1 Practice Midterm 1

The following generalization of the alternating series test is useful:

The series $\sum_{n=1}^{\infty} a_n b_n$ converges if the partial sums $A_N = \sum_{n=1}^{N} a_n$ are bounded (i.e. if there exists M such that $|A_N| < M$ for all N) and the sequence b_n is monotonically decreasing and $b_n \to 0$ as $n \to \infty$.

1. Find the region of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}-1}{\sqrt{n}+1} z^n$$

2. For the following series indicate whether they converge absolutely, conditionally, or diverge. Explain your answer.

$$(a) \quad \sum_{n=1}^{\infty} (e^{i\pi n + 1/n} - (-1)^n), \qquad (b) \quad \sum_{n=1}^{\infty} \frac{i^n}{n}$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{z^n + (-1)^n}{n}, |z| = 1 \qquad (d) \quad \sum_{n=1}^{\infty} \frac{1 + n^{-1/2}}{n} z^n, |z| = 1$$

3. The function $f(z)$ is analytic at $z = z_0$. Prove that the limit

$$\lim_{z \to z_0} \frac{f(z)^2 - f(z_0)^2}{z - z_0}$$

exists. Find the limit.

4. Consider the function $f(z) = \sqrt{\frac{\bar{z}}{z}}$ in the upper-half plane $Im(z) > 0$. Does it have a limit as $z \to 0$?

5. Suppose an entire function $f(z)$ has the form $f(z) = u(x) + iv(y)$. Find the most general form of such function.

6. Justify your answers.

   (a) Does the function $\frac{z}{|z|}$ have a limit as $z \to 0$?

   (b) The function $log(z)$ is defined on the upper half plane as $log(z) = \ln(r) + i\theta$ if $z = re^{i\theta}$. Does the function $\log(z) - \ln|z|$ have a limit when $z \to z_0 \neq 0 \in \mathbb{R}$?

   (c) Does it have a limit as $z \to 0$?

   (d) Is it continuous in the upper half plane?

7. Compute the integral $\int_C f dz$ for $f(z) = \frac{x+iy}{x^2+y^2}$, $C = \{z(t) = \cos t + i \sin t | 0 \leq t \leq \pi\}$.

8. Compute the improper integral $\int_C f dz$ if it converges. If diverges, explain why.
$$f(z) = (x - iy)e^{iy}, \quad C = \{z(t) = 1 + it | 0 \leq t \leq \infty\}$$

9. Find the integral $\int_C (x^3 - iy^3 + 3ix^2y - 3xy^2) dz$. Here C is any path (choose it at your convenience) connecting the origin and $s + it$.

The following combinations of problems will give you a good idea what to expect on the exam.

$$1, 3, 4, 5, 6$$

$$1, 2, 4, 5, 6$$

$$2, 4, 5, 7, 8$$

...etc.