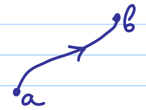
Week 3.

Entire fncns:

Proposition: If $f = F'$ where F is analytic on the smooth curve C, then

$$\int_C f\,dz = F(b) - F(a)$$



Proof: $F'(z) = f(z)$, choose a parameterization of C: $C = \{s(t)\}_{t=0}^{1}$.

$$\int_C F'\,dz \overset{def}{=} \int_0^1 F'(s(t))\,s'(t)\,dt \overset{\text{chain rule}}{=}$$

$$= \int_0^1 \frac{d}{dt} F(s(t))\,dt = F(s(1)) - F(s(0))$$

$$= F(b) - F(a),$$

h/w exercise: prove the chain rule for complex differentiation ∎

Def. $C = \{s(t)\}_{t=0}^{1}$ is closed if $s(0) = s(1)$, it is simple if $s(t_1) = s(t_2)$ implies $t_1 = t_2$ (no self-intersections).

The main goal today is to prove

Thm. If f is an entire function there exists another entire fncn F such that

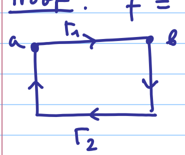
$$F'(z) = f(z)$$

for all $z \in \mathbb{C}$.

Proof 1. Let R be a rectangle $\Gamma = \partial R$ be its boundary (∂R is the notation for the boundary of a region)

Lemma 1 $\int_\Gamma f\,dz = 0$ if $f = \alpha + \beta z$

Proof. $f = F'$, $F' = \alpha z + \frac{\beta}{2} z^2$,



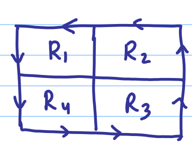
$$\int_\Gamma f\,dz = \int_{\Gamma_1} f\,dz + \int_{\Gamma_2} f\,dz$$

$$= (F(b) - F(a)) + (F(a) - F(b))$$

$$= 0$$ ∎

Lemma 2. $\int_\Gamma f\,dz = 0$ if f is an entire function.

Proof.



(stand. orientation is counter-clock)

$\Gamma_i = \partial R_i$

$$I = \int_\Gamma f\,dz = \sum_{i=1}^{4} \int_{\Gamma_i} f\,dz$$

(integration over common boundaries appear twice, one in one direction, one in the other, as a result they do not contribute to the sum)

Let $R^{(1)}$ be one of R_i, $i = 1, 2, 3, 4$ s.t. $\left|\int_{\Gamma^{(1)}} f\,dz\right| \geq \left|\int_{\Gamma_i} f\,dz\right|$ for all $i = 1, 2, 3, 4$

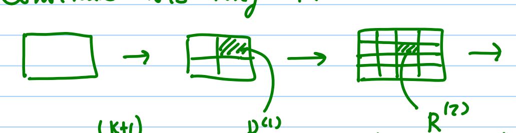
then

$$|I| \leq \sum_{i=1}^{4} \left|\int_{\Gamma_i} f\,dz\right| \leq 4\left|\int_{\Gamma^{(1)}} f\,dz\right|$$

or

$$\left|\int_{\Gamma^{(1)}} f\,dz\right| \geq \frac{|I|}{4}$$

Continue bisecting R



Let $R^{(k+1)}$ be one of $R_1^{(k)}, R_2^{(k)}, R_3^{(k)}, R_4^{(k)}$

with maximal $\left|\int_{R_i^{(k)}} f\,dz\right|$. By induction we will have

$$\left|\int_{R^{(k)}} f\,dz\right| \geq \frac{I}{4^k},$$

Now the strategy is to prove that the l.h.s. of this equation $\to 0$ faster than 4^k as $k \to 0$.

We have $R \supset R^{(1)} \supset R^{(2)} \supset \dots$, let $z_0 = \bigcap_{i=1}^{\infty} R^{(k)}$ (a point) (prove it, as an exercise)

Since f is entire, it is analytic in a nbd of z_0 and $\Rightarrow$

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)\varepsilon_z$$

where $\varepsilon_z \to 0$ as $z \to z_0$

By the previous lemma

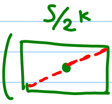
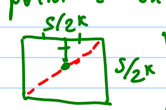
$$\int_{\Gamma^{(k)}} f\,dz = \int_{\Gamma^{(k)}} (z - z_0)\,\varepsilon_z\,dz$$

We want to estimate this integral from above.

Let s be length of the largest side of R.

