MEASURES AND THEIR RANDOM REALS

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Abstract. We study the randomness properties of reals with respect to arbitrary probability measures on Cantor space. We show that every non-recursive real is non-trivially random with respect to some measure. The probability measures constructed in the proof may have atoms. If one rules out the existence of atoms, i.e. considers only continuous measures, it turns out that every non-hyperarithmetical real is random for a continuous measure. On the other hand, examples of reals not random for a continuous measure can be found throughout the hyperarithmetical Turing degrees.

1. Introduction

Most studies on algorithmic randomness focus on reals random with respect to the uniform distribution, i.e. the $(1/2, 1/2)$-Bernoulli measure, which is measure theoretically isomorphic to Lebesgue measure on the unit interval. The theory of uniform randomness, with all its ramifications (e.g. computable or Schnorr randomness) has been well studied over the past decades and has led to an impressive theory.

Recently, a lot of attention focused on the interaction of algorithmic randomness with recursion rheory: What are the computational properties of random reals? In other words, which computational properties hold effectively for almost every real? This has led to a number of interesting results, many of which will be covered in forthcoming books by Downey and Hirschfeldt and Nies [19].

While the understanding of "holds effectively" varied in these results (depending on the underlying notion of randomness, such as computable, Schnorr, or weak randomness, or various arithmetic levels of Martin-Löf randomness, to name only a few), the meaning of "for almost every" was usually understood with respect to Lebesgue measure. One reason for this can surely be seen in the fundamental relation between uniform Martin-Löf tests and descriptive complexity in terms of (prefix-free) Kolmogorov complexity: A real is not covered by any Martin-Löf test (with respect to the uniform distribution) if and only if all of its initial segments are incompressible (up to a constant additive factor).

However, one may ask what happens if one changes the underlying measure. It is easy to define a generalization of Martin-Löf tests which allows for a definition of randomness with respect to arbitrary *computable* measures. In fact, this can be done in a uniform way, with the help of an *optimal, continuous, left-enumerable semimeasure* [12].

For arbitrary measures, the situation is more complicated. Martin-Löf [16] studied randomness for arbitrary Bernoulli measures, Levin [12, 13, 14] studied arbitrary measures on 2^ω, while Gács [5] generalized Levin's approach to a large class of topological spaces. If turns out that much of the theory (i.e. existence of a uniform test, the connection with descriptive complexity etc.) can be preserved under reasonable assumptions for the underlying space.

In this paper, we will study the following question: Given a real $x \in 2^\omega$, with respect to which measures is x random? This can be seen as an inverse or dual to the usual investigations in algorithmic randomness: Given a (computable) measure μ (where μ usually is the uniform distribution), what are the properties of a μ-random real?

Of course, every real x is trivially random with respect to a measure μ which assigns some positive mass to it (as a singleton set), i.e. $\mu\{x\} > 0$. But one can ask if there is a measure for which this is not the case and x is still random. It turns out that this is possible precisely for the non-recursive reals. Furthermore, one could ask whether there exists a measure μ such that x is μ-random and μ does not have atoms at all, i.e. $\mu\{y\} = 0$ for all y. The answer to this question shows an unexpected correspondance between the randomness properties of a real and its complexity in terms of *effective descriptive set theory*: If x is not Δ^1_1, then there exists a non-atomic measure with respect to which x is random.

These results seem to motivate a further investigation. What is the exact classification of all reals (inside Δ^1_1) which are not random w.r.t. some continuous measure? If we look at n-randomness ($n \geq 2$), what is the size and structure of the reals that are not n-random for some continuous measure? The latter question will be addressed in separate paper [22].

2. Measures on Cantor Space

In this section we first quickly review the basic notions of measure on the Cantor space 2^ω. The space of probability measures on 2^ω is compact Polish, and we can devise a suitable effective representation of measures in terms of Cauchy sequences with respect to a certain metric. This will enable us to code measures via binary sequences, so we can use them as oracles in Turing machine computations. The goal is to extend Martin-Löf's notion of randomness to arbitrary measures by requiring that a Martin-Löf test for a measure is uniformly enumerable in (a representation of) the measure. This will be done in Section 3.

2.1. **The Cantor space as a metric space.** The *Cantor space* 2^ω is the set of all infinite binary sequences, also called *reals*. The usual metric on 2^ω is defined as follows: Given $x, y \in 2^\omega$, $x \neq y$, let $x \cap y$ be the longest common initial segment of x and y (possibly the empty string $\emptyset$). Define

$$d(x,y) = \begin{cases} 2^{-|x \cap y|} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Given a set $A \subseteq 2^\omega$, we define its *diameter* $d(A)$ as

$$d(A) = \sup\{d(x,y) : \; x, y \in A\}.$$

Endowed with this metric, 2^ω is a compact Polish space. A countable basis is given by the *cylinder sets*

$$N_\sigma = \{x : \; x \upharpoonright n = \sigma\},$$

where σ is a finite binary sequence of length n. We will occasionally use the notation $N(\sigma)$ in place of N_σ to avoid multiple subscripts. $2^{<\omega}$ denotes the set of all finite binary sequences. If $\sigma, \tau \in 2^{<\omega}$, we use $\sqsubseteq$ to denote the usual prefix partial ordering.

