Week 6.

Note Title 7/26/2009

Computation of definite integrals using residues.

Improper integrals:

- $$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R_1\to\infty}\int_a^{R_1} f\,dx + \lim_{R_2\to\infty}\int_{R_2}^{a} f\,dx$$
- $$\int_a^b f(x)\,dx = \left(f \text{ has a discontinuity at } a<c<b\right) = \lim_{\varepsilon_1\to+0}\int_{c+\varepsilon_1}^{b} f\,dx + \lim_{\varepsilon_2\to+0}\int_a^{c-\varepsilon_2} f\,dx$$

Principal values of such integrals:

$$\text{v.p.}\int_{-\infty}^{+\infty} f\,dx = \lim_{R\to\infty}\int_{-R}^{R} f\,dx,$$

$$\text{v.p.}\int_a^b f\,dx = \lim_{\varepsilon\to+0}\left(\int_{c+\varepsilon}^{b} f\,dx + \int_a^{c-\varepsilon} f\,dx\right)$$

Examples: ① $\text{v.p.}\int_{-\infty}^{+\infty} x\,dx = \lim_{R\to\infty}\frac{x^2}{2}\Big|_{-R}^{R} = 0$,

$\int_{-\infty}^{+\infty} x\,dx$ diverges

② $\text{v.p.}\int_{-1}^{1}\frac{dx}{x} = \lim_{\varepsilon\to+0}(\ln\varepsilon - \ln\varepsilon) = 0$

$\int_{-1}^{1}\frac{dx}{x}$ diverges

③ $\text{v.p.}\int_{-\infty}^{+\infty}\frac{dx}{x^2+1} = \lim_{R\to\infty}\arctan(x)\Big|_{-R}^{R} = \pi$

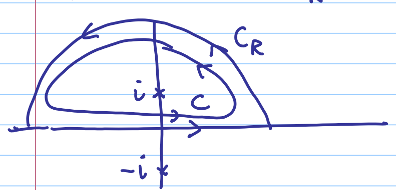
$$\text{v.p.}\int_{-\infty}^{+\infty}\frac{dx}{x^2+1} = \int_{-\infty}^{+\infty}\frac{dx}{x^2+1}$$

In general: if the improper integral converges, its pr. value also converges.

Example 1 $I = \int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^2}$

converges $\Rightarrow$ $I = \text{v.p.}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^2}$

$$\int_{-R}^{R}\frac{dx}{(x^2+1)^2} = -\int_{C_R} + \oint_C$$



$$\left|\int_{C_R}\frac{dx}{(x^2+1)^2}\right| \le \int_0^{\pi}\frac{|dz|}{|z^2+1|^2} \le$$

$|a+b| \ge \big||a|-|b|\big| \Rightarrow$

$\Rightarrow |z^2+1| \ge |z|^2-1 = R^2-1$

$$\le \frac{\pi R}{(R^2-1)^2} \xrightarrow{R\to 0} 0$$

$$\Rightarrow I = -\lim_{R\to\infty}\int_{C_R} + \oint_C = \oint_C,$$

$$\oint_C\frac{dz}{(z^2+1)^2} = 2\pi i\,\operatorname*{Res}_{z=i}\left(\frac{1}{(z^2+1)^2}\right)$$

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2}\,\frac{1}{(z-i)^2} = \frac{1}{(2i)^2(z-i)^2} - \frac{2}{(2i)^3}\,\frac{1}{z-i} + \dots$$

$\text{Res} = -\frac{i}{4}$,

$$I = \frac{2\pi}{4} = \frac{\pi}{2}$$

General strategy: Assume Q has no real roots

$$\int_{-\infty}^{+\infty}\frac{P\,dx}{Q} = \langle \text{converges absolutely when } \deg P \le \deg(Q)-2 \rangle$$

$$= \text{v.p.}\int_{-\infty}^{+\infty}\frac{P\,dx}{Q} = -\lim_{R\to\infty}\int_{C_R}\frac{P\,dx}{Q} + \oint_C$$

$$\left|\int_{C_R}\frac{P\,dx}{Q}\right| \le \frac{C}{R} \to 0$$

↑ when $\deg(P) \le \deg(Q)-2$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{P\,dx}{Q} = \oint_C \frac{P\,dx}{Q} = 2\pi i \sum_j \operatorname{Res}_{z=z_j}\left(\frac{P}{Q}\right)$$

z_j are : $Q(z_j)=0$, $\operatorname{Im} z_j > 0$.

