one can use more advanced

notion of Ricci curvature for metric spaces developed by Sturm and Lott-Villani). Similarly Properties 4), 5) are related to the potential of the Kähler metric and Monge-Ampère equation respectively for collapsing manifolds X_t .

Affine structure in 3) is motivated by the SYZ conjecture which claims that as t is small then X_t should look as a (special) Lagrangian torus fibration (i.e. “real integrable system”).

In all known examples B is a topological manifold. If all X_t are simply-connected then B is in fact a topological sphere S^n . For example for the quintic 3-fold we have $B \simeq S^3$, and the singular subset B^{sing} is in fact a trivalent graph inside of this sphere.

FIGURE: TETRAHEDRON



The volume form for X_t ensures that B^{sm} is oriented. Hence one has the local action of $SL(n, \mathbf{Z}) \times \mathbf{R}^n$ on B^{sm} .

The above conjecture is different from the original SYZ conjecture. It can be deduced from known results in the case of maximal degeneration of abelian varieties. It was checked by Gross and Wilson in the non-trivial case of $K3$ surfaces. General case is still open.

12.2 Semiflat spaces

It is natural to ask whether one can reconstruct the family X_t from the limiting data 1)-5), at least as t is sufficiently small. This reconstruction can be useful from the point of view of mirror symmetry. Indeed, an appropriate

modification of SYZ conjecture suggested in 0011041 says that the maximally degenerate mirror dual family X_t^\vee collapses as $t \rightarrow 0$ to the same limiting data 1)-5) with the only exception that the \mathbf{Z} -affine structure is the dual one (with respect to the metric). Hence a solution to the reconstruction problem would give a way for constructing the mirror dual family.

As first approximation to the complex structure on X_t we consider the *semiflat space* X_t^{sf} which is the total space of the torus bundle over B^{sm} with the fiber over $b \in B^{sm}$ given by the torus $T_b/\varepsilon T_b^{\mathbf{Z}}$. Here T_b is the tangent space, $T_b^{\mathbf{Z}}$ is the lattice defined by the \mathbf{Z} -affine structure, and $\varepsilon = (\log(1/|t|))^{-1/2}$. We will ignore ε in what follows. Then X_t^{sf} carries the canonical complex structure given by the splitting of the tangent space into the “vertical” and “horizontal” subspaces. Then choosing affine coordinates x_1, \dots, x_n on B and “angle” coordinates $\theta_1, \dots, \theta_n$ on the torus fiber we define in the natural way the complex coordinates z_1, \dots, z_n on the total space such that $x_i = \operatorname{Re}(z_i)$, $1 \leq i \leq n$. Locally the torus fibration over B^{sm} is modelled by the “tropical map” $(\mathbf{C}^*)^n \rightarrow \mathbf{R}^n$, $(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$. The Kähler metric is given by $\partial\bar{\partial}\pi^*h$, where π is the projection to B^{sm} and h is the potential of the Riemannian metric (see Condition 4)). The metric is flat along the fibers, hence the name “semiflat” for the total space. It is non-compact Calabi-Yau with the holomorphic volume form locally given by the formula $\Omega^{n,0} = \bigwedge_{1 \leq i \leq n} \frac{dz_i}{z_i}$.

Notice that the space B can be considered as a manifold with singular Monge-Ampère metric. Singularities prevent us from using the real version of Yau’s theory. One needs its analog for Alexandrov spaces.

Affine structure gives us a representation $\pi_1(B^{sm}) \rightarrow SL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$. Equivalently, this representation can be thought of as a pair:

- a) Local system on B^{sm} .
- b) Cohomology class $\delta \in H^1(B^{sm}, T^{\mathbf{Z}} \otimes \mathbf{R})$.

The data b) can be interpreted as a torsor over $T^{\mathbf{Z}} \otimes \mathbf{R}$. Notice that we can also rotate torus fibers. This indicates that we need an additional data consisting of the torsor over a local system of tori $T^{\mathbf{Z}} \otimes i\mathbf{R}/2\pi\mathbf{Z}$. This torsor can be encoded in the cohomology class in $H^1(B^{sm}, T^{\mathbf{Z}} \otimes i\mathbf{R}/2\pi\mathbf{Z})$.

The above-mentioned conjecture about mirror dual family can be illustrated in the following local observation. Consider $\mathbf{R}^n \oplus (\mathbf{R}^n)^*$ (sum of the tangent and cotangent spaces at a point of B^{sm}). Let π_1 and π_2 be the natural projections on the direct summands. Then we have a Lagrangian submanifold L given by the graph of dh . Monge-Ampère equation implies that the pull-backs to L of the Euclidean volume forms on \mathbf{R}^n and $(\mathbf{R}^n)^*$

coincide.

An important question is about the behavior of the \mathbf{Z} -affine structure near B^{sing} . Simplest case of singularity (so-called “focus-focus singularity”) can be obtained by removing from \mathbf{R}^2 the sector bounded by rays $x = 0, y \geq 0$ and $x = y, x \geq 0$ and gluing the boundary rays by the transformation $(x, y) \mapsto (x + y, y)$. We get a topological manifold with \mathbf{Z} -affine structure which is singular at $(0, 0)$. Its monodromy about the origin is given by the Jordan unipotent block. This is also the monodromy of a generic elliptically fibered $K3$ surface about a singular fiber (A_1 -Kodaira singularity). A higher-dimensional version is given by the product of \mathbf{R}^{n-2} with the standard non-singular \mathbf{Z} -affine structure and the above singular one. This produces a codimension two singularity, which is described by a pair: primitive vector $v \in T_p^{\mathbf{Z}}$ and primitive covector $v^* \in (T_p^{\mathbf{Z}})^*$ such that $(v^*, v) = 0$. Then the monodromy is given by a transvection $u \mapsto u + (v^*, u)v$.

Let us summarize parameters for the \mathbf{C} -structure in the semiflat space X :

- 1) Local system of lattices $\Gamma \rightarrow B^{sm}$.
- 2) \mathbf{Z} -affine structure given by a closed 1-form $\alpha \in \Gamma(B^{sm}, \Gamma \otimes \Omega^{1,cl})$ such that at any point $b \in B^{sm}$ the map $v \mapsto \alpha(v)$ gives rise to an isomorphism $T_b B^{sm} \simeq \Gamma_b \otimes \mathbf{R}$.
- 3) Twisting class $\beta \in H^1(B^{sm}, i\mathbf{R}/2\pi\mathbf{Z})$. The latter can be thought of as “ B -field” or “imaginary part” of the Kähler form.

Notice that 1) and 2) come for free from a real integrable system. Recall that the latter is given by a smooth map $\pi : (Y, \omega_Y) \rightarrow B$ from $2n$ -dimensional symplectic manifold Y to an n -dimensional manifold B such that the generic fiber is a smooth Lagrangian torus. Then we set B^{sm} to be the union of non-degenerate fibers, $B^{sing} = B - B^{sm}$. Fibers of the local system Γ are given by $\Gamma_b = H^1(\pi^{-1}(b), \mathbf{Z})$. The closed form α is defined by the period map $\gamma \mapsto \int_{\gamma} \omega_Y$, where $\gamma \in \Gamma_b^{\vee} = H_1(\pi^{-1}(b), \mathbf{Z})$. We will assume for simplicity that the integer part of the monodromy of the affine structure belongs to the subgroup $SL(n, \mathbf{Z}) \subset GL(n, \mathbf{Z})$.

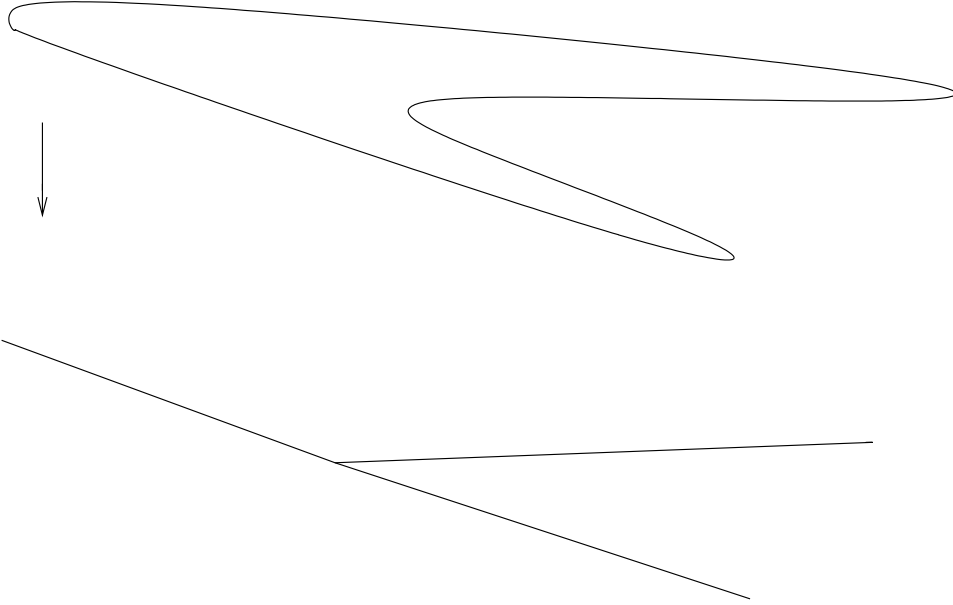
Classical mechanics gives huge number of examples of real integrable systems (the name comes of course from the Liouville integrability theorem), e.g. Euler tops. Many of them have B^{sing} of real codimension one. For example, if Y is a toric variety then fibers of the real integrable system are often fibers of the moment map of the symplectic torus action. They degenerate over the boundary of the corresponding convex polytope. On the other hand, collapsing Calabi-Yau manifolds give rise to real integrable systems

for which B^{sing} has real codimension at least two. As we will discuss in the next section, in this case one can “modify” a semiflat space and construct a “compactified” manifold over B . Moreover this modification is essentially unique since it is determined by the cohomology class $[\Omega_{X^{sf}}^{n,0}] \in H^n(X^{sf}, \mathbf{C})$.

12.3 Modification of semiflat structure

The idea of modification of the semiflat space X^{sf} come from Homological Mirror Symmetry (HMS). Recall that it claims an equivalence of A_∞ -categories $D^b(X) \simeq Fuk(X^\vee, \omega_{X^\vee})$ (more precisely one should add the B -field to the RHS). Then X can be thought of as a moduli of skyscraper sheaves in $D^b(X)$. If we assume SYZ picture then there is a singular torus fibration $\pi^\vee : X^\vee \rightarrow B$. Then the vertical tangent bundle is the degeneration of the complex structure. In order to have the right number of parameters we need to consider pairs (*Lagrangian fiber, $U(1)$ -local system on it*). These objects of the Fukaya category correspond to skyscraper sheaves and hence their moduli space is X^{sf} . This picture has to be modified because there are torus fibers which are not objects of the Fukaya category. The obstruction (called m_0) is related to the presence of pseudo-holomorphic discs with the boundary on the torus fibers. If we restore the parameter t then we see that in the limit $t \rightarrow 0$ those discs becomes trees on B with edges which are \mathbf{Z} -affine lines (in the dual affine structure). External vertices of the trees belong to B^{sing} (by the maximum principle).

FIGURE : DISCS AND THEIR PROJECTIONS WHICH ARE TREES

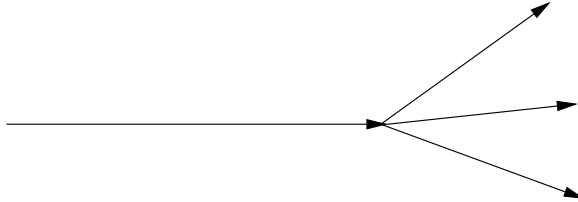


Therefore we have an open subset $U \subset X^{sf}$ corresponding to the “honest” objects of the Fukaya category as well as the trees on the base B we should be used in order to “modify” X^{sf} . Under some mild assumptions one can prove that the union of all trees is a countable union of hypersurfaces in B (“walls”).

In order to construct trees and walls we should start with the local picture near B^{sing} . We assume that around B^{sing} the local monodromy of the \mathbf{Z} -affine structure is given by a transvection $\gamma \mapsto \gamma + (\mu^*, \gamma)\mu$ where $\gamma, \mu \in \Gamma = T^{\mathbf{Z}}, \mu^* \in T^{\mathbf{Z}}$ and $(\mu^*, \mu) = 0$ (here (x^*, y) is the canonical pairing between the monodromy lattice and its dual). In other words, we have special direction (invariant line) in the dual affine structure which corresponds to μ^* . Furthermore we have a hyperplane orthogonal to μ^* . If two such hyperplanes meet each other we have two pairs $\mu_i^*, \mu_i, i = 1, 2$ which describe each hyperplane. Assume that $(\mu_i^*, \mu_j) \neq 0$ if $i \neq j$. Then we have a canonical splitting $T^{\mathbf{Z}} \otimes \mathbf{Q} \simeq \mathbf{Q}^2 \oplus \mathbf{Q}^{n-2}$, where \mathbf{Q}^2 is the RHS is spanned by μ_1^*, μ_2^* . Then everything is reduced to the two-dimensional situation. We have an internal vertex of the oriented tree with the incoming edges μ_1^*, μ_2^* and outgoing edges in any direction $n_1\mu_1^* + n_2\mu_2^*, n_1, n_2 \geq 0$. This comes from the corresponding picture for pseudo-holomorphic discs. An outgoing ray $n_1\mu_1^* + n_2\mu_2^*$ defines a hyperplane foliated by the rays which are parallel

to this one. Then we continue by induction. All possible intersection of edges of trees give a union of hyperplanes. Those are the walls.