This extends in a natural way to $2^{<\omega} \cup 2^\omega$. Thus, $x \in N_\sigma$ if and only if $\sigma \sqsubset x$. Finally, given $U \subseteq 2^{<\omega}$, we write N_U to denote the open set induced by U, i.e. $N_U = \bigcup_{\sigma \in U} N_\sigma$.

2.2. **Probability measures.** A *probability measure* on 2^ω is a *countably additive, monotone function* $\mu : \mathcal{F} \to [0,1]$, where $\mathcal{F} \subseteq \mathcal{P}(2^\omega)$ is σ-algebra and $\mu(2^\omega) = 1$. μ is called a *Borel probability measure* if $\mathcal{F}$ is the Borel σ-algebra of 2^ω. It is a basic result of measure theory that a probability measure is uniquely determined by the values it takes on an algebra $\mathcal{A} \subseteq \mathcal{F}$ that generates $\mathcal{F}$. It is not hard to see that the Borel sets are generated by the algebra of *clopen sets*, i.e. finite unions of basic open cylinders. Normalized, monotone, countably additive set functions on the algebra of clopen sets are induced by any function $\rho : 2^{<\omega} \to [0,1]$ satisfying

$$\rho(\epsilon) = 1 \text{ and for all } \sigma \in 2^{<\omega},\ \rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1), \tag{2.1}$$

where ϵ denotes the empty string. Then $\mu(N_\sigma) = \rho(\sigma)$ yields an monotone, additive function on the clopen sets, which in turn uniquely extends to a Borel probability measure on 2^ω. In the following, we will deal exclusively with Borel probability measures, and hence we will identify such measures with the underlying function on cylinders satisfying (2.1), and write, in slight abuse of notation, $\mu(\sigma)$ instead of $\mu(N_\sigma)$. Besides, if in the following we speak of *measures*, we will always mean Borel probability measures.

The *Lebesgue measure* $\mathcal{L}$ on 2^ω is obtained by distributing a unit mass uniformly along the paths of 2^ω, i.e. by setting $\mathcal{L}(\sigma) = 2^{-|\sigma|}$. A *Dirac measure*, on the other hand, is defined by putting a unit mass on a single real, i.e. for $x \in 2^\omega$, let

$$\delta_x(\sigma) = \begin{cases} 1 & \text{if } \sigma \subset x, \\ 0 & \text{otherwise.} \end{cases}$$

If, for a measure μ and $x \in 2^\omega$, $\mu(\{x\}) > 0$, then x is called an *atom* of μ. Obviously, x is an atom of δ_x. A measure that does not have any atoms is called *continuous*.

2.3. **The space of probability measures on 2^ω.** We denote by $\mathcal{P}(2^\omega)$ the set of all probability measures on 2^ω. $\mathcal{P}(2^\omega)$ can be given a topology (the so-called *weak topology*) by defining $\mu_n \to \mu$ if $\mu_n(B) \to \mu(B)$ for all Borel sets B whose boundary has μ-measure 0. This is equivalent to requiring that $\int f d\mu_n \to \int f d\mu$ for all bounded continuous real-valued functions f on 2^ω. In Cantor space, the weak topology is also generated by sets of the form

$$M_{\sigma,p,q} = \{\mu \in \mathcal{P}(2^\omega) :\ p < \mu(N_\sigma) < q\}, \tag{2.2}$$

where $\sigma \in 2^{<\omega}$ and $p < q$ are rational numbers in the unit interval. The sets $M_{\sigma,p,q}$ form a *subbasis* of the weak topology.

It is known that if X is Polish and compact metrizable, then so is the space of all probability measures on X (see for instance [8]). Therefore, $\mathcal{P}(2^\omega)$ is compact metrizable and Polish. A compatible metric is given by

$$d_{\text{meas}}(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} d_n(\mu, \nu),$$

where

$$d_n(\mu.\nu) = \frac{1}{2} \sum_{|\sigma|=n} |\mu(N_\sigma) - \nu(N_\sigma)|.$$

We want to find an *effective presentation* of $\mathcal{P}(2^\omega)$ reflecting its properties. We follow the framework of Moschovakis [17]. A countable, dense subset $\mathcal{D} \subseteq \mathcal{P}(2^\omega)$ is given by the set of measures which assume positive, rational values on a finite number of rationals, i.e. $\mathcal{D}$ is the set of measures of the form

$$\nu_{\delta,Q} = \sum_{\sigma\in\Delta} Q(\sigma)\delta_{\sigma^\frown 0^\omega},$$

where Δ is a finite set of finite strings (representing dyadic rational numbers) and $Q : \Delta \to [0,1] \cap \mathbb{Q}$ such that $\sum_{\sigma\in\Delta} Q(\sigma) = 1$.