We have:

1. $|z - z_0| \leq$ (the largest distance from a point z_0 inside $R^{(k)}$ to a point z on $\partial R^{(k)} = \Gamma^{(k)}$)

$= \Big($  $\Big) \leq \Big($  $\Big) = \frac{s}{2^k}\sqrt{2}$

2. $\varepsilon_z \to 0$ as $z \to z_0$ $\Rightarrow$

$\Rightarrow$ for $\forall \varepsilon > 0$, $\exists \delta$, s.t. $|\varepsilon_z| < \varepsilon$ for all $|z - z_0| < \delta$

$\Rightarrow$ for any $\varepsilon > 0$ $\exists N$, s.t. $|\varepsilon_z| < \varepsilon$ for all $|z - z_0| < \frac{\sqrt{2}\,s}{2^k}$ and for all $k > N$

3. $\int_{\Gamma^{(k)}} |dz| \overset{def}{=} \int_0^1 \left|\frac{d\gamma(t)}{dt}\right| dt =$ (length of $\Gamma^{(k)}$) $\leq 4 \cdot \frac{s}{2^k}$

Combining 1., 2., 3. with $\left|\int_\Gamma f\,dz\right| \leq \int_\Gamma |f|\,|dz|$

we have

$$\left|\int_{\Gamma^{(k)}} \varepsilon_z (z - z_0)\,dz\right| \leq \varepsilon \cdot \frac{\sqrt{2}\,s}{2^k} \cdot \frac{4s}{2^k} = \varepsilon\frac{4\sqrt{2}\,s^2}{4^k}$$

Combining with $(*)$, we have:

for any $\varepsilon > 0$, $\exists N$ s.t. for all $k > N$

$$\frac{|I|}{4^k} \leq \left|\int_{\Gamma^{(k)}} f\,dz\right| \leq \varepsilon\,\frac{4\sqrt{2}\,s^2}{4^k}$$

$\Rightarrow$ $|I| \leq \varepsilon\, 4\sqrt{2}\,s^2$ for any $\varepsilon > 0$

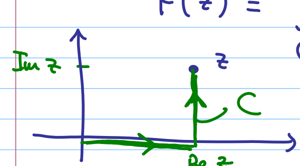
$\Rightarrow$ $|I| = 0$ $\Rightarrow$ $\int_\Gamma f\,dz = 0$

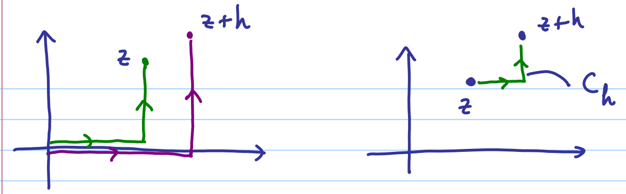
∎

Now, we can prove the theorem.

Define

$$F(z) = \int_C f(\zeta)\,d\zeta,$$





$$F(z+h) = F(z) + \int_{C_h} f\,dz$$

$$\left(\text{because} \quad \int_{C_1} f\,dz + \int_{C_2} f\,dz = \int_{C_1 \cdot C_2} f\,dz \right)$$

$$\Rightarrow \quad \frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{C_h} f(\zeta)\,d\zeta$$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{C_h} (f(\zeta) - f(z))\,d\zeta$$

Because f is continuous, for any $\varepsilon > 0$ and sufficiently small $|h|$,

$$|f(\zeta) - f(z)| < \varepsilon$$

on C_h. $\Rightarrow$

Exe: $\int_{C_h} dz = h$

for any $\varepsilon > 0$ and suffiently small h

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \le \frac{1}{|h|}\, \varepsilon \cdot \underbrace{\text{length}(C_h)}_{= |\operatorname{Re} h| + |\operatorname{Im} h|}$$

$$\le \varepsilon ,$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f'(z)$$

$$\Rightarrow \quad f = F' \qquad \blacksquare$$

Corollary (Cauchy (closed curve) thm for entire functions).

$\int_C f\,dz = 0$ if C is closed simple and f is entire.

Indeed, $f = F'$, $\int_C f\,dz = F(a) - F(a) = 0$

In-class problems:

1. Prove that $f(z) = \exp(P(z))$ is an entire function

2. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ where $|c_k| \le \frac{(1000)^k}{k!}$ is f an entire function?

3. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ where $|c_k| \le \frac{k^k}{k!}$ is f an entire function?

4. Describe the region of convergence of $\sum_{k=0}^{\infty} \left(\frac{1+z}{1-z} \right)^k$ (in "z-plane")

Important facts about entire functions:

1. Entire functions are infinitely differeniable.

2. For each $a \in \mathbb{C}$ the Taylor series of an entire function $f(z)$ is equal to its Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

for all $z \in \mathbb{C}$.

Next week we will prove these statements in more general setting: for fncns analytic in a domain.