FIGURE: TWO-DIMENSIONAL WCF PICTURE



Outside of the walls we have the semiflat space X^{sf} , which parametrizes Lagrangian tori with $U(1)$ -local systems. This semiflat structure has to be modified on the walls (there are no objects of the Fukaya category parametrized by points of walls). Change of coordinates when we cross the wall is defined by a pair μ^*, μ is given by an automorphism of X^{sf} which on the monomials corresponding to lattice points looks like this:

$$z^{\lambda^*} \mapsto F_{wall}(z^{\mu^*})^{(\lambda^*, \mu)} z^{\lambda^*}.$$

Here $F(x)$ is a series in one variable which is often meromorphic (or even rational). We must ensure that changes of coordinates corresponding to different walls are compatible. For example in the figure below the composition of all changes of coordinates for the half-plane $x > 0$ must coincide with the one for the half-plane $x < 0$.

This compatibility is the wall-crossing formula (WCF for short).

There are several questions about the suggested procedure of modification of X^{sf} . First, one has to prove that infinite products which appear in WCF converge. Second, since the inductive procedure of constructing the walls starts with B^{sing} we should check the compatibility there. By our assumption about the monodromy, everything reduces to the two-dimensional question. If we take the simplest focus-focus singularity with the monodromy $(x_1, x_2) \mapsto (x_1 + x_2, x_2)$ then on the complex coordinates we have a transformation $(z_1, z_2) \mapsto (z_1 z_2, z_2)$, where $x_i = \log|z_i|, i = 1, 2$. We have two walls which are formed by two opposite rays. We may assume they are positive and negative y -axes. Then the change of coordinates from the second to first

quadrant (crossing the wall $y > 0$) is given by the formula

$$(z_1, z_2) \mapsto (z_1(1 + z_1^{-1}), z_2),$$

while the change from the fourth to the third quadrant (crossing the wall $y < 0$) is given by the formula

$$(w_1, w_2) \mapsto (w_1(1 + w_2^{-1}), w_2),$$

where $w_i = z_i^{-1}$, $i = 1, 2$.

Then the compatibility of changes of coordinates ensures that if we go along the closed path about the origin then the composition of all automorphisms is the identity map. In our case indeed we have

$$(z_1, z_2) \mapsto \left(\frac{z_1}{1 + z_2}, z_2 \right) \mapsto \left(z_1(1 + z_2^{-1}) \frac{z_2}{1 + z_2}, z_2 \right) = (z_1, z_2).$$

Thus we obtain an algebraic surface $\mathbf{C}_{a,b}^2 - \{ab = 1\}$ which is symplectic with the symplectic form $\omega^{2,0} = \frac{da \wedge db}{ab-1}$. Here $z_2 = ab - 1$, and the symplectic form is in fact a holomorphic volume form which gives a Calabi-Yau structure.

Remark 12.3.1 *All the automorphisms preserve the volume form. Furthermore we can replace the sign and consider transformations which involve $1 - z_2$ instead of $1 + z_2$, etc. This corresponds to the possibility to “twist” the formulas for automorphisms by a cocycle with values in $\{\pm 1\}$.*

Functions F_{wall} should satisfy some equations imposed by the compatibility conditions. In the above example it is

$$F(z_2^{-1})z_2 \frac{1}{F(z_2)} = 1,$$

where F is understood as a formal series. If we assume that F is rational then we see that it has to be a polynomial of degree 1. This explains the above $F = 1 + x$. If the monodromy is conjugate to a unipotent 2×2 block with the above-diagonal entry $k \in \mathbf{Z}_{\geq 0}$, then F will be a polynomial of degree k .

Summarizing, we have modified X^{sf} to a complex manifold X (under the non-trivial assumption of convergency of all series). Moreover, since the automorphisms preserve the holomorphic volume form then $[\Omega_X^{n,0}] = [\Omega_{X^{sf}}^{n,0}]$.

This is a kind of Torelli theorem statement: the restriction of the holomorphic volume form on the semiflat part uniquely determines the complex structure on X . Notice that there is no Torelli theorem in non-compact case. Notice also that $\dim H^n(X^{sf}, \mathbf{C})$ coincides with the dimension of the moduli space of pairs $(X, \Omega_X^{n,0})$ (moduli space of “gauged” or “fat” Calabi-Yau manifolds). Several times mentioned convergency is not known in general. So far it can be achieved (again under some assumptions) only over a non-archimedean field (e.g. Novikov field). Notice also that the Monge-Ampère condition of the metric does not play any role in the above considerations. Hence we can choose any metric such that locally in affine coordinates $g_{ij} = \partial_i \partial_j h$, where h is a convex function.

12.4 Hyperkähler case

A special class of Calabi-Yau manifolds is formed by those which are hyperkähler. The latter means that we have a Riemannian manifold with the metric whose holonomy belongs to the group $Sp(n) \subset SU(2n) \subset SO(4n, \mathbf{R})$. Hyperkähler manifolds (HK-manifolds for short) are studied in few compact cases (e.g. the Hilbert scheme of K3 surfaces). The situation is substantially less understood in non-compact case. For example one does not know an analog of Torelli theorem. In HK case one has a family of complex structures $J_\zeta, \zeta \in \mathbf{CP}^1$. Moreover one has a holomorphic symplectic form $\omega^{2,0}$. Then the holomorphic volume form $\Omega_\zeta^{2n,0} = \wedge^n \omega_\zeta^{2,0}$. All that can be packaged in the concept of *twistor space*. Twistor space is given by a complex analytic map $\pi : X \rightarrow \mathbf{CP}^1$ such that the preimage $\pi^{-1}(\zeta)$ of a point ζ is a complex manifold (X, J_ζ) endowed with a holomorphic symplectic form $\omega_\zeta^{2,0}$ which has poles of order 1 at $\zeta = 0, \infty$ (i.e. there exist $\lim_{\zeta \rightarrow 0} \zeta \omega_\zeta^{2,0}$ and $\lim_{\zeta \rightarrow \infty} \zeta^{-1} \omega_\zeta^{2,0}$) and the limits define symplectic structures on the fibers of π at 0 and ∞ . In addition it is required that x is endowed with an antiholomorphic involution σ which is compatible with the involution $\zeta \mapsto -\frac{1}{\bar{\zeta}}$ on \mathbf{CP}^1 .

Equivalently one can formulate first part of the above structure by saying that X is a Poisson manifold, π is a Poisson map (with trivial Poisson structure on \mathbf{CP}^1), and fiber of π are symplectic leaves. Notice that the complex manifold X is never algebraic. Finally π is required to have many sections with normal bundle isomorphic to the direct sum $\bigoplus_{1 \leq i \leq 2n} \mathcal{O}(1)$, where $2n = \dim_{\mathbf{C}} X$ and moreover X should be a σ -invariant part of the space of all such sections.

Let us return to the story with collapse and assume that we have a collapsing family of HK manifolds. Then, assuming the conjecture discussed previously, we have a limiting metric space (B, g_B) which is endowed with a family of \mathbf{Z} -affine structures parametrized by $t \in \mathbf{R}$. This structure is described in terms of *complex* integrable systems. The latter is defined as a complex analytic map $\pi : (X, \omega_X^{2,0}) \rightarrow B$ such that the generic fiber of π is a complex torus (i.e. it is isomorphic to $\mathbf{C}^n/\mathbf{Z}^{2n}$). Let us assume that the fiber is a polarized abelian variety (but the base B can be non-algebraic). Then we also obtain a class in $H^2(\pi^{-1}(b), \mathbf{Z})$ of an ample bundle on the fiber $\pi^{-1}(b)$. Notice that “generic” means “outside of the real codimension 2 singular subset B^{sing} ”. Indeed, in complex case the fiber can degenerate on a complex codimension 1 subvariety $B^{sing} \subset B$. Then, as in the real case we have a local system Γ over $B^{sm} = B - B^{sing}$ with the fiber $\Gamma_b = H^1(\pi^{-1}(b), \mathbf{Z})$. Again, similarly to the real story we have (this time holomorphic) closed γ -valued 1-form α such that $(\alpha, \gamma) = \int_\gamma \omega^{2,0}$ for any local section γ of Γ . Polarization on the fibers is given by symplectic structure $\omega : \wedge^2 \Gamma \rightarrow \mathbf{Q}$ which is compatible with Kähler metric. It induces a canonical metric g_B on B^{sm} . If we fix (locally) a trivialization $\Gamma_b \simeq \mathbf{Z}^{2n}$ then $\alpha = (\alpha_1, \dots, \alpha_{2n})$ and g_B is Kähler with the Kähler form

$$\omega_B^{1,1} = \frac{1}{2\pi i} \sum_{i,j} \omega^{ij} \alpha_i \wedge \bar{\alpha}_j,$$

where the coefficients ω^{ij} are defined by the polarization. Writing locally $\alpha_i = dz_i$ for some holomorphic functions $z_i, 1 \leq i \leq 2n$ we obtain an embedding $B \rightarrow \mathbf{C}^n$ as a holomorphic Lagrangian submanifold of a symplectic *affine* space (this corresponds to what is called special geometry of B^{sm}). The family of real \mathbf{Z} affine structures on B^{sm} (it appears in the hypothetical picture of HK collapse) is naturally derived from a family of closed 1-forms $\alpha_\chi = \text{Re}(\chi\alpha)$ which in turn comes from a holomorphic family of symplectic structures $\chi\omega^{2,0}, \chi \in \mathbf{C}^*$.

Proposition 12.4.1 *For any of these \mathbf{Z} -affine structures the corresponding metric g_B is Monge-Ampère.*

We can take a “twist” from $H^1(B^{sm}, \Gamma \otimes i\mathbf{R}/2\pi\mathbf{Z})$ representing the corresponding torsor and obtain a semiflat family X_χ^{sf} as described previously. These semiflat complex manifolds are holomorphic symplectic with symplectic forms $\omega_{X_\chi}^{2,0}$. Then we can use the WCF as above and modify these semi-

flat spaces. As a result we obtain a family of complex symplectic manifolds $(X_\chi, \omega_{X_\chi}^{2,0})$. More precisely, this is true under some convergency conditions.

Definition 12.4.2 *Complex integrable system* $\pi : (Y, \omega_Y^{2,0}) \rightarrow B$ is called *exact* (or *Seiberg-Witten type*) if there is $Z \in \Gamma(B^{sm}, \mathcal{O}_B \otimes \Gamma)$ such that $dZ = \alpha$, where α is the closed 1-form obtained by integration of $\omega_Y^{2,0}$ over integer first homology of fibers of π .

The function Z is called *central charge*. In the case of exact integrable systems the manifold B^{sm} is locally embedded (via Z) as a Lagrangian submanifold into a *vector space* parallel to $\Gamma_b \otimes H^1(\pi^{-1}(b), \mathbf{C})$ (for a general complex integrable system it is embedded only in the affine space).

Example 12.4.3 (*Seiberg-Witten curve*)

Let $B = \mathbf{C}$ be a complex line endowed with a complex coordinate u . We denote by $X^0 = T^*(\mathbf{C} \setminus \{0\})$ the cotangent bundle to the punctured line. We endow it with the coordinates $(x, y), y \neq 0$ and the symplectic form

$$\omega^{2,0} = dx \wedge \frac{dy}{y} .$$

There is a projection $\pi^0 : X^0 \rightarrow B$ given by

$$\pi^0(x, y) = \frac{1}{2}(x^2 - y - \frac{c}{y}) ,$$

where c is a fixed constant. Fibers of π^0 are punctured elliptic curves

$$y + \frac{c}{y} = x^2 - 2u .$$

We denote by X the compactification of X^0 obtained by the compactifications of the fibers. We denote by $\pi : X \rightarrow B$ the corresponding projection. Then $Z_u \in H^1(\pi^{-1}(u), \mathbf{C})$ is represented by a meromorphic 1-form $\lambda_{SW} = \frac{x dy}{y}$ (Seiberg-Witten form). The form λ_{SW} has zero residues, hence it defines an element of $H^1(\pi^{-1}(u), \mathbf{C})$ for any $u \in B^{sm}$, where $B^{sm} = B \setminus \{b_-, b_+\}$ consists of points where the fiber of π is a non-degenerate elliptic curve.