Another type of integrals:

$$I = \int_{-\infty}^{+\infty} \frac{\cos\alpha x}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{ix\alpha}}{x^2+1}\,dx +$$

$\alpha > 0$

$$+ \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{-ix\alpha}}{x^2+1}\,dx = I_1 + I_2 \quad (I_2 = \overline{I_1})$$

$$I_1 = \frac{1}{2}\text{v.p.}\int_{-\infty}^{+\infty} \frac{e^{ix\alpha}}{x^2+1}\,dx = -\lim_{R\to\infty} \frac{1}{2}\int_{C_R} + \frac{1}{2}\oint_C$$

$$\left|\int_{C_R} \frac{e^{iz\alpha}}{z^2+1}\,dz\right| \le \int_{C_R} \frac{e^{-\operatorname{Im} z\,\alpha}}{|z^2+1|}\,|dz| \le \frac{\pi}{R} \to 0 \quad R\to 0$$

$$\operatorname{Im} z\Big|_{C_R} = R\sin\varphi \ge 0, \quad 0\le\varphi\le\pi, \qquad |z^2+1| \ge R^2-1$$

$$I_1 = \frac{1}{2}\oint_C \frac{e^{iz\alpha}}{z^2+1}\,dz = \frac{1}{2}\cdot 2\pi i\,\frac{e^{-\alpha}}{2i} =$$

$$= \frac{\pi}{2}e^{-\alpha},$$

Similarly $I_2 = \frac{\pi}{2}e^{-\alpha}$

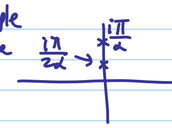
$$\Rightarrow \boxed{I = \pi e^{-\alpha}}$$

More general: or $\sin\alpha x$

$$\int_{-\infty}^{+\infty} \frac{P\cos(\alpha x)}{Q}\,dx, \quad \deg P \le \deg Q - 2$$

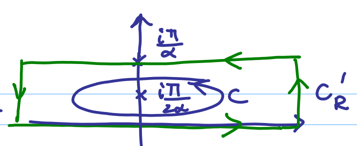
can be computed in the same way.

Example 2. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{ch(\alpha x)}$ $\left\{\begin{array}{l}\text{simple}\\ \text{pole}\end{array}\right.$ $\frac{i\pi}{2\alpha} \rightarrow$



$$ch\left(\alpha\left(x+\frac{i\pi}{\alpha}\right)\right) = -ch(\alpha x),$$

Consider



$$\oint_C \frac{dz}{ch(\alpha z)} = \int_{-R}^{R} \frac{dx}{ch(\alpha x)} +$$

$$+ \int_{R+i\frac{\pi}{\alpha}}^{-R+\frac{i\pi}{\alpha}} \frac{dz}{ch(\alpha z)} + \int_{C_R} + \int_{C_R'}$$

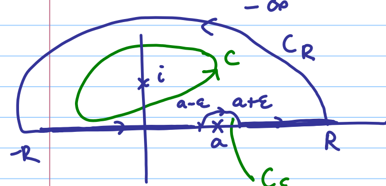
$$\left|\int_{C_R}\right|, \left|\int_{C_R'}\right| \to 0 \quad R\to\infty$$

$$\int_{R+\frac{i\pi}{\alpha}}^{-R+i\frac{\pi}{\alpha}} \frac{dz}{ch(\alpha z)} = -\int_{-\infty}^{+\infty} \frac{dx}{ch(\alpha x + i\pi)} = \int_{-\infty}^{+\infty} \frac{dx}{ch\,\alpha x}$$

$$\Rightarrow \oint_C \frac{dz}{ch(\alpha z)} = 2\int_{-\infty}^{+\infty} \frac{dx}{ch\,\alpha x} = 2I$$

$$2I = 2\pi i \operatorname{Res}_{z=i\frac{\pi}{2\alpha}} \frac{1}{ch(\alpha z)} = 2\pi i\,\frac{1}{\alpha\,sh(i\frac{\pi}{2})} = \frac{2\pi}{\alpha}$$

Example 3 $\displaystyle\text{v.p.}\int_{-\infty}^{+\infty} \frac{1}{(x-a)(x^2+1)}\,dx = I$



$$I = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\oint_C - \int_{C_R} - \int_{C_\varepsilon}\right)$$

$$\left|\int_{C_R}\right| \le \frac{\pi}{R^2} \to 0$$

$$\int_{C_\varepsilon} \frac{dz}{(z-a)(z^2+1)} = -\int_0^{\pi} \frac{\varepsilon i e^{i\varphi}\,d\varphi}{\varepsilon e^{i\varphi}\left((a+\varepsilon e^{i\varphi})^2+1\right)}$$

$$\xrightarrow{\varepsilon\to 0} -i\int_0^{\pi} \frac{d\varphi}{a^2+1} = -\frac{\pi i}{a^2+1},$$

$$I = \pi i \frac{1}{a^2+1} + 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z-a)(z^2+1)} =$$

$$= \pi i \frac{1}{a^2+1} + 2\pi i \frac{1}{(i-a)(2i)}$$

$$= \pi i \frac{1}{a^2+1} + \frac{\pi i}{i-a} ,$$

"Multivalued" functions