A straightforward calculation shows that for $\mu, \nu \in \mathcal{D}$, the following two relations are recursive:

$$d_{\text{meas}}(\mu,\nu) < q \quad \text{and} \quad d_{\text{meas}}(\mu,\nu) \leq q,$$

for rational $q \geq 0$.

We fix an effective enumeration of $\mathcal{D} = \{\nu_0, \nu_1, \dots\}$. $\mathcal{D}$ is called a *recursive presentation*. We can invoke the basic machinery of *(effective) descriptive set theory* to obtain a recursive surjection

$$\pi : \omega^\omega \to \mathcal{P}(2^\omega)$$

and a Π^0_1 set $P \subseteq \omega^\omega$ such that $\pi|_P$ is one-to-one and $\pi(P) = \mathcal{P}(2^\omega)$. (See Moschovakis [17], 3E.6.) This is achieved via an *effective Lusin scheme*, a family of sets mirroring the tree structure of ω^ω. Every element in P is an intersection of nested F_s, corresponding to a path through ω^ω. This path in turn represents a *Cauchy sequence* of measures in $\mathcal{D}$.

However, since $\mathcal{P}(2^\omega)$ is compact, we would like to have a representation that is not only closed but *effectively compact*, that is, a Π^0_1 subset of 2^ω. We can obtain such a representation, but we have to give up injectivity for this.

Proposition 2.1. *There exists a recursive sequence (r_n) and a continuous surjection*

$$\pi : [T] \to \mathcal{P}(2^\omega),$$

where $T \subset \omega^{<\omega}$ is the full (r_n)-branching tree, i.e. every node in T of length n has exactly r_n immediate successors.

For details on this argument, see [21].

3. Randomness and Transformations of Measures

3.1. **Random Reals.** We define randomness of reals for arbitrary measures as a straightforward extension of Martin-Löf's test notion. The basic idea is to require the test to be *enumerable* (Σ^0_1) in the representation of the measure.

Definition 3.1. Let μ be a probability measure on 2^ω, and let p_μ be a representation of μ, i.e. $p_\mu \in P$ and $\pi(p_\mu) = \mu$.

(1) A *p_μ-Martin-Löf test* is a sequence $(U_n)_{n\in\mathbb{N}}$ of subsets of $2^{<\omega}$ such that (U_n) is uniformly recursively enumerable in p_μ and for each n,

$$\sum_{\sigma\in U_n} \mu(N_\sigma) \leq 2^{-n}.$$

(2) A real $x \in 2^\omega$ is *random for* μ, or simply μ*-random*, if x is not covered by any p_μ-Martin-Löf test $(U_n)_{n\in\mathbb{N}}$, i.e. for any such test it holds that

$$x \notin \bigcap_{n\in\mathbb{N}} N_{U_n}$$

Informally, a real x is μ-random if it is not contained in any effectively presented (in p_μ) G_δ set of μ-measure zero.

Of course, every real x is trivially μ-random if it is a μ-atom. We will say x is non-trivially μ-random if it is μ-random and $\mu\{x\} = 0$.

The concept can be relativized in the usual way. Given two reals $x, z \in 2^\omega$, x is called μ*-random relative to* z if no p_μ-Martin-Löf test $\{U_n\}$ *uniformly enumerable in* $p_\mu \oplus z$ covers x.

A high useful fact about Martin-Löf randomness is the existence of a universal test. A p_μ-test (U_n) is *universal* if it covers every real that is covered by some p_μ-test. The existence of a universal p_μ-test is a straightforward generalization of the unrelativized result. Gács [5] obtained much more general results.

One might argue that this definition of randomness is subject to a certain arbitrariness, as it depends on a particular representation of a measure. In fact, for any real x and any measure μ one can find a Cauchy sequence of measures in $\mathcal{D}$ such that a standard binary encoding ρ_μ of this sequence computes x, hence x is not random for μ with respect to the representation ρ_μ.

However, we are only interested in results of the type "For which reals does there exist a measure for which x (non-trivially) looks random?" – i.e. can x look random at all? If there exists such a (representation of a) measure, there is good reason to say that x has *some* random content. Besides, it is not the aim of this paper to provide a most general solution to the problem of defining randomness for arbitrary measures. The goal is to exhibit an interesting connection between the randomness properties of reals (with respect to *some* representation) and its logical complexity.

Moreover, our results will work for *any* representation of $\mathcal{P}(2^\omega)$ that is a Π^0_1 subset of Cantor space. And since $\mathcal{P}(2^\omega)$ is compact, a suitable framework for effective measure theory should resort to an effective representation with the same topological properties. Finally, our representation P is obtained by following the canonical correspondances for Polish spaces used in (effective) descriptive set theory.

We will from now on identify a measure with its represenation $p_\mu \in P$.

A straightforward argument yields that the generalized notion of Martin-Löf randomness introduced in Definition 3.1 coincides with the original concept for computable measures. This is due to the following observation.