Besides of this example there are plenty of others. E.g. all Hitchin systems with regular singularities are exact. Local Calabi-Yau 3-folds give rise to exact integrable systems.

We claim that with an exact complex integrable system and any $b \in B$ and $\gamma \in \Gamma_b$ one can associate an integer $\Omega_b(\gamma) \in \mathbf{Z}$ (DT-invariant). More precisely we can prove this under the assumption that near B^{sing} the \mathbf{Z} -affine structure looks as a product of the standard 2-dimensional focus-focus singularity and the complementary non-singular affine space. There is also some finiteness of trees assumption (see below). Here is the procedure (basically, it is the same as the above-discussed modification of the semiflat space).

1) If $b \in B^{sing}$ we set Γ_b to be the homology lattice for a nearby non-singular point and $\Omega_b(\gamma) = 1$ if γ is the primitive generator of the monodromy-invariant direction. For all other γ the DT-invariant is trivial. This formula is motivated by the study of 2-dimensional case.

2) For $b \in B^{sm}$ and fixed $\gamma \in \Gamma_b$ we consider all oriented trees T with external vertices at B^{sing} passing through b in the direction γ of the edge and such that:

a) all edges of T are integer lines with respect to the dual affine structure for some α_χ (recall that we have a Kähler metric on the base);

b) internal vertices of T belong to the walls $\{(b, \mu) | \frac{Z_b(\mu)}{\chi} \in \mathbf{R}_{>0}\}$.

In other words we consider union of all walls for the affine structures corresponding to all rescaled $\frac{\omega^{2,0}}{\chi}, \chi \in \mathbf{C}^*$. The dimension of such union is $n = \dim_{\mathbf{C}} B^{sm}$. Hence a generic point b belongs to such a wall. Each wall is “foliated” by the above trees. In order to compute $\Omega_b(\gamma)$ we move along each tree from B^{sing} toward to B , performing WCF at each internal point. If the number of trees passing through b in the direction γ is finite, there are finitely many wall-crossing formulas on the way, and hence we arrive to the finite number $\Omega_b(\gamma) \in \mathbf{Z}$. Finally we arrive to the function $\Omega : tot(\Gamma) \rightarrow \mathbf{Z}$ which is constructible and satisfies the wall-crossing formulas. We expect that it also satisfies the Support Property defined in the lecture on the stability data in graded Lie algebras.

Question 12.4.4 *Is there a Calabi-Yau threefold (or 3CY category) around?*

Finally we remark that one can use quantized symplectomorphism and obtain quantized complex symplectic manifold (not a quantized integrable system) depending on a parameter $q = 1 + \dots$. Because of the admissibility property one can take q to be a root of 1.

13 Twistor families and Stokes data

13.1 Complex integrable systems and families of complex symplectic manifolds

We discussed in the previous lectures that the collapsing HK manifolds give rise to complex integrable systems. Our procedure of construction the mirror dual was based on that complex integrable system only. Thus, having a complex integrable system $\pi : (Y, \omega^{2,0}) \rightarrow B$ we can define the mirror dual to Y in two steps: first by constructing a semiflat dual and then by modifying it by means of wall-crossing formulas. The data we are dealing with consist of:

- 1) Local system of lattices $\Gamma \rightarrow B^{sm}$.
- 2) Torsor over $\Gamma^\vee \otimes i\mathbf{R}/2\pi\mathbf{Z}$.
- 3) Holomorphic closed 1-form $\alpha \in \Omega^{1,cl}(B^{sm}, \Gamma \otimes \mathcal{O}_{B^{sm}})$.

If fibers of the complex integrable system are abelian varieties then the corresponding polarizations give rise to a Kähler metric on B^{sm} . Taking the moduli space of Lagrangian torus fibers endowed with $U(1)$ -local systems we get X^{sf} , a semiflat manifold.

The above construction can be upgraded so one can construct a non-compact twistor family. For that one rescales the holomorphic symplectic form $\omega^{2,0} \mapsto \frac{\omega^{2,0}}{\zeta}, \zeta \in \mathbf{C}^*$. Since the construction of the semiflat manifold can be performed for each ζ we obtain a family of semiflat manifolds over \mathbf{C}^* . Let $R > 0$ be a sufficiently large number. Then choosing locally a splitting of the tangent space to X^{sf} into horizontal and vertical we can introduce a torsor which corresponds to holomorphic local coordinates $z_i^{(\zeta)}, 1 \leq i \leq n$ such that

$$d \log z_i^{(\zeta)} = R(\zeta^{-1}\alpha_i + \zeta\bar{\alpha}_i) + \sqrt{-1}d\theta_i,$$

where α_i are the coordinates of α in the chosen trivialization, and θ_i are flat coordinates on the vertical tori. Then we have a family of complex manifolds (X^{sf}, J_ζ) which are symplectic as long as fibers of π are abelian varieties. This family of complex structures has limits as $\zeta \rightarrow 0, \infty$. The total space of the family also carries a complex structure (just add $\frac{d\zeta}{\zeta}$ to the above 1-forms). Moreover it carries an antiholomorphic involution which lifts $\zeta \mapsto -\frac{1}{\zeta}$ on \mathbf{CP}^1 . The fibers at $\zeta = 0, \infty$ are dual integrable systems, i.e. their non-degenerate fibers are dual abelian varieties with based point, i.e. $Pic_0(\pi^{-1}(b))$. Hence it is a twistor space.

Remark 13.1.1 *In the last lecture we used a different symbol χ for rescaling the symplectic structure. In fact $Re(d\log z_i^{(\zeta)})$ gives the same affine structure, but the imaginary part corresponds to some twist of X_χ^{sf} . The parameter R^1 corresponds to the non-archimedean parameter t in the Novikov field. This means that unless we can prove the convergence of the series, for each ζ we construct a symplectic manifold over a non-archimedean field $\mathbf{C}((\frac{1}{R}))$.*

Next step is the modification of the semiflat twistor family. But now the wall-crossing formulas will depend on ζ because the walls will depend on the parameter. Indeed, the walls are defined as sets of pairs $\{(\zeta, \widehat{u}) \in \mathbf{C}^* \times X^{sf}\}$ such that for $u = \pi(\widehat{u}) \in B$ and some $\gamma \in \Gamma_u$ one has

$$Re(\zeta^{-1}Z_u(\gamma) + \zeta\overline{Z_u(\gamma)}).$$

Notice that $|\zeta^{-1} + \overline{\zeta}| \geq 2$, hence $Arg(\zeta^{-1} + \overline{\zeta}) = -Arg(\zeta)$. Thus we can rewrite the definition of the wall in the form

$$Re(\zeta^{-1} + \overline{\zeta})Z_u(\gamma) = 0.$$

For fixed $u \in B$ the walls become rays in the ζ -plane. The modification procedure goes exactly as before, but now the wall-crossing formulas are applied to new collection of walls. In the end we have a twistor space, which is a family $X_\zeta, \zeta \in \mathbf{CP}^1$ of complex manifolds (here we assume the convergency of the modifying automorphisms). Outside of the union of walls we have $X_\zeta = X_\zeta^{sf}$ (i.e. there are no modifications). This picture leads an explicit formulas for the modification of natural coordinates on X^{sf} . Furthermore, Gaiotto-Moore-Neitzke wrote explicit integral equations for the modified coordinates (which they called Darboux coordinates). It goes as follows. First, in the local picture, for any $\gamma = (\gamma^i)_{1 \leq i \leq 2n} \in \Gamma \simeq \mathbf{Z}^{2n}$ we have holomorphic semiflat coordinates

$$z_\gamma^{sf} = exp \left(R(\zeta^{-1}z(\gamma) + \zeta\overline{Z(\gamma)} + \sqrt{-1} \sum_i \gamma^i \theta_i) \right).$$

We would like to define a holomorphic σ -invariant section $s(\zeta)$ of the map $X \rightarrow \mathbf{CP}^1$. Outside of the walls we can identify X with X^{sf} , hence $s(\zeta)$ is a holomorphic map to X^{sf} . But when we cross a wall, the section gets changed by a transformation, so we can write

$$z_\gamma^{sf}(s(\zeta)) = z_\gamma^{sf}(\zeta)(1 + \dots) := z_\gamma^{sf}(\zeta)G_\gamma(\zeta).$$

Gaiotto-Moore-Neitzke suggested the following form for the correction:

$$G_\gamma(\zeta) = \exp\left(\frac{-1}{4\pi i} \sum_\mu \Omega_u(\mu) \langle \gamma, \mu \rangle \int_{t \in \mathbf{R}_{<0} Z_u(\mu)} \frac{dt}{t} \frac{t + \zeta}{t - \zeta} \log(1 - z_\gamma^{sf}(s(\zeta)))\right).$$

They also proved the following result.

Proposition 13.1.2 *The modified holomorphic coordinates $z_\gamma = z_\gamma^{sf} G_\gamma$ satisfy WCFs.*

The modified coordinates also enjoy the following properties:

- a) $z_\gamma z_\mu = z_{\gamma+\mu}$.
- b) $\overline{z_\gamma}(\zeta) = z_{-\gamma}(-1/\overline{\zeta})$.

The integral equation for G_γ is a special case of the so-called Thermodynamical Bethe Ansatz (TBA) equation.

Question 13.1.3 *How to quantize these formulas?*

Under some assumptions on the DT-invariants $\Omega_u(\gamma)$ and for sufficiently large R one has a unique solution of TBA equations derived by Gaiotto-Moore-Neitzke. The answer is given in terms of iterated Bessel function. Alternatively one can work over the non-archimedean field $\mathbf{C}((1/R))$.

13.2 Irregular singularities and Stokes phenomenon

Let $K = \mathbf{C}\{z\}[z^{-1}]$ be the field of germs of meromorphic functions at $z = 0$. It can be considered as a subfield of the field of Laurent series $\mathbf{C}((z))$. Both fields are differential fields with respect to the action of the linear map $\frac{d}{dz}$, so one can speak about D -modules over K . More pedantically

Definition 13.2.1 *A D -module over K is given by a vector space $M \simeq K^n$ endowed with a connection $\nabla := \nabla_{\partial/\partial z} : M \rightarrow M$.*

After a choice of isomorphism $M \simeq K^n$ we can write $\nabla_{\partial/\partial z} = \frac{d}{dz} + z^{-N} \sum_{i \geq 0} A_i z^i$, where $A_i \in \text{Mat}(n, \mathbf{C})$, and the series is convergent in a neighborhood of $z = 0$. There is a gauge group action on the space of connections, which on $A(z) := z^{-N} \sum_{i \geq 0} A_i z^i$ takes the form

$$A(z) \mapsto gA(z)g^{-1} + g^{-1} \frac{dg}{dz}, g \in GL(n, K).$$

The following problem arises:

Classify D -modules up to gauge equivalence.

This problem was solved by Deligne and Malgrange. It consists of two steps:

- a) Formal classification.
- b) Non-formal classification.

13.2.1 Formal classification

In this case we will speak about *formal D -modules*, meaning D -modules over the field $\mathbf{C}((z))$. The idea of formal classification goes back to Turritin. Then there are two cases.

1) *Regular singularities.*

In this case ∇ is gauge equivalent to the connection with $N = 1$, i.e. the connection matrix has the pole of order 1 at $z = 0$. Let us write the connection as $\nabla = \frac{d}{dz} + z^{-1}A_0$. The constant matrix A_0 is not an invariant of the gauge equivalence class, but the monodromy $T = \exp(2\pi i A_0)$ is the only invariant.

2) *Irregular singularities*

Definition 13.2.2 *Polar Puiseux polynomial (PPP for short) is a finite sum $P(z) = \sum_{\lambda \in \mathbf{Q}_{<0}} c_\lambda z^\lambda$, $c_\lambda \in \mathbf{C}$, $z \in \mathbf{R}_{>0}$ considered up to the following equivalence: $P(z)$ is equivalent to $P(e^{2\pi i m} z)$, where $m \in \mathbf{Z}$, $i = \sqrt{-1}$.*

We will denote the space of all PPPs by \mathcal{P} .

For a fixed $P(z) \in \mathcal{P}$ we can choose the minimal $N \geq 1$ $c_\lambda \neq 0$ and such that for all summands with $c_\lambda \neq 0$ we have $\lambda \in \frac{1}{N}\mathbf{Z}$. Then there are $\mathbf{Z}/N\mathbf{Z}$ expressions equivalent to $P(z)$.

A formal D -module M can be split over $\mathbf{C}((z))$ into a finite sum

$$M \simeq \bigoplus_{P_\alpha \in \mathcal{P}} M_\alpha,$$

where

$$M_\alpha \simeq e^{P_\alpha(z)} \mathbf{C}((z)) \otimes_{\mathbf{C}((z))} N_\alpha.$$

Here N_α is the D -module with regular singularities (which is by Riemann-Hilbert correspondence is the same as $\mathbf{C}[T, T^{-1}]$ -module).

For each $P_\alpha \in \mathcal{P}$ we have a canonical N -fold covering of the circle S^1 (circle of directions at $z = 0$). Each formal D -module M_α is then determined by a local system on this covering.

13.2.2 Non-formal classification

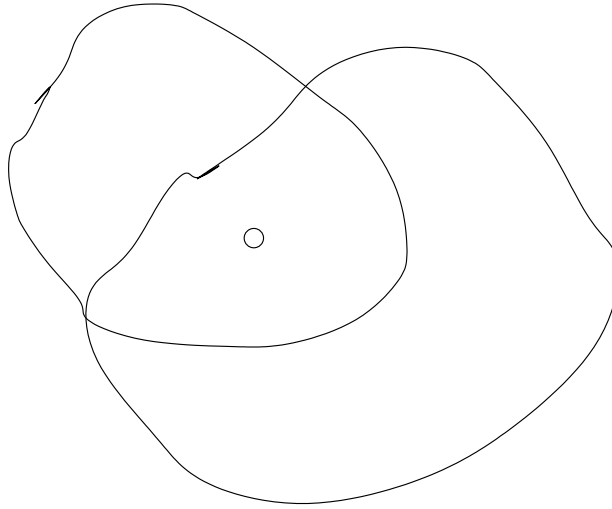
Let us extend scalars from K to $\mathbf{C}((z))$. Then a D -module M gives rise to a formal D -module $M_{form} \simeq M \otimes_K \mathbf{C}((z)) \simeq \bigoplus_{P_\alpha \in \mathcal{P}} M_\alpha$, as above. We may assume that all summands M_α are non-trivial.

This decomposition gives rise to a collection of smooth curves in \mathbf{C} around $z = 0$ such that connected components of this collection are in one-to-one correspondence with the set of PPP polynomials P_α in the sum. The curves are “graphs of multivalued functions”. Each function depends on a “very small” parameter $\varepsilon > 0$. Namely, we define the curve as the map $S^1 \rightarrow \mathbf{C}$ given by

$$r_\alpha(\varphi) = |\exp(P_\alpha(\varepsilon e^{i\varphi}))|.$$

Intersection point corresponds to the condition that two branches have the same growth in this direction. This is another way to encode the usual story with Stokes rays.

FIGURE (CURVES ABOUT ORIGIN AND STOKES SECTORS)



In each Stokes sector all branches of e^{P_α} are totally ordered: $e^{P_\alpha} > e^{P_\beta}$ if $|e^{P_\alpha}| > |e^{P_\beta}|$ for all z which are sufficiently close to zero and belong to this sector. On the boundary of the sector two solutions grow with the same speed. These are Stokes rays. Thus a D -module over K defines a local

system $E \rightarrow S^1$ such that the fiber E_φ for φ which is not a direction of a Stokes ray is filtered. Terms of the filtration correspond to e^{P_α} (i.e. solutions grow slower than e^{P_α} in the direction φ). The filtration changes when we cross the Stokes line. Let $E_{\leq 0}^L \subset E_{\leq 1}^L \subset \dots$ is the filtration for angles slightly smaller than φ and $E_{\leq 0}^R \subset E_{\leq 1}^R \subset \dots$ is a similar filtration for angles slightly bigger than φ (“left” and “right” filtrations). Then an “elementary” change of filtration at the i -th terms is described by the conditions:

- a) $E_{\leq j}^L = E_{\leq j}^R, j \neq i.$
- b) $E_{\leq i+1}^L/E_{\leq i-1}^L \simeq E_{\leq i+1}^R/E_{\leq i-1}^R$, and the subspaces $E_{\leq i}^L/E_{\leq i-1}^L$ and $E_{\leq i}^R/E_{\leq i-1}^R$ are complementary to each other.

The change of filtration can be encoded as a permutation of the branches of solutions. It is a subgroup of the symmetric group.

Theorem 13.2.3 *D-modules over K with irregular singularities are in one-to-one correspondence with the above-described Stokes data. In particular Stokes data form an abelian category, which is the category of finite-dimensional modules over a smooth algebra.*

Taking associated graded spaces for the filtrations we recover the formal classification of D -modules.

Example 13.2.4 *Let $P_\alpha = \frac{c_\alpha}{z}, c_\alpha \in \mathbf{C}, 1 \leq \alpha \leq N$. Then the collection of curves about $z = 0$ is just a collection of intersecting N circles. Corresponding category of irregular D -modules is equivalent to the category of representations of the quiver with N vertices and such that there is one loop at each vertex i which acts as invertible operator and for any two different vertices i, j there are two arrows $i \rightarrow j$ and $j \rightarrow i$. Then objects of the category are $\mathbf{C}[T_i, T_i^{-1}]$ -modules, $i \in I$. One can see that the braid group acts by automorphisms of this algebra. The action is non-trivial. It comes from the interpretation of the quiver as a configuration of points on the complex line. Then moving the points we get an action of the fundamental group on the above quiver algebra.*

In a similar way, the collection $P_\alpha = \frac{c_\alpha}{z^{1/2}}, c_\alpha \in \mathbf{C}, 1 \leq \alpha \leq N$ gives a category of finite-dimensional modules over $\mathbf{C}\langle x^{\pm 1}, y \rangle$. Here is an explanation of this. Because of two branches of the square root, the curve about the origin will have one self-intersection. We assign a vector space V_1 to the internal loop and a vector space V_2 to the outer loop. At the intersection

point the filtration $V_1 \subset V_1 \oplus V_2$ splits and the complement filtration comes from the filtration $V_2 \subset V_1 \oplus V_2$. Therefore $V_1 \simeq V_2$. Hence the corresponding quiver has one invertible arrow x which encodes this isomorphism as well as an arrow y which encodes an arbitrary map $V_2 \rightarrow V_2$.

We expect that all categories of Stokes data arising in this way have cohomological dimension ≤ 1 .

Similar geometry appears in a different topic related to Cecotti-Vafa theory of solitons in 2-dimensions. In that case we have a holomorphic function $f : X \rightarrow \mathbf{C}$. Assume that all its critical points are isolated, Morse, and all critical values are different. Then for any two distinct critical values c_i, c_j we consider gradient lines (in some metric) of the function $Re(\frac{f}{c_i - c_j})$. They are projected into straight segments joining c_i and c_j . Counting the number of segments is the same as a counting of the number of vanishing special Lagrangian spheres. These numbers n_{ij} are related to Stokes data. Simplest wall-crossing formula reads as $c \mapsto c + ab$ and corresponds to the crossing the wall where the critical values a, b, c belong to the same string line. Equivalently, the numbers n_{ij} define stability data on the Lie algebra $gl(n, \mathbf{C})$, where n is the number of critical values of f , and the above wall-crossing formula is just the wall-crossing formula from Section 3.5.

14 Application to Hitchin systems

14.1 Reminder on complex integrable systems of Seiberg-Witten type

Recall that a *complex integrable system* is a holomorphic map $\pi : X \rightarrow B$ where $(X, \omega_X^{2,0})$ is a holomorphic symplectic manifold, $\dim X = 2 \dim B$, and the generic fiber of π is a Lagrangian submanifold, which is a polarized abelian variety. We assume (in order to simplify the exposition) that the polarization is principal. The fibration π is non-singular outside of a closed subvariety $B^{sing} \subset B$ of codimension at least one. It follows that on the open subset $B^{sm} := B \setminus B^{sing}$ we have a local system $\mathbf{\Gamma}$ of symplectic lattices with the fiber over $b \in B^{sm}$ equal to $\Gamma_b := H_1(X_b, \mathbf{Z})$, $X_b = \pi^{-1}(b)$ (the symplectic structure on Γ_b is given by the polarization).

Furthermore, the set B^{sm} is locally (near each point $b \in B^{sm}$) embedded as a holomorphic Lagrangian subvariety into an affine symplectic space parallel to $H_1(X_b, \mathbf{C})$. Namely, let us choose a symplectic basis $\gamma_i \in$

$\Gamma_b, 1 \leq i \leq 2n$. Then we have a collection of holomorphic closed 1-forms $\alpha_i = \int_{\gamma_i} \omega_X^{2,0}, 1 \leq i \leq 2n$ in a neighborhood of b . There exists (well-defined locally up to an additive constant) holomorphic functions $z_i, 1 \leq i \leq 2n$ such that $\alpha_i = dz_i, 1 \leq i \leq 2n$. They define an embedding of a neighborhood of b into \mathbf{C}^{2n} as a Lagrangian submanifold. The collection of 1-forms α_i gives rise to an element $\delta \in H^1(B^{sm}, \mathbf{\Gamma}^\vee \otimes \mathbf{C})$. We assume that $\delta = 0$. This assumption is equivalent to an existence of a section $Z \in \Gamma(B^{sm}, \mathbf{\Gamma}^\vee \otimes \mathcal{O}_{B^{sm}})$ such that $\alpha_i = dZ(\gamma_i), 1 \leq i \leq 2n$.

Definition 14.1.1 We call Z the central charge of the integrable system.

In this case we will speak about *complex integrable systems with central charge* (or *Seiberg-Witten integrable systems*, see example below).

For such an integrable system, for every point $b \in B^{sm}$ we have a symplectic lattice Γ_b endowed with an additive map $Z_b : \Gamma_b \rightarrow \mathbf{C}$. It gives rise to a continuous family of stability data on graded Lie algebras \mathfrak{g}_{Γ_b} with central charges Z_b .

14.2 Hitchin systems

Given a smooth compact complex curve C we consider the moduli stack $Bun_C(n, 0)$ of rank n degree zero topologically trivial holomorphic vector bundles. Then $T^*Bun_C(n, 0) \simeq M_{Higgs}$, where M_{Higgs} is the moduli stack of Higgs bundles, i.e. pairs $(E, \varphi \in \Gamma(C, \Omega_C^1 \otimes End(E)))$, where $E \rightarrow C$ is the holomorphic vector bundle satisfying the above conditions. Section φ can be thought of as a coordinate in the cotangent direction. Equivalently one can consider coherent sheaves E' on T^*C such that $Supp(E')$ is a finite covering of C and $E = \pi_* E'$. Here $\pi: T^*C \rightarrow C$ is the natural projection. Typically, E' is a line bundle over the spectral curve $\Sigma \subset T^*C$. There is a natural map from M_{Higgs} to the space B of spectral curves. Hitchin proved the following result.

Theorem 14.2.1 This map defines a complex integrable system.

Now we can pick up a special torsor over $Hom(\Gamma, \mathbf{C}^*)$. It is defined by the characters $R^{cl} \rightarrow S^1$ of the quasi-classical limit of the quantum torus, i.e. R^{cl} is generated by $e_\gamma^{cl}, \gamma \in \Gamma$ subject to the relations

$$e_{\gamma_1}^{cl} e_{\gamma_2}^{cl} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} e_{\gamma_1 + \gamma_2}^{cl}.$$

Conjecture 14.2.2 *Applying the general machinery discussed in the previous lectures, with the above-mentioned choice of the torsor, we obtain a family $X_\zeta, \zeta \in \mathbf{C}^*$ of complex symplectic manifolds such that each manifold is diffeomorphic to the smooth part of $\text{Rep}(\pi_1(C), GL(n, \mathbf{C}))/GL(n, \mathbf{C})$. Hence it is a finite scheme over \mathbf{Z} .*

Moreover, we claim that there is a domain $U_\zeta \subset X_\zeta$ which can be identified with a domain in $\text{Hom}(\Gamma, \mathbf{C}^*)$ and such that the above-discussed construction of X_ζ identifies the corresponding domains in $\text{Hom}(\Gamma, \mathbf{C}^*)$ via cluster transformations. In the case of $SL(2, \mathbf{C})$ Hitchin systems this was studied in detail by Gaiotto-Moore-Neitzke. The data consist of a smooth projective curve C with finitely many marked points $x_i, 1 \leq i \leq n$ and PPP P_i attached to every marked point. Then the space of Higgs bundles is the total space of $SL(2, \mathbf{C})$ local systems with prescribed irregular behavior at the marked points. The base of the corresponding complex integrable system can be identified with the space of quadratic differentials with prescribed singular behavior at the marked points. Let us consider for simplicity the case of poles of first order. A quadratic differential defines a flat metric on $C - \{x_{i_1 < i_2 \leq n}\}$. Generically a trajectory joins singular points. This gives rise to a triangulation of C with vertices at the marked points $x_i, 1 \leq i \leq n$. Corresponding coordinates on the moduli space of local systems are Penner's (a.k.a Fock and Goncharov) coordinates.