Proposition 3.2. *For any measure* $\mu \in \mathcal{P}(2^\omega)$*, the following are equivalent.*

(1) $p_\mu \in P$ *is computable.*
(2) There exists a computable function $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}$ *such that for all* $\sigma \in 2^{<\omega}$, $n \in \mathbb{N}$,

$$|g(\sigma, n) - \mu(N_\sigma)| \leq 2^{-n}.$$

Lebesgue measure $\mathcal{L}$ is arguably the most prominent (computable) measure. $\mathcal{L}$-random reals are the most studied ones by far, and we will, for sake of consistency with most literature,use the name *Martin-Löf random reals* for them.

Reals that are random with respect to some computable probability measure have been called *proper* [28] or *natural* [18].

Obviously no recursive real can be non-trivially random with respect to anymeasure. The following observation yields that trivially random reals with respect to computable measures are precisely the recursive reals.

Proposition 3.3 (Levin, 1970)**.** *If μ is a computable measure and $\mu\{x\} > 0$ for some $x \in 2^\omega$, then x is computable.*

Proof. Suppose $\mu\{x\} > c > 0$ for some computable μ and rational c. Let g be a computation function for μ, i.e. g is recursive and for all σ and n, $\|g(\sigma, n) - \mu(\sigma)\| \leq 2^{-n}$. Define a recursive tree T by letting $\sigma \in T$ if and only if $g(\sigma, |\sigma|) \geq c - 2^{-|\sigma|}$. By definition of T and the fact that μ is a (probability) measure, it holds that for sufficiently large m,

$$|\{\sigma : \ \sigma \in T \ \wedge \ |\sigma| = m\}| \leq \frac{1}{c - 2^{-m}}.$$

But this means that every infinite path through T is isolated, i.e. if x is an infinite path through T, there exists a string σ such that for all $\tau \sqsupseteq \sigma$, $\tau \in T$ implies $\tau \sqsubset x$. Furthermore, every isolated path through a recursive tree is recursive, and hence x is recursive. □

As regards non-recursive reals, Levin proved that, from a computability theoretic point of view, randomness with respect to a computable probability measure is computationally as powerful as Martin-Löf randomness. This was independently shown by Kautz [7].

Theorem 3.4 (Levin, 1970; Kautz 1991)**.** *A non-recursive real which is random with respect to some computable measure is Turing equivalent to a Martin-Löf random real.*

The proof of Theorem 3.4 uses the fact that reductions induce continuous (partial) mappings from 2^ω to 2^ω. Such mappings transform measures.

3.2. **Transformation of Measures.** Let μ be a Borel measure on 2^ω, and let $f : 2^\omega \to 2^\omega$ be a μ-measurable function, that is, for all measurable $A \subseteq 2^\omega$, $f^{-1}(A)$ is μ-measurable, too. Such f induces a new measure μ_f, often referred to as the *image measure*, on 2^ω by letting

$$\mu_f(A) = \mu(f^{-1}(A)).$$

Any nonatomic measure can be transformed into Lebesgue measure $\mathfrak{L}$ this way. Recall that a function $f : 2^\omega \to 2^\omega$ is a *Borel automorphism* if it is bijective and for any $A \subseteq 2^\omega$, A is a Borel set if and only if $f(A)$ is.

Theorem 3.5 (see [6])**.** *Let μ be a nonatomic probability measure on 2^ω.*

(1) There exists a continuous mapping $f : 2^\omega \to 2^\omega$ such that $\mu_f = \mathfrak{L}$.
(2) There exists a Borel automorphism h of 2^ω such that $\mu_h = \mathfrak{L}$.

The simple idea to prove (1) is to identify Cantor space with the unit interval $[0, 1]$ and define $f(x) = \mu[0, x]$, that is, to let f equal the distribution function of μ.

To prove Theorem 3.4, Levin showed that this idea works in the effective case, too. Furthermore, one can even deal with the presence of atoms, as long as the measure is computable. Roughly speaking, a computable transformation transforms a computable measure into another computable measure. At the same time, a computable transformation should not affect randomness properties, so a sequence random with respect to some measure should, when transformed by a computable mapping, be random with respect to the transformed measure. This is the fundamental *conservation of randomness* property.

In the following we will use (relativized) conservation of randomness to transform Lebesgue measure to render a given real random.

4. Randomness with Respect to Arbitrary Measures

The Levin-Kautz result implies that the set of reals that are non-trivially random with respect to a computable measure are contained in the set of Martin-Löf random Turing degrees. But what about the reals (non-trivially) random with respect to an arbitrary measure? In this section we will show that these coincide with the non-recursive reals. Every non-recursive real is non-trivially random with respect to some measure.

Theorem 4.1. *For any real $x \in 2^\omega$, the following are equivalent:*

(i) There exists a probability measure μ such that x is not a μ-atom and x is μ-random.
(ii) x is not recursive.

Proof. (i) $\Rightarrow$ (ii): If x is recursive and μ is a measure with $\mu(\{x\}) = 0$, then we can obviously construct a μ-test that covers x, by computing (recursively in μ) the measure of initial segments of x, which tends to 0. More formally, given n, compute (recursively in μ) a length l_n for which $\mu(N_{x \upharpoonright l_n}) < 2^{-n}$. Define a μ-test (U_n) by letting $U_n = \{x \upharpoonright l_n\}$.