Change of a triangulation gives rise to a cluster transformation of these coordinates. They come from a general theory discussed in previous lectures. Namely, to a triangulated surface one can associate a quiver with potential. Hence one has a 3CY category arising from the dimer model. This interpretation gives us a way (at least theoretically) to compute all DT-invariants $\Omega(\gamma)$.

Proposition 14.2.3 *The “categorical” invariants $\Omega(\gamma)$ coincide with those computed via gradient trees and wall-crossing formulas applied to the Hitchin complex integrable system.*

Thus we have a family X_ζ of affine schemes over \mathbf{Z} . Since all fibers can be identified (it is a local system) we can take a function $f \in \mathcal{O}(X_\zeta)$ and study its behavior as $\zeta \rightarrow 0$ with $\text{Arg}(\zeta) := \varphi$ being fixed. An example of such function is given by the trace of the monodromy of the corresponding “zeta-connection” $\zeta \frac{d}{dz} + A(z)$. Along each ray with the fixed φ the function grow as $e^{c/\zeta}$. Therefore we have a filtration of $\mathcal{O}(X_\zeta)$ by the order of

exponential growth. This filtration is defined over \mathbf{Z} . It can be thought of as an infinite-dimensional analog of the Stokes filtration arising in the non-commutative Hodge structures. This filtration appears in the resurgence theory of Voros. The resurgence properties of the monodromy were studied by Carlos Simpson in the 90's. The theory of exponential Hodge structures and exponential integrals provide gives a new point of view on this phenomenon. In particular, the wall-crossing formulas which appeared in the work of Dillinger-Delabaere-Pham on WKB asymptotics of the complex Shrödinger operator with polynomial potential is a special case of the wall-crossing formulas discussed above.

15 Black holes moduli spaces and attractor flows

15.1 Local Calabi-Yau manifolds vs compact ones

Complex integrable systems with polarization correspond to non-compact Calabi-Yau threefolds. For example, there is a well-known paper by Diaconescu-Donagi and Pantev that all $A-D-E$ Hitchin systems can be realized as families of intermediate Jacobians of some explicitly constructed non-compact CY threefolds. It is also known after the work of Donagi and Markman that in the case when a Calabi-Yau threefold X is compact, its intermediate Jacobian $J(X)$ is a compact non-algebraic torus (recall that $J(X)$ is a double coset with respect to $H^3(X, \mathbf{Z})$ on one side and $F^2 H^3 := H^{2,1} \oplus H^{3,0}$ on the other side. In the compact case the metric on $J(X)$ is hyperbolic. Thus one can say that complex integrable systems appear in the limit of hyperbolic geometry.

Let \mathcal{M}_X be the moduli space of complex structures on X , and \mathcal{L}_X be the moduli space of “gauged” Calabi-Yau manifolds, i.e. the moduli space of pairs $(\tau, \Omega_\tau^{3,0})$, where $\tau \in \mathcal{M}_X$ and $\Omega_\tau^{3,0}$ is a holomorphic 3-form. Since the latter is defined up to a non-zero complex factor, we have a projection $\mathcal{L}_X \rightarrow \mathcal{M}_X$ with fibers \mathbf{C}^* . Let $\widetilde{\mathcal{M}}_X$ be the universal cover of \mathcal{M}_X and $\widetilde{\mathcal{L}}_X$ be the pullback of the above \mathbf{C}^* -bundle. We have the lattice $\Gamma = H^3(X, \mathbf{Z}) \simeq \mathbf{Z}^{2b+2}$ which does not depend on a complex structure. This gives rise to maps

$$\widetilde{\mathcal{L}}_X \rightarrow \Gamma \otimes \mathbf{C} \rightarrow \Gamma \otimes \mathbf{R},$$

where the last map is the projection Re by the real part of the holomorphic volume form. The middle vector space is complex symplectic (it is $H^3(X, \mathbf{C})$ endowed with Poincaré pairing). The first map defines a local embedding of $\tilde{\mathcal{L}}_X$ as a complex Lagrangian cone. The composition map is a local homeomorphism. Pulling back the standard \mathbf{Z} -affine structure from $\Gamma \otimes \mathbf{R} \simeq H^3(X, \mathbf{R})$ we endow $\tilde{\mathcal{L}}_X$ with the induced \mathbf{Z} -affine structure. Dividing by the fundamental group $\pi_1(\mathcal{M}_X)$ we obtain a space, which is believed can be decomposed into a finitely many polyhedral cones.

15.2 Moduli spaces of supersymmetric black holes

Our space-times is Minkowski $\mathbf{R}^{3,1}$, while the back-ground for the supersymmetric string theory is $\mathbf{R}^{3,1} \times X$, where X is a Calabi-Yau 3-fold. More precisely, instead of X one should compactify by a superconformal field theory with the central charge $c = 3$ (the value of central charge corresponds to the dimension of X). Moduly space of such SCFTs is a disjoint union $\sqcup_X \mathcal{M}_X \times \mathcal{M}_{X^\vee}$, where X^\vee is the mirror dual Calabi-Yau manifold. Since the compactifying Calabi-Yau X can depend on a point of the space-time $\mathbf{R}^{3,1}$, this physical picture leads to a map $\mathbf{R}^{3,1} \rightarrow \mathcal{M}_X \times \mathcal{M}_{X^\vee}$. The map must satisfy some differential equations. Around 2000 F. Denef suggested a class of time-independent solutions to these equations. They come from certain maps $\phi : \mathbf{R}^3 - \{x_1, \dots, x_n\} \rightarrow \tilde{\mathcal{L}}_X$. Such a map defines a metric, which is singular at the points $x_i, 1 \leq i \leq n$ (positions of black holes). The condition on ϕ is the following one: $\Re \circ \phi$ is a harmonic map for a Euclidean metric on \mathbf{R}^3 . Here Re is the previously discussed projection to $\Gamma \otimes \mathbf{R} = H^3(X, \mathbf{R})$. The latter is a symplectic vector space, but when we are talking about harmonic maps we endow it with any flat metric. Denef suggested the following ansatz:

$$\phi(x) = \sum_{1 \leq i \leq n} \frac{\gamma_i}{|x - x_i|} + z_\infty.$$

Here $\gamma_i \in \Gamma$ are fixed integer vectors (charges of black holes) and $z_\infty \in \Gamma \otimes \mathbf{R}$ is a fixed vector (conditions at infinity). There are some compatibility conditions:

$$\left\langle \sum_i \gamma_i, z_\infty \right\rangle = 0,$$

and for any $1 \leq i \leq n$ we have

$$\lim_{x \rightarrow x_i} \langle \gamma_i, \phi(x) \rangle = 0.$$

Here $\langle \bullet, \bullet \rangle$ is the integer symplectic structure on Γ . One can easily see that we have $3n$ unknowns $x_i \in \mathbf{R}^3$ subject to $(n-1)$ independent equations. Modding out parallel translations we see that the moduli space of black holes $BHM(\{\gamma_i\}, z_\infty)$ has dimension $2n-2$. Furthermore, Jan de Boer with collaborators found that $BHM(\{\gamma_i\}, z_\infty)$ carries a closed 2-form

$$\omega = \frac{1}{2} \sum_{i \neq j} \langle \gamma_i, \gamma_j \rangle dvol,$$

where $dvol$ is the volume form (area) of the two-dimensional sphere swept by $(x_i - x_j)/|x_i - x_j|$. The volume form is normalized in such a way that the total area of the sphere is 1.

Since ω is closed it defines the cohomology class $[\omega] \in H^2(BHM, \mathbf{Z})$.

Conjecture 15.2.1 *The form ω is non-degenerate and hence defines a symplectic structure on the moduli space $BHM(\{\gamma_i\}, z_\infty)$.*

Notice that the space BHM is not compact and the points x_i cannot be very far apart for fixed γ_i and z_∞ . (ETO PRAVDA???) Instead of maps to the universal covering $\tilde{\mathcal{L}}_X$ of the Lagrangian cone of moduli of gauged CY manifolds, we consider maps $\mathbf{R}^3 - \{x_i\} \rightarrow \Gamma_{\mathbf{R}} = \Gamma \otimes \mathbf{R}$. The problem now can be formulated without any reference to black holes, etc. Namely, we have a skew-symmetric integer matrix (α_{ij}) (that was $\alpha_{ij} = \langle \gamma_i, \gamma_j \rangle$) and a collection of real numbers v_i that was $-\langle \gamma_i, v_\infty \rangle$. For any $1 \leq i \leq n$ we have the equation

$$\sum_{j \neq i} \frac{\alpha_{ij}}{|x_i - x_j|} = v_i.$$

The necessary condition for this system of equations to have a solution is $\sum_{1 \leq i \leq n} v_i = 0$.

We denote by $\mathcal{N}_n := \mathcal{N}((\alpha_{ij}), (v_i), n)$ the moduli space of solutions (i.e. the space of solutions with the natural topology modulo translations $x_i \mapsto x_i + c$). Clearly $\dim \mathcal{N}_n = 2n-2$. In the space of vectors $v = (v_i)$ for which the moduli space \mathcal{N} is nonempty, we consider a collection of walls W_S , which is singled out for any proper subset S of $\{1, \dots, n\}$ by the condition $\sum_{i \in S} v_i = 0$. Complements of the walls define a decomposition of \mathbf{R}^{n-1} into the union of chambers which was not studied by mathematicians (but it was considered by physicists which studied analytic continuation of Feynman amplitudes in QFT). In particular the combinatorics of the chambers is not known (e.g. the number of connected components).

There are several sources of non-compactness of \mathcal{N}_n . In particular, points can go to infinity within the wall. Another possibility is to collide a point x_i with x_j (in fact only a collision of at least three points together creates a singularity). Then the boundary $\partial\mathcal{N}_n$ is the union of strata of the type $\mathcal{N}_k \times D_S$, where $k < n$, $|S| = n - k$ and D_S consists of configuration of points (x_i) such that

$$\sum_{j \in S} \frac{\alpha_{ij}}{|x_i - x_j|} = 0$$

modulo the natural action of group $\mathbf{R}_{>0}^\times \times \mathbf{R}^3$ (rescalings and shifts).

One can show that the symplectic form ω is $S0(3)$ -invariant, hence it descends on the quotient of \mathcal{N}_n (even including the boundary $\partial\mathcal{N}_n$).

Proposition 15.2.2 *The volume $Vol_n := \int_{\mathcal{N}_n} \omega^{n-1}$ is locally constant outside the walls. When crossing the wall W_S we have the following wall-crossing formula*

$$\Delta Vol_n = Vol_k \cdot Vol_{n-k},$$

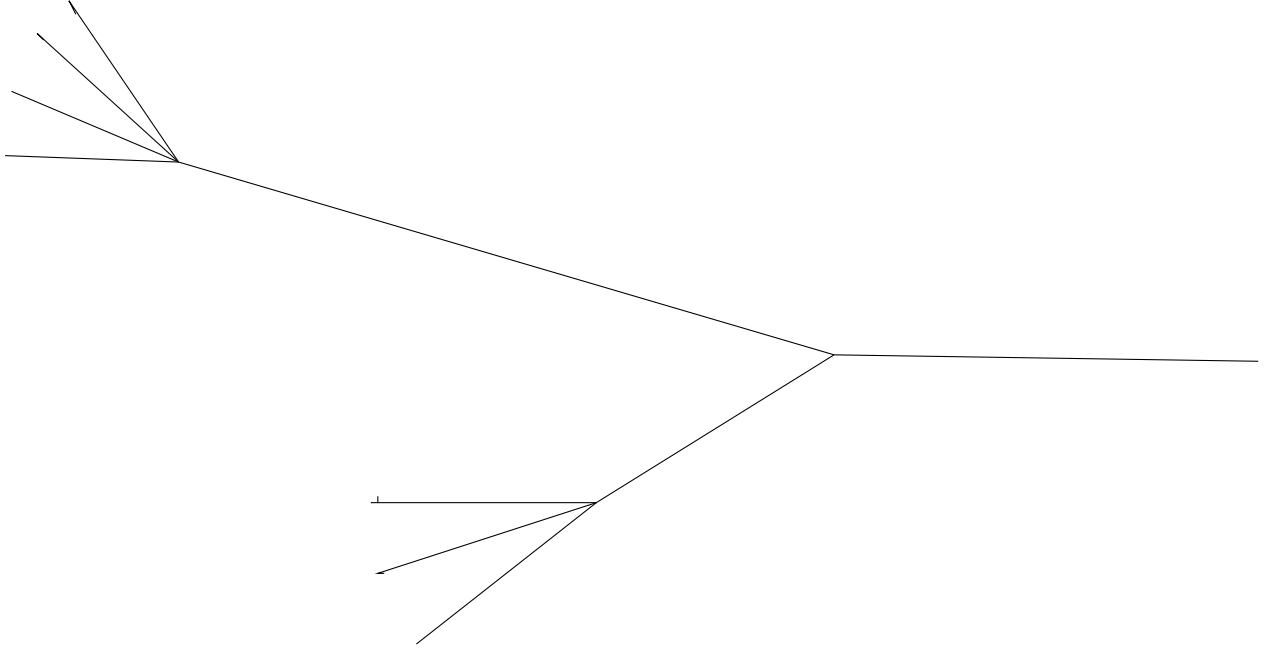
where $k = |S|$.