(ii) $\Rightarrow$ (i): The idea of the proof is the following: The Kucera-Gacs result [11, 4] that every real is reducible to a Martin-Löf random real actually yields that every Turing degree $\geq \mathbf{0}'$ contains a Martin-Löf random real (so for every real x, $x \oplus \emptyset'$ is Turing equivalent to a Martin-Löf random real). It is not hard to see that the result relativizes, i.e. every real $\geq_{\mathrm{T}} y'$ relative to y is Turing equivalent to some Martin-Löf random real R relative to y. We will use the Posner-Robinson Theorem [20] to obtain such a real y for given x.

The (relative) Turing equivalence to a random real allows us to transform the uniform measure $\mathcal{L}$ in a sufficiently controlled manner. More precisely, we can obtain a Π^0_1 class of (representations of) measures, all of which are good candidates for a measure that renders x random. We will use a compactness argument to show that at least one member μ of the class has the property that the Martin-Löf random real R is still $\mathcal{L}$-random *relative to* p_μ (here p_μ is viewn as a real, not a measure). Then, x has to be random with respect to measure defined by μ, since otherwise an Martin-Löf μ-test could be effectively transformed to a Martin-Löf $\mathcal{L}$-test relative to p_μ which R would fail.

We start with the relativized version of Kucera's result. Given a real $z \in 2^\omega$, we write $\equiv_{\mathrm{T}(z)}$ to denote Turing equivalence relative to z (similarly for $\leq_{\mathrm{T}(z)}$ and $\geq_{\mathrm{T}(z)}$).

Theorem 4.2 (Kučera [11]). *For y and z in 2^ω, if $y \geq_{\mathrm{T}(z)} z'$ for some $z \in 2^\omega$, then y is Turing equivalent relative to z to some real R that is Martin-Löf random relative to z.*

For the proof of Theorem 4.2 is based on a careful but straightforward relativization of an argument by Kučera [11] that ensure an effective non-zero lower bound on the μ-measure of sections of Π^0_1-classes that contain μ-random reals with respect to a measure μ.

We use the following notation: Given a measure μ, a real $z \in 2^\omega$, and a universal μ-Martin-Löf test $(U_n)_{n\in\mathbb{N}}$ relative to z, we denote by $P^{\mu,z}_n$ the complement of the nth level of (U_n), i.e. $P^{\mu,z}_n = 2^\omega \setminus N(U_n)$.

Lemma 4.3 (Kučera [11])**.** *Let μ be a probability measure on 2^ω. If $A \subseteq 2^\omega$ is $\Pi^0_1(z)$, then there exists a function $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}^{>0}$ computable in $\mu \oplus z$ such that for any $\sigma \in 2^{<\omega}$ and any $n \in \mathbb{N}$ it holds that*

$$P^{\mu,z}_n \cap A \cap N_\sigma \neq \emptyset \quad \Rightarrow \quad \mu(A \cap N_\sigma) \geq g(\sigma, n).$$

As indicated above we will use the Posner-Robinson Theorem to obtain a real C relative to which the given non-recursive real x is above the jump of g.

Theorem 4.4 (Posner and Robinson [20])**.** *If $x \in 2^\omega$ is noncomputable, then there is a $z \in 2^\omega$ such that $z \oplus g \geq_{\mathrm{T}} g'$.*

Given a non-recursive real x, we can use Theorem 4.2 and the the Posner-Robinson Theorem 4.4 to obtain

> a real R which is Martin-Löf random relative to some $z \in 2^\omega$ and which is $\mathrm{T}(z)$-equivalent to x.

There are Turing functionals Φ and Ψ recursive in z such that

$$\Phi(R) = x \quad \text{ and } \quad \Psi(x) = R.$$

We will use the functionals to define a set of measures M. If Φ were total and invertible, there would be no problem to define the desired measure, as one could simply 'pull back' Lebesgue measure using Φ^{-1}. In our case we have to use Φ and Ψ to control the measure. We are guaranteed that this will work *locally*, since Φ and Ψ are inverses on R and x. Therefore, given a string σ (a possible initial segment of x) we will single out strings which appear to be candidates for initial segments of an inverse real.

Let $w(\epsilon) = \epsilon$. If we see that for some s, $\Psi[s](\sigma) \supset w(\sigma^-)$, where σ^- is σ minus the last bit, let $w(\sigma) = w(\sigma^-)^\frown i$, with $i \in \{0, 1\}$ such that $w(\sigma^-)^\frown i \subseteq \Psi[s](\sigma)$. Whenever $w(\sigma)$ gets defined, let

$$\mathrm{Pre}(\sigma) = \{\tau \in 2^{<\omega} : \Phi(\tau) \sqsupseteq \sigma \;\;\&\;\; \tau \sqsupseteq w(\sigma)\}.$$

Note that $\mathrm{Pre}(\sigma)$ is r.e. in z.

If we want to define a measure μ with respect to which A is random, we have to satisfy two requirements:

(1) The measure μ will *dominate* an image measure induced by Φ. This will ensure that any Martin-Löf random real is mapped by Φ to a μ-random real.
(2) The measure must *not be atomic* on x.

To meet these requirements, we restrict the values of μ in the following way (if $\mathrm{Pre}(\sigma)$ is defined):

$$\mathcal{L}(N_{\mathrm{Pre}(\sigma)}) \leq \mu(N_\sigma) \leq \mathcal{L}(N_{\Psi(\sigma)}). \tag{4.1}$$

The first inequality ensures that (1) is met, whereas the second guarantees that μ is non-atomic on the domain of Ψ (since if $\Psi(y)\downarrow$, then $\lim_n |w(y \upharpoonright n)| = \infty$).

Let M be the set of all measures that satisfy (4.1). We show that M is $\Pi^0_1(z)$.

Since the weak topology is generated (as a subbasis) by sets of the form (2.2), each configuration given by (4.1) kills off one or more branches in the tree of measures

T_P. Since the configurations (4.1) are a $\Sigma_1^0(z)$ event, there is a computable subtree of T_P consisting of 'admissible' measures. Let M be the set of infinite paths through this subtree, which is Π_1^0.

To see that M is not empty, note that the consistency requirements 4.1 yield valid, non-empty intervals for any pair of (not necessarily effective reductions) reductions sending x to R and vice versa. In particular, they will be valid for any extension of Φ to a total function, which in turn defines a probability measure contained in M.

Finally, we want to argue that x is random with respect to one of the measures in M. If $\mu \in M$, then any test (U_n) for μ transforms into a test for $\mathcal{L}$ (relative to z) by enumerating the Φ preimage of (U_n). If (U_n) covers x, $(\Phi^{-1}(U_n))$ covers R. However, we cannot immediately conclude that x is μ-random, since $(\Phi^{-1}(U_n))$ is only a $\mathcal{L}$-test relative to $z \oplus p_\mu$. We therefore have to argue that there is one measure in M whose representation p_μ does not, when given as an additional oracle to a $\mathcal{L}$-test, affect the $\mathcal{L}$-randomness of R. This follows from the following basis theorem for Π_1^0-classes. It was independently obtained by Downey, Hirschfeldt, Miller, and Nies [2].

Theorem 4.5. *Let $z \in 2^\omega$, and let $T \subseteq 2^{<\omega}$ be an infinite tree recursive in z. Then, for every real R which is Martin-Löf random relative to z, there is an infinite path y through T such that R is Martin-Löf random relative to $z \oplus y$.*

Proof. Given $z \in 2^\omega$ and some $\tau \in 2^{<\omega}$, let $(U_n^\tau)_{n\in\mathbb{N}}$ denote a universal Martin-Löf test relative to z and τ (which is still uniformly enumerable in C). We enumerate a Martin-Löf test $(V_n)_{n\in\mathbb{N}}$ relative to z as follows: enumerate a string σ into V_n if N_σ is contained in $N(U_n^\tau)$ for all $\tau \in T$ with $|\tau| = |\sigma|$ (note that there are only finitely many such τ).

If R is Martin-Löf random relative to z, there has to be some n such that $R \notin N_{V_n}$. This means that for every m, $R \upharpoonright m$ is not enumerated in V_n, hence for every m there is a $\tau \in T$ of length m such that N_τ is not contained in $N(U_n^\tau)$. Consequently, there is an infinite subtree of T of nodes τ which do not enumerate an initial segment of R into $N(U_n^\tau)$. Applying König's Lemma yields an infinite path y through this subtree.

Obviously, R is random relative to $z \oplus y$, because otherwise, due to the "use principle" (see for example 25), for every n there would be an initial segment τ of y such that $R \in N_{U_n^\tau}$, a contradiction. □

We can now complete the proof of Theorem 4.1. Let $\mu \in M$ be such that R is Martin-Löf random relative to $z \oplus p_\mu$. Assume there is a μ-test $(V_n)_{n\in\mathbb{N}}$ that covers x. We define a new test U_n by enumerating $\mathrm{Pre}(\sigma)$ for every string σ that is enumerated into V_n. Since $\Phi(R) = x$ and $\Psi(x) = R$, there must be infinitely many $\sigma_n \sqsubset x$ such that $\mathrm{Pre}(\sigma)$ is defined and some $\tau_n \sqsubseteq \sigma_n$ is enumerated in to (V_n). Hence (U_n) covers R, and since μ satisfies the measure condition (4.1), the Lebesgue measure of N_{U_n} is bounded by $\mu(N(V_n))$. Thus (U_n) is a Martin-Löf $\mathcal{L}$-test relative to $z \oplus p_\mu$. But this contradicts the fact that R is Martin-Löf random relative to $R \oplus p_\mu$. □

Note that there is a μ that makes x random whose complexity does not exceed that of x by much. The tree determining M is computable in $x \oplus \emptyset'$, and it takes another jump to find a path through the tree which conserves randomness for some

R Turing-equivalent to x. Hence, there is a measure recursive in x'' for which x is random.