In fact volume is just the first term in the quasi-classical formula for the index. It was conjectured by Manschot-Pioline-Sen that spaces \mathcal{N}_n carry Spin structure. Then one can consider the Dirac operator on the line bundle which the tensor product of the Spin line bundle and the pre-quantum line bundle for the symplectic form ω . Then MPS conjectured (and checked in examples) that the generating function of the index satisfy the wall-crossing formulas which coincide with the wall-crossing for 3CY categories discussed in these lectures.

Question 15.2.3 *What is the relationship of the MPS formulas with COHA?*

Let us return to the relation of the moduli space of black holes with the maps $\phi : \mathbf{R}^3 - \{x_i\} \rightarrow \mathcal{M}_X$ discussed above. In the simplest case $n = 1$ the image of ϕ looks as an interval between z_∞ and γ_1 . For $n \geq 1$ we have a finite tree with the root at z_∞ and external legs having fixed directions $\gamma_i, 1 \leq i \leq n$. Here we lift the trees to the universal covering $\tilde{\mathcal{L}}_X$ and use the affine structure lifted from $H^3(X, \mathbf{R}) = \Gamma_{\mathbf{R}}$. The lifted trees are not compact, and their external legs “go to infinity”.

FIGURE: ROOTED TREE WITH MARKED ROOT AND MARKED EXTERNAL LEGS



An edge e of the tree has integer direction $\gamma_e \in \Gamma$, and these directions satisfy the balancing conditions at each internal vertex v :

$$\sum_{e \rightarrow v} \gamma_e = \sum_{v \rightarrow e} \gamma_e,$$

where $v \rightarrow e$ (resp. $e \rightarrow v$) denotes incoming to v (resp. outgoing from v) edges of the tree. Furthermore, the condition $\langle v, \gamma_e \rangle = 0$ is satisfied for all incoming and outgoing to v edges e .

Conjecture 15.2.4 *For any $z_\infty \in \Gamma_{\mathbf{R}}$ and $\gamma \in \Gamma$ such that $\langle z_\infty, \gamma \rangle = 0$ there exist finitely many rooted trees in $\Gamma_{\mathbf{R}}$ with root at z_∞ and the root edge equal to γ which have integer edges and satisfy the balancing conditions at the internal vertices.*

One reason for the Conjecture is the existence of the function $vol : \tilde{\mathcal{L}}_X \rightarrow \mathbf{R}_{>0}$ such that $vol(X, \Omega_X^{3,0}) = \sqrt{|\Omega^{3,0} \wedge \overline{\Omega^{3,0}}|}$. Then one can show that along integer edges of the tree the function vol is concave: $vol'' < 0$.

It follows from the finiteness conjecture that with any $p \in \mathcal{L}_X$ and $\gamma \in \Gamma$ we can associate a ‘‘BPS invariant’’ $\Omega_p(\gamma) \in \mathbf{Z}$, and the collection of these invariants satisfy our wall-crossing formulas. Since \mathcal{L}_X carry \mathbf{Z} -affine structure,

we can speak about integer points (i.e. those which are projected to $\gamma \in \Gamma_{\mathbf{R}}$. We call them *attractor points*. Besides of attractor points we have so-called *conifold points*, which are boundary points where the affine structure becomes singular. Each attractor point can be thought of as a “direction to infinity” in \mathcal{L}_X . Then one has the following result (up to some minor things to check).

Conjecture 15.2.5 *Solutions of the wall-crossing formulas are in one-to-one correspondence with \mathbf{Z} -valued functions at the generalized attractor points.*

Indeed, we we can reconstruct $\Omega_p(\gamma)$ for any point p by an inductive procedure along the above-described tropical trees passing through p in the direction γ . At each internal vertex we use our wall-crossing formulas, while at the external vertices (which are generalized attractor points) we have the given BPS invariants. Since the number of tropical trees pssing through p is finite by the above Conjecture, we hav a finite procedure which computes $\Omega_p(\gamma)$.

The final object which can be reconstructed from the collection $\{\Omega_p(\gamma)\}$ satisfying WCFs is a complex symplectic manifold M . The numbers $\Omega_p(\gamma)$ are encoded in the geometry of this manifold. Probably M is defined over \mathbf{Z} (and this would explain why $\Omega_p(\gamma) \in \mathbf{Z}$. Phycisists expect that the is even a bigger space, which is a quaternion-Kähler manifold (i.e. a Riemannian manifold with the holonomy $Sp(1)Sp(n)$). The metric satisfies the Einstein equation but the Ricci curvature is negative. There is a twistor space description of QK manifolds. The twistor space is a contact complex manifold of dimension $2b + 3$. In order to upgrade M (which is a sort of “divisor at infinity for the QK manifold) one needs and extra parameter. In physics it corresponds to a prsence of $M5$ -brane, and leads to a prediction for the function $\Omega_p : \Gamma \times \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Z}$. We hope that the previously discussed quantum DT-invariants give rise to the needed extra parameter. At least at the formal level one can connect contact deformations with the quantum ones.

16 Wall-crossing formulas for the Hall algebra of lattices

This is an example of the wall-crossing formulas in the framework of Arakelov geometry. Let us start with motivations. Consider the projective line \mathbf{P}^1

over the finite field \mathbb{F}_q . Every coherent sheaf on \mathbf{P}^1 is a direct sum of a vector bundle and a torsion sheaf. Thus the K_0 group of $\text{Coh}(\mathbf{P}^1)$ is \mathbf{Z}^2 and the map $cl : K_0 \rightarrow \mathbf{Z}^2$ is $E \mapsto (rk E, deg E)$. The central charge is $Z : \mathbf{Z}^2 \rightarrow \mathbf{C}$, $Z(r, d) = -d + ir$. In particular, torsion sheaves are mapped to $\mathbf{R}_{<0}$.

Now we are going to use some analogies well-known from Arakelov geometry. More generally, the analog of \mathbf{P}^1 is $\text{Spec}(\mathbf{Z}) \cup \{\infty\}$, torsion sheaves correspond to finite abelian groups, vector bundles correspond to finite rank lattices endowed with hermitian forms. Then if E is such a lattice that the analog of degree is $deg(E) = \log(vol(E \otimes \mathbf{R}/E))$. Now degree is a real number. The analog of the moduli space of vector bundles of a given rank n and degree zero will be the double coset \mathcal{Z}_n of $SL(n, \mathbf{R})$ over $SO(n, \mathbf{R})$ and $SL(n, \mathbf{Z})$, i.e. $SO(n, \mathbf{R}) \backslash SL(n, \mathbf{R}) / SL(n, \mathbf{Z})$. Let $(E, (\bullet, \bullet))$ be a metrized lattice (i.e. the finite rank lattice $E \simeq \mathbf{Z}^n$ endowed with hermitian form (\bullet, \bullet)).

Definition 16.0.6 We call it semistable if for any non-trivial metrized sublattice $F \subset E$ we have

$$\frac{deg(F)}{rk(F)} \leq \frac{deg(E)}{rk(E)}.$$

We call the metrized lattice stable if under the same assumptions the inequality is strict.

Recall that we have already discussed the factorization formula, which claims that $\sum(\text{all objects}) = \prod_{\text{slopes}} \sum(\text{semistables with fixed slope})$. There is a similar formula in the framework of metrized lattices.

Let us introduce variables $v_n = vol(Z_n)/2 = \frac{vol(SL(n, \mathbf{R})/SL(n, \mathbf{Z}))}{vol(O(n))}$. It can be computed explicitly as a function of $n \geq 1$: $v_n = \frac{\zeta(2) \dots \zeta(n)}{vol(S^0) vol(S^1) \dots vol(S^{n-1})}$, where S^d is the unit sphere in \mathbf{R}^d and hence $vol(S^d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and $\Gamma(x)$ is the Gamma-function.

Let $u_n, n \geq 1$ denotes a similar volume of the subspace of semistable lattices. One can compute these numbers for small n :

$$\begin{aligned} v_1 &= u_1 = 1/2 \\ v_2 &= \frac{\zeta(2)}{4\pi}, u_2 = \frac{\zeta(2) - \pi/2}{4\pi}. \end{aligned}$$

Of course one has $0 < u_n \leq v_n$. Numerical tests show that for large n the numbers u_n and v_n are very close.

Using the analogy between metrized lattices and vector bundles on \mathbf{P}^1 we will call the former *vector bundles on $\overline{Spec}(\mathbf{Z}) := Spec(\mathbf{Z}) \cup \{\infty\}$* . We will use similar terminology for semistable bundles. One can show that every vector bundle over $\overline{Spec}(\mathbf{Z})$ admits a Harder-Narasimhan filtration. To be more precise, one has to define the category of vector bundles, i.e. to define morphisms. All that can be defined as well as the notion of short exact sequence (notice that the category of vector bundles is not abelian).

The collection of variables (u_n) and (v_n) are related to each other:

$$v_n = \sum_{k \geq 1} \sum_{d_1, \dots, d_k \geq 1, \sum_i d_i = n} u_{d_1} \dots u_{d_k} c_{d_1 \dots d_k},$$

where

$$c_{d_1 \dots d_k} = \int \dots \int_{-\infty < t_1 < \dots < t_k < +\infty, \sum_i d_i t_i = 0} \exp\left(-\sum_{i < j} t_i t_j d_i d_j\right) \prod_i d_i \prod_i dt_i \delta\left(\sum_i d_i t_i\right).$$

From this formula one computes

$$c_{d_1 \dots d_k} = \frac{\prod_i d_i}{\sum_i d_i} \prod_{1 \leq i \leq k} \frac{1}{d_1 + \dots + d_k} \prod_{j \geq 2} \frac{1}{d_j + \dots + d_k}.$$

These formulas show that conversely variables u_n can be reconstructed from v_n . These formulas can be naturally interpreted in terms of the quantum torus. Namely, let us consider a semigroup \mathcal{M}_+ which is the union of horizontal rays $r_n = \{(x, y) \in \mathbf{R}^2 | x \geq 0, y = n\}, n = 1, 2, 3, \dots$. Then we define the algebra $C_0^\infty(\mathcal{M}_+)$ of compactly supported smooth functions endowed with the convolution product

$$(f_1 * f_2)(z) = \int_{z_1 + z_2 = z} f_1(z_1) f_2(z_2) \exp(\langle z_1, z_2 \rangle) d\mu(z_1),$$

where $\langle z_1, z_2 \rangle = -x_2 y_1 + x_1 y_2$ and $d\mu(z)$ is some measure invariant with respect to the shifts $x \mapsto x + a, a \geq 0$. Let us set $v_d = I(x + id), u_d = J(x + id)$. Then

$$I = J + J * J + J * J * J + \dots$$

Here we use the order convolution product, i.e.

$$(f_1 * f_2 * \dots * f_k)(z) = \int_{z_1 + \dots + z_k = z, \text{Arg}(z_1) > \text{Arg}(z_2) > \dots} \prod_i f_i(z_i) \exp\left(\sum_{i < j} \langle z_i, z_j \rangle\right) d\mu.$$

For $z = x + id$ we define $J_{\leq 0}(z)$ as $J(x + id)$ if $x < 0$ and zero otherwise. Similarly we define $I_{\leq 0}$. Then

$$I_{\leq 0} = J_{\leq 0} + J_{\leq 0} * J_{\leq 0} + J_{\leq 0} * J_{\leq 0} * J_{\leq 0} + \dots$$

One can show that $I_{\leq 0}(x + id)$ is non-zero only for $x \leq 0$, where it can be written as $I_{\leq 0}(x + id) = \sum_{0 \leq j \leq d-1} a_{jd} \exp(jx)$. Using these coefficients we can define

$$\Phi(x, y) = \sum_{jd} a_{jd} x^j y^{d-j} \in \mathbf{R}[[x, y]].$$

Let us also define

$$I(t) = \sum_d v_d t^d, J(t) = \sum_d t^d.$$

Then the function Φ satisfies the following partial differential equation:

$$x\partial/\partial x(x\partial/\partial x + y\partial/\partial y)\Phi = (-x\partial/\partial x J(x))\Phi,$$

with the conditions

$$\Phi(x, x) = J(x), \Phi(0, y) = I(y).$$

Series $I(t)$ and $J(t)$ diverge, similarly to the series which appear in WCF. Reason for that is the appearance of $\text{vol}(O(d))$ in the denominators. They can be regularized with the help of Barnes G -function. The latter informally is the infinite product

$$G(z + 1) = \prod_{1 \leq n < \infty} (z + n)^n.$$

More precisely

$$G(z + 1) = (2\pi)^{z/2} e^{-\frac{z(z+1)+cz^2}{2}} \prod_{1 \leq n < \infty} \left(1 + \frac{z}{n}\right)^n e^{-\frac{z+cz^2}{n}}.$$

Then $G(n + 1) = 0!1!\dots(n - 1)!$. We can introduced also complex and real versions of the function G :

$$G_{\mathbf{C}}(z) = G(z + 1) \exp\left(-\frac{\log 2\pi}{2} z^2 - \frac{\log 2\pi}{2} z\right),$$

$$G_{\mathbf{R}}(z) = G(z/2+1)G\left(\frac{z-1}{2}+1\right) \exp\left(-\frac{\log \pi}{4} z^2 - \frac{\log \pi}{4} z - \frac{3}{2}\zeta'(-1) - \frac{\log 2}{24} + \frac{\log \pi}{4}\right).$$

Then

$$G_{\mathbf{C}}(n) = \frac{1}{\text{vol}(U(n))}, G_{\mathbf{R}}(n) = \frac{1}{\text{vol}(O(n))}.$$

We define

$$V(s) = \frac{G_{\mathbf{R}}(s)}{\zeta(s+1)\zeta(s+2)\dots}.$$

Then $V(d) = v_d, d = 1, 2, \dots$. Hence $V(s)$ is an analytic continuation of v_d and for large d v_d behaves as $\exp(\frac{1}{4}d^2 \log d)$. Let us define $F(s) = \sum_{d \geq 1} V(d)t^d$. This is the same as the contour integral $\int \frac{V(s)t^s ds}{\tan(\pi s)}$. The contour encircles the positive x -axis. This series is not convergent but it is an entire function in $\log t$. Also $V(s)V(-s)$ is periodic with period 1. All the above can be formulated over any number field.

Remark 16.0.7 *Riemann Hypothesis implies that the regularized series $\sum_d v_d t^d$ behaves as $o(1/\sqrt{t})$.*

The question is: how to write down the wall-crossing formulas.

17 Some open problems and speculations

17.1 Chern-Simons theory with complex gauge group and new invariant of $3d$ manifolds

Probably one can associate a (critical) COHA with any 3CY category endowed with orientation data (the latter notion was first introduced in 0811.2435) whose objects form an ind-Artin stack. The potential in this case is a (partially) formal function. Any compact oriented 3-dimensional C^∞ manifold X gives an example of such a category. Namely, let us consider the triangulated category $D_{\text{constr}}^b(X)$ of complexes of sheaves with locally constant cohomology. This category has a t -structure with the heart equivalent to the category of finite-dimensional complex representations of the fundamental group $\pi_1(X, x_0), x_0 \in X$. For a given $n \geq 0$ the stack $\text{Rep}_n(X)$ of representations of dimension n is an Artin stack of finite type over \mathbf{C} . Locally (in analytic topology) we can represent $\text{Rep}_n(X)$ as the set of critical points of the (real) Chern-Simons functional:

$$CS_{\mathbf{R}}(A) = \int_X \text{Tr} \left(\frac{dA \cdot A}{2} + \frac{A^3}{3} \right),$$

where $A \in \Omega^1(X) \otimes \text{Mat}(n, \mathbf{C})$, modulo action of the gauge group. It looks plausible that the corresponding 3CY category admits orientation data. Therefore we obtain a topological invariant of X given by the motivic DT-series in one variable. For $X = S^3$ the invariant coincides with the motivic DT-series for the quiver $A_1 = Q_0$ endowed with the trivial potential (essentially it is the quantum dilogarithm). For $X = (S^1)^3$ it is given by the quantum MacMahon function discussed above in the example of the quiver Q_3 . One can also compute the invariant e.g. for $X = S^1 \times S^2$, but in general the answer is not known. Another interesting problem is to include knots and links in this picture. Then one would expect a theory of “motivic knot invariants” with values in an appropriate category of motives.

17.2 Holomorphic Chern-Simons and manifolds with G_2 -holonomy

Similarly to the above we consider holomorphic Chern-Simons functional for $\bar{\partial}$ -connections on C^∞ complex vector bundles on a compact complex 3CY manifold endowed with a non-zero holomorphic 3-form. In this case *holomorphic Chern-Simons functional* is defined modulo the abelian subgroup of periods of the holomorphic 3-form. Also for both C^∞ and holomorphic Chern-Simons functionals one can try to define a *rapid decay* version of COHA. Then one takes into account Stokes data (it the same as gluing data in the work of Katzarkov-Kontsevich-Pantev on non-commutative Hodge theory). It is achieved by counting gradient lines of the real part of $\exp(i\phi)CS_{\mathbf{C}}$ for various $\phi \in \mathbf{R}/2\pi\mathbf{Z}$ which connect different critical points of the holomorphic Chern-Simons functional $CS_{\mathbf{C}}$. The story is similar to the case of complexified real Chern-Simons functional discussed in the previous subsection. All that goes beyond the formalism of 3CY categories, as the gradient lines are trajectories in the space of non-flat connection in the case of a real oriented 3-dimensional manifold, or non-holomorphic $\bar{\partial}$ -connections in the complex case. Geometrically gradient lines correspond to self-dual non-unitary connections on the 4-dimensional Riemannian manifold $X \times \mathbf{R}$ in the case of real Chern-Simons, or on the $Spin(7)$ -manifold $X_{\mathbf{C}} \times \mathbf{R}$ in the case of complex Chern-Simons, with appropriate boundary conditions at infinity. In the real Chern-Simons case it was recently studied by Witten. The resulting structure should be thought of as an exponential mixed Hodge structure of infinite rank.

Let us discuss some details and explain why holomorphic Chern-Simons can be related to the $2d/4d$ wall-crossing formulas recently studied by Gaiotto-Moore-Neitzke.

Let X be a complex projective Calabi-Yau threefold. We assume that the cohomology class $[\omega^{1,1}]$ of its Kähler form is “sufficiently large”, so the Fukaya category $\mathcal{F}(X, \omega)$ is well-defined (here $\omega = \text{Im}(\omega^{1,1})$ is the corresponding symplectic form). We will denote the derived category of coherent sheaves by $D^b(X)$. We will assume that the rational Hodge structure on $H^3(X, \mathbf{Q})$ is indecomposable. It is equivalent to the following assumption: the image of $K_0(D^b(X))$ in the Deligne homology $H_D^\bullet(X)$ is a countable subgroup. We will explain this condition later.

Let us fix a holomorphic volume form $\Omega^{3,0} \in H^3(X, \mathbf{C})$. It defines a homomorphism $Z : \Gamma := H^3(X, \mathbf{Z}) \rightarrow \mathbf{C}, \gamma \mapsto \int_\gamma \Omega^{3,0}$. It should be thought of as a central charge for Bridgeland stability condition on $\mathcal{F}(X, \omega)$. The definition of the stability condition is still conjectural for various reasons, e.g. because we do not know whether every cohomology class $\gamma \in \Gamma$ is represented by a special Lagrangian submanifold (i.e. by a stable object for the stability condition). We are going to ignore some technical difficulties and assume that our theory of Donaldson-Thomas invariants can be applied both to $\mathcal{F}(X, \omega)$ and $D^b(X)$.

We would like to speak about families of objects of $D^b(X)$ parametrized by holomorphic curves. A family parametrized by curve C is understood as an element of $D^b(X \times C)$.

Definition 17.2.1 *We say that two objects E_0 and E_1 of $D^b(X)$ are equivalent if there is a family of objects parametrized by a curve such that E_0 and E_1 belong to this family.*

One can check that in this way we obtain an equivalence relation. In particular, we can speak about objects equivalent to the zero object of $D^b(X)$. Notice that if C is rational then the above equivalence is the same as numerical equivalence. But in general the equivalence relation is stronger. The reason for introducing the above equivalence relation is the following. The group $K_0(D^b(X))$ is in general very large. We define a countable group $\widehat{\Gamma}$ as the quotient of $K_0(D^b(X))$ by the subgroup generated by the isomorphism classes of objects equivalent to zero. Then the Chern character gives rise to a homomorphism $ch : \widehat{\Gamma} \rightarrow H^{ev}(X, \mathbf{Z})$. Notice that (up to the torsion) the group $H^{ev}(X, \mathbf{Z})$ is isomorphic to the topological K -group $K_{top}^0(X)$. We

will assume that the map ch factors through $K_{top}^0(X)$. Let $\widehat{\Gamma}_0$ be the kernel of the map ch . It is a countable abelian group. Isomorphism classes of the topologically trivial vector bundles belong to the $\widehat{\Gamma}_0$. Then we have an exact sequence

$$0 \rightarrow K_{top}^1(X) \rightarrow \widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_0 \rightarrow 0.$$

Here $K_{top}^1(X)$ is the topological K^1 -group, which is (up to the torsion) isomorphic to $H^{odd}(X, \mathbf{Z}) = H^3(X, \mathbf{Z}) \simeq \Gamma$. Geometrically, the above extension $\widehat{\Gamma}_1$ comes from a change of a topological (or smooth, if we work in the smooth category) trivialization of a topologically trivial vector bundle on X . Indeed, restricting ourselves to a single vector bundle (similarly we can work with perfect complexes), we may assume that isotopy classes of changes of the trivialization are parametrized by homotopy classes of maps $X \rightarrow U(N)$, where $U(N)$ is the unitary group. For sufficiently large N it is the same as $K_{top}^1(X)$. In order to simplify the exposition we will assume that cohomology groups of X do not have torsion, hence we can identify $H^{ev}(X, \mathbf{Z})$ with $K_{top}^0(X)$ and $H^{odd}(X, \mathbf{Z})$ with $K_{top}^1(X)$.

We can define the holomorphic Chern-Simons functional $CS_{\mathbf{C}} : \widehat{\Gamma}_1 \rightarrow \mathbf{C}$ in the following way. Let $E \rightarrow X$ be a topologically trivial trivialized vector bundle endowed with a $(0, 1)$ -connection. Then we set

$$CS_{\mathbf{C}}(a) = \int_X Tr \left(\frac{\bar{\partial}a \cdot a}{2} + \frac{a^3}{3} \right) \wedge \Omega^{3,0}.$$

As we explained above, the change of trivialization amounts to the change of the value of $CS_{\mathbf{C}}$ by an element of $K_{top}^1(X)$. Identifying the latter with $\Gamma = H^3(X, \mathbf{Z})$ (up to the torsion) we can identify it with the space of periods $\int_{\gamma} \Omega^{3,0}, \gamma \in \Gamma$. Thus the value of $CS_{\mathbf{C}}$ is defined up to a period.

More pedantically, the definition of $CS_{\mathbf{C}}$ can be given in terms of connected components of the space $\mathcal{A}_X^{0,1}$ of $(0, 1)$ -connections on vector bundles on X (better to consider a generalization to the case of complexes, but we will not do that). Since the space of connections on a fixed vector bundle is affine, the space $\mathcal{A}_X^{0,1}$ is homotopically equivalent to the space \mathcal{B}_X of vector bundles on X . Therefore the space of connected components $\pi_0(\mathcal{A}_X^{0,1})$ is an abelian group isomorphic to $K_{top}^0(X) = H^{ev}(X, \mathbf{Z})$.

We would like to use the analogy with the above-discussed toy-model example with Y being the space of $(0, 1)$ -connections on vector bundles on X and f being the holomorphic Chern-Simons functional. Then $\mathcal{A}_X^{0,1} =$

$\sqcup_{\alpha \in H^{ev}(X, \mathbf{Z})} Y_\alpha$ is the decomposition into the union of connected components (topological types of vector bundles).