5. Randomness with Respect to Continuous Measures

In this section we investigate what happens if one replaces the property "*random with respect to an arbitray probability measure*" (which turned out to hold for any non-recursive real) with *randomness for continuous probability measures*. First, we give an explicit construction of a non-recursive real which no longer has this property.

5.1. **A real not continuously random.**

Theorem 5.1. *There exists a non-recursive real which is not random with respect to any continuous measure.*

Proof. Consider the halting problem $\emptyset'$. Denote by $\emptyset'[s]$ the approximation to $\emptyset'$ enumerated after s steps. Now let $s_0 := 0$ and define

$$s_{n+1} := \max\{\min\{s : \emptyset'[s] \upharpoonright n+1 = \emptyset' \upharpoonright n+1\}, s_n + 1\}.$$

Finally, let $S := \{s_n : n \geq 0\}$. We claim that the characteristic sequence of S is not random with respect to any continuous measure.

Let $\mu \in \mathcal{P}(2^\omega)$ be continuous. Since 2^ω is compact, given n, we can, for every rational $\varepsilon > 0$, compute (recursively in μ) a number $l(\varepsilon)$ such that

$$\forall \sigma \in \{0,1\}^{l(\varepsilon)}\, (\mu N_\sigma \leq \varepsilon).$$

We use this to uniformly enumerate the nth level of a μ-Martin-Löf test $(V_n)_{n\in\mathbb{N}}$ that covers S.

First, compute $n_0 = l(2^{-n-1})$ and $n_1 = l(2^{-n-1}/n_0)$. Enumerate $S[n_1] \upharpoonright n_0$ into V_n. Furthermore, for all k such that $s_k < n_0$, enumerate the string $S[n_1] \upharpoonright s_k{}^\frown 0^{n_1 - s_k}$ into V_n.

Note that $S \upharpoonright n_0$ can change at most n_0 many times. Observe that, if $\emptyset'$ changes at a position m at time t, it will move the 'marker' s_m and with it all the markers s_k, $k > m$ to a position $\geq t$, thereby 'inserting' long blocks of zeroes into S. So, if $S \upharpoonright n_0$ changes after time n_1, it will insert a block of at least $n_1 - n_0$ zeroes into S. These changes are covered through the test by the strings of the form $S[n_1] \upharpoonright s_k{}^\frown 0^{n_1 - s_k}$. Therefore, S is covered by V_n.

Finally, note that the measure of V_n is at most 2^{-n}, since

$$\begin{aligned}\sum_{v \in V_n} \mu N_v &= \mu N_{S[n_1]\upharpoonright n_0} + \sum_{s_k < n_0} \mu N_{S[n_1]\upharpoonright s_k{}^\frown 0^{n_1 - s_k}} \\ &\leq 2^{-n-1} + n_0 \frac{2^{-n-1}}{n_0} = 2^{-n}.\end{aligned}$$

□

5.2. **Classifying the not continuously random reals.**

Theorem 5.1 suggests the following problem: classify the reals which are random with respect to a continuous measure, and to obtain bounds on the complexity of a real which is not continuously random. Denote by NCR the set of all not continuously random reals.

The structure of the Martin-Löf test given in the proof of Theorem 5.1 is rather simple, in particular, each level consists of only finitely many strings and hence each level of the cover is a clopen set.

This indicates it may be possible to construct more complicated examples of non-continuously random reals, where one has to use the full power of Martin-Löf tests.

However, an observation by Kjos-Hanssen and Montalban [9] provides examples of not continuously random reals all through the hyperarithmetical Turing degrees.

We start by observing that, for every continuous measure μ, a countable subset of 2^ω has μ-measure zero. This follows directly from the countable additivity of measures.

For countable Π^0_1 classes, we can strengthen this to effective μ-measure zero, and hence no countable Π^0_1 class contains a continuosly random real.

Theorem 5.2 (9)**.** *If $A \subseteq 2^\omega$ is a countable and Π^0_1, then no member of A can be random for a continuous measure.*

Proof. This is an immediate consequence of Lemma 4.3. □

Kreisel [10] showed that every member of a countable Π^0_1 class is hyperarithmetic, i.e. contained in Δ^1_1. Furthermore, he showed that members of countable Π^0_1 classes (also called *ranked points*) can be found all the way through the hyperarithmetical Turing degrees. Later, Cenzer, Clote, Smith, Soare, and Wainer [1] showed that ranked points appear at each hyperarithmetical level of the Turing jump.

Corollary 5.3. *For every recursive ordinal β there exists an $x \in \mathrm{NCR}$ such that $x \equiv_{\mathrm{T}} \emptyset^{(\beta)}$.*

We would like to obtain an upper bound on the complexity of reals in NCR; in particular, we would like to know whether NCR is countable. We start with the following simple observation.