The set of critical points $Crit_\alpha(CS_{\mathbf{C}}) \subset Y_\alpha$ coincides with the set of holomorphic vector bundles of fixed topological type α . The union $Crit(CS_{\mathbf{C}}) = \sqcup_\alpha Crit_\alpha(CS_{\mathbf{C}})$ is the set of holomorphic vector bundles on X . Its (bounded) triangulated envelope is $D^b(X)$. We have: $Crit_\alpha(CS_{\mathbf{C}}) = \sqcup_i Z_{\alpha,i}$, where the index i runs through a finite set. Our assumption that the image of $K_0(D^b(X))$ in $H_D^\bullet(X)$ is discrete ensures that the set of connected components of each universal covering $\tilde{Z}_{\alpha,i} \subset \tilde{Y}_\alpha$ is a torsor over $K_{top}^1(X)$ (this is true for any subspace Z of Y_α which satisfies the condition that the image $\pi_1(Z) \rightarrow \pi_1(Y_\alpha)$ is trivial). Let us explain the last point. Notice that elements $\alpha \in K_{top}^0(X)$ are in one-to-one correspondence with connected components of the Deligne cohomology group $H_D^\bullet(X)$. For any α we have: $\pi_1(H_D^\bullet(X)) \simeq \pi_1(Y_\alpha) \simeq K_{top}^1(X) = H^3(X, \mathbf{Z})$. We denote by \tilde{Y}_α the universal covering of Y_α . Since the differential $dCS_{\mathbf{C}}$ is well-defined on Y_α , the restriction of the function $CS_{\mathbf{C}}$ on each Y_α is defined up to a constant, and its lift to \tilde{Y}_α is well-defined. The composition

$$\pi_1(Z_{\alpha,i}) \rightarrow \pi_1(K_0(D^b(X))) \rightarrow \pi_1(H_D^\bullet(X))$$

coincides with the composition

$$\pi_1(Z_{\alpha,i}) \rightarrow \pi_1(Y_\alpha) \rightarrow \pi_1(H_D^\bullet(X)),$$

where the last arrow is in fact isomorphism. Then the image of $K_0(D^b(X))$ in $H_D^\bullet(X)$ is discrete iff the induced homomorphism of fundamental groups is trivial. But this is exactly the condition of the triviality of the image $\pi_1(Z_{\alpha,i}) \rightarrow \pi_1(Y_\alpha)$, which according to the toy-model example guarantees that the torsor is well-defined. Another explanation of this condition comes from the fact that the union of the images of $K_0(D^b(X))$ in $H_D^\bullet(X)$ over all complex structures must be a complex Lagrangian submanifold in total space of the complex integrable system over the base $\mathcal{L}_X \subset H^3(X, \mathbf{C})$ with the fiber H_D^\bullet . Since it covers \mathcal{L}_X over a generic point we see that for the generic fiber the image of $K_0(D^b(X))$ in it must be discrete. Applying the considerations from the toy-model example, we see that for any $i \neq j$ we can define Γ_{ij} as the set of homotopy equivalent paths from a point in $Z_{\alpha,i}$ to a point in $Z_{\alpha,j}$ (inside of Y_α). This set does not depend on a particular choice of points inside of the components. Furthermore, as we have explained, Γ_{ij} is a torsor over $\Gamma = K_{top}^1(X) = H^3(X, \mathbf{Z})$. Furthermore, we have an additive

map $Z_{ij} : \Gamma_{ij} \rightarrow \mathbf{C}$ defined as the integral of the 1-form $dCS_{\mathbf{C}}$ over any path which represents the element of Γ_{ij} .

Then we notice that the gradient lines for $CS_{\mathbf{C}}$ can be interpreted as G_2 -holonomy connections on the 7-dimensional manifold $X_7 := X \times \mathbf{R}$. Extending $CS_{\mathbf{C}}$ to the derived category, we can consider (complexes) of coherent sheaves supported to holomorphic curves in X . Holomorphic curves give rise to 3-dimensional associative cycles on X_7 . Gradient lines of $CS_{\mathbf{C}}$ can be interpreted as associative 4-dimensional cycles interpolating between such 3-dimensional cycles. Then, replacing $CS_{\mathbf{C}}$ by $e^{i\varphi}CS_{\mathbf{C}}, \varphi \in \mathbf{R}/2\pi\mathbf{Z}$, we see that for a discrete subset of φ 's the interpolating associative cycle can “bubble”, developing a 3-dimensional special Lagrangian submanifold. This relates the gradient trajectories of $CS_{\mathbf{C}}$ with SLAGS on X .

Under our assumptions, to every $\gamma \in \Gamma$ we can assign an integer $\Omega(\gamma)$, which is a DT-invariant of semistable objects of the Fukaya category $\mathcal{F}(X, \omega)$. Stability condition is determined by a choice of complex structure on X as well as by a choice of the holomorphic volume form $\Omega^{3,0}$. Similarly, a choice of Kähler form $\omega^{1,1}$ determines an asymptotic stability condition on $D^b(X)$. But even without stability condition we can speak about Cohomological Hall algebra \mathcal{H} . We have $\mathcal{H} = \bigoplus_{\mu \in H^{ev}(X, \mathbf{Z})} \mathcal{H}_{\mu}$. In the framework of holomorphic Chern-Simons theory we have the set $S := \{z_i\}$ of critical values of $CS_{\mathbf{C}}$. For each $z_i \in S$ we can define the corresponding COHA $\mathcal{H}(z_i)$, e.g. in the Betti realization as a relative cohomology $H^{\bullet}(CS_{\mathbf{C}}^{-1}(D_i), CS_{\mathbf{C}}^{-1}(z_i - \varepsilon))$, where D_i is the disc of radius ε with the center at z_i .

17.3 Categorification of critical COHA

The critical cohomology groups $H_c^{\bullet, crit}(X, f)$ are related to a certain 2-periodic triangulated category. Namely, for any closed subset $X^s \subset f^{-1}(0)$ let us define the category of matrix factorizations supported on X^s in the following way:

$$MF_{X^s}(f) := D_{X^s}^b(\text{Coh}(f^{-1}(0))) / \text{Perf}_{X^s}(f^{-1}(0)),$$

where the subscript X^s denotes the category of bounded complexes of coherent sheaves on the closed subscheme $f^{-1}(0)$ (resp. of perfect complexes on $f^{-1}(0)$), with cohomology sheaves supported on the closed subset X^s . Then there is a Chern character homomorphism

$$ch : K_0(MF_{X^s}(f)) \rightarrow (H_c^{ev, crit}(X, f))^{\vee}.$$

One has also an equivariant version $MF_{X^s, G}(f)$ of the above category and of the Chern character (here G is an algebraic group acting on X and preserving X^s and f). We expect that the multiplication on the critical COHA comes from a monoidal structure on the direct sum of categories

$$\bigoplus_{\gamma \in \mathbf{Z}_{\geq 0}^I} \bigoplus_{z \in \mathbf{C}} MF_{M_\gamma^s, G_\gamma}(W_\gamma^{-1}(z)).$$

The correspondences M_{γ_1, γ_2} should be upgraded to functors between different summands. The monoidal structure could be thought of as a categorification of the critical COHA (which is itself a categorification of DT-invariants).

17.4 Donaldson 4d theory, Borcherds automorphic forms

We suggested in 0811.2435 that our wall-crossing formulas should be related to those in the Donaldson theory of $4d$ manifolds with $b_2^+ = 1$ as well as with Borcherds hyperbolic Kac-Moody algebras and multiplicative automorphic forms. It was partially motivated by the product structure of WCFs as well as the observation that the “chamber” structure of the space of stability conditions is similar (and sometimes coincides) with the Weyl cameras structure for some root system. Also, after a choice of stability condition, each ray in the upper-half plane generates an algebra which resembles the universal enveloping algebra associated with a positive root of a Kac-Moody Lie algebra. More about this see in Section 5.2 of our 1006.2706. Since Donaldson theory depends on a choice of the gauge group, the latter should be somehow encoded in our story. In case of a complex projective surface S the expected relationship should combine the well-known description of Seiberg-Witten theory via complex integrable systems with our DT-theory. In this case the local 3CY is given by the total space of the anticanonical bundle of S . The challenging question (which also goes back to 0811.2435) is: how to relate the wall-crossing formulas in Donaldson theory with the theory of stability data on graded Lie algebras? There is another (maybe unrelated) question. Recall that there exists a physical interpretation of Donaldson theory due to Witten and others. Also, there is a more recent series of papers of Gaiotto-Moore-Neitzke and others on the physical meaning of our WCFs. One can ask: how to interpret physically the WCF associated with stability data on an arbitrary graded Lie algebra?

17.5 Stability conditions on curves and moduli of abelian differentials

We pointed out this question in 0811.2435.

Geometry similar to the one on the space of stability conditions appears in the theory of moduli spaces of holomorphic abelian differentials in the work of Kontsevich and Zorich. The moduli space of abelian differentials is a complex manifold, divided by real “walls” of codimension one into pieces glued from convex cones. It also carries a natural non-holomorphic action of the group $GL^+(2, \mathbf{R})$. There is an analog of the central charge Z in the story. It is given by the integral of an abelian differential over a path between marked points in a complex curve. We conjectured in 0811.2435 that the moduli space of abelian differentials associated with a complex curve with marked points, is isomorphic to the moduli space of stability structures on the Fukaya category of this curve. This can be made more precise if one considers the Fukaya category of the corresponding local Calabi-Yau threefold instead. This threefold is the total space of the conic bundle over the curve. The Fukaya category in question takes into consideration only compact Lagrangian submanifolds. It is \mathbf{Z} -graded (not $\mathbf{Z}/2$ -graded as the Fukaya category of the curve), hence one can apply the theory developed in these lectures. On the other hand, this category is closely related to the Fukaya category of the curve. Counting of DT-invariants for the Calabi-Yau threefold then becomes a geometric problem of counting certain geodesics on the curve.

17.6 Resurgence and WCF

There is a striking similarity between our wall-crossing formulas and identities for the Stokes automorphisms in the theory of WKB asymptotics of solutions of a second order Schrödinger operator $-\hbar^2(d/dx)^2 + V(x)$ with polynomial potential on \mathbf{P}^1 developed by Voros and others. It was motivated in turn by the work of Écalle on general properties of resurgent functions. Very briefly, the idea is to work with the Laplace transform of divergent series. Such a transform can admit an analytic continuation to a multivalued function with infinitely many poles (but finitely many in each compact subset). WKB series are resurgent in this sense with respect to the small parameter (Planck constant). It is instructive to think that the argument of the Planck constant is a point of the circle, while the absolute value is a

formal parameter. WKB solutions have locally the form $\exp(S(x)/\hbar)u(x)$, where $S(x) = \int \sqrt{V(x)}dx$. They analytically depend on the point of $x \in \mathbf{P}^1$, but can jump when the argument of Planck constant \hbar passes through some critical directions. The critical directions are derived from the geometry of the potential. More precisely, there is a spectral curve $y^2 = V(x)$ in the story, and the critical directions correspond to certain geodesics of the quadratic differential defined by the 1-form $\sqrt{V(x)}dx$.

There is an underlying graded Lie algebra of “alien derivatives” which plays an important role in the theory. The grading is given by the integer first homology of the spectral curve (or rather the relative homology with respect to the divisor of ramification points). Stokes automorphisms which take care about the change of the WKB series (or rather its analytic continuation) when we cross the critical directions are elements of the associated group. There are relations between the Stokes automorphisms which from our point view correspond to the wall-crossing formulas in the case of Hitchin integrable systems.

In order to see this, in the above story we replace \mathbf{P}^1 by an arbitrary curve, and the second order operator depending on a small parameter by a path in the space of connections. Then from the point of view of ordinary differential equations, we study the behavior of flat sections (or monodromy) of the “ ζ -connection” on a curve. Zeta connection locally looks like $\zeta \frac{d}{dz} + A(z)$ as $\zeta \rightarrow 0$. This general problem for the behavior of the monodromy was considered by Carlos Simpson at the beginning of 90’s. It is easy to see the relation of this story to our discussion about complex integrable systems. The limit $\zeta \rightarrow 0$ “compactifies” the moduli space of connections by a divisor of Higgs bundles. The latter is a complex integrable system studied by Hitchin. Let $(X, \omega^{2,0})$ be its total space. Then, as we discussed above, we can construct the mirror dual to the family of symplectic structures on X given by $Re(\omega^{2,0}/\zeta)$ (more precisely we should take into consideration $Im(\omega^{2,0}/\zeta)$ as a B -field. The construction uses the modification of the naive “semiflat” mirror dual by means of the wall-crossing formulas (they take into account pseudo-holomorphic discs with boundary on Lagrangian torus fibers of the Hitchin system). As a result one get a family $\mathcal{M}_\zeta, \zeta \in \mathbf{C}^*$ of complex symplectic manifolds. In fact, they form a local system with non-trivial monodromy about $\zeta = 0$. We expect that it can be “compactified” at $\zeta = 0$ by a dual complex integrable system (fibers are dual abelian varieties). Moreover, we expect that \mathcal{M}_ζ are schemes of finite type over \mathbf{Z} . The above mentioned trace of the monodromy is an example of a function from $\mathcal{O}(\mathcal{M}_\zeta)$. Another example

is the value of a flat section at a given point. The exponential behavior of such functions as $\zeta \rightarrow 0$ determines the Stokes filtration on the infinite-dimensional vector bundle on \mathbf{C}^* with the fiber $\mathcal{O}(\mathcal{M}_\zeta)$. The resurgence properties of the WKB solutions of the equation $\zeta \frac{d}{dz} f + A(z)f = 0$ studied in 90's is then entirely controlled by this “non-commutative exponential Hodge structure”. The wall-crossing formulas being spelled out in the language of this “Betti realization” are equivalent to the Stokes automorphism relations in the theory of Voros resurgence.