Proposition 5.4. *The set* NCR *of all not continuously random reals is Π^1_1.*

Proof. We have

$$x \in \mathrm{NCR} \Leftrightarrow (\forall y)[y \text{ represents a measure } \mu \text{ and } \mu \text{ is continuous} \\ \rightarrow \text{ some } \mu\text{-test covers } x].$$

'y represents a measure' is obviously arithmetic, and due to compactness of 2^ω, a measure is μ is continuous if and only if

$$(\forall n)(\exists l)(\forall \sigma)[|\sigma| = l \rightarrow \mu(N_\sigma) \leq 2^{-n}].$$

Hence 'μ is continuous' is arithmetic in y. Furthermore 'some test covers x' can be expressed as

$$(\exists e)[(\forall n)[(\forall s) \sum_{\sigma \in W_{\{e\}^y_s(n)}} p_c(y) N_\sigma \leq 2^{-n} \ \wedge \ (\exists \sigma)[\sigma \in W_{\{e\}^y_s(n)} \ \wedge \ \sigma \sqsubset x]],$$

which is arithmetic in x and y. □

Furthermore, it is not hard to see that NCR does not have a perfect subset.

Proposition 5.5. NCR *does not have a perfect subset.*

Proof. Assume $X \subseteq \mathrm{NCR}$ is a perfect subset represented by a perfect tree T_X with $[T_X] = X$. We devise a measure μ by setting $\mu(N_\epsilon) = 1$, and define inductively

$$\mu(N_{\sigma^\frown i}) = \begin{cases} \mu(N_\sigma) & \text{if } \sigma^\frown(1-i) \notin T, \\ \frac{1}{2}\mu(N_\sigma) & \text{otherwise.} \end{cases}$$

i.e. we distribute the measure uniformly over the infinite paths through T.

Obviously, μ is continuous, and since all its mass is concentrated on X (x is the *support* of μ, i.e. the smallest closed set whose complement has measure zero), X must contain a μ-random real. (The set of μ-random reals is always a set of μ-measure 1.) □

The *Perfect Subset Property* refers to the principle that every set in a pointclass is either countable or contains a perfect subset. The Perfect Subset Property for $\mathbf{\Pi}^1_1$ is not provable in ZFC. Gödel's showed that if $V = L$, then there exists an uncountable Π^1_1 set without a perfect subset. Mansfield [15] and Solovay [26] showed that any Σ^1_2 set without a perfect subset is contained in the constructible universe L.

Corollary 5.6. NCR *is contained in Gödel's constructible universe L.*

The upper bound L appears indeed very crude, and an analysis of the proof technique of Theorem 4.1 together with a recent result by Woodin [27] yields the countability of NCR. Even more, Corollary 5.3 is in some sense optimal: Every real outside Δ^1_1 is random with respect to some continuous measure.

In the proof of Theorem 4.1, the decisive property which ensured the non-trivial μ-randomness of x was (4.1):

$$\mathcal{L}(N_{\mathrm{Pre}(\sigma)}) \leq \mu(N_\sigma) \leq \mathcal{L}(N_{\Psi(\sigma)}).$$

Here, the second inequality guarantees that μ is not a μ-atom. Although we know $\mu(\sigma)$ will converge to 0 if we consider longer and longer initial segments $\sigma \sqsubset x$, this has not to be the case for reals other than x on which $\Phi \circ \Psi$, as we are dealing with Turing reductions. If, however, our reductions are total, i.e. *truth-table*, we have that, we can show that the resulting measure μ will be continuous. We include this observation in the following "recursion theoretic" characterization of continuous randomness.

Proposition 5.7. *Let x be a real. For any $z \in 2^\omega$, the following are equivalent.*

(i) x is random relative to z for a continuous measure μ recursive in z.
(ii) x is random relative to z for a continuous dyadic measure ν recursive in z.
(iii) There exists a functional Φ recursive in z which is an order-preserving homeomorphism of 2^ω such that $\Phi(x)$ is Martin-Löf random relative to z.
(iv) x is truth-table equivalent relative to z to a Martin-Löf random real relative to z.

Here *dyadic* measure means that the measure is of the form $\mu(N_\sigma) = m/2^n$ with $m, n \in \mathbb{N}$. The theorem can be seen as an effective version of the *classical isomorphism theorem* for continuous probability measures (see for instance [8]).[1]

Proof. We give a proof for $z = \emptyset$. It is easy to see that it relativizes.

(i) ⇒ (ii): Let x be μ-random. We show how we can, recursively in μ, change μ into a dyadic measure ν such that x is random with respect to ν, too. The construction is similar to Schnorr's rationalization of martingales [24]. We define ν

[1] The theorem suggests that for continuous randomness representational issues do not really arise, since there is always a measure with a computationally minimal representation.

by recursion on the full binary tree $2^{<\omega}$. Initially, let $\nu(\epsilon) = 2$. Given $\nu(\sigma)$, choose $\nu(\sigma^\frown 0)$ and $\nu(\sigma^\frown 1)$ as dyadic rationals such that

$$\mu(\sigma^\frown i) < \nu(\sigma^\frown i) \leq \mu(\sigma^\frown i) + 2^{-(|\sigma|+1)}, \qquad i \in \{0, 1\},$$

and

$$\nu(\sigma^\frown 0) + \nu(\sigma^\frown 1) = \nu(\sigma).$$

This is possible since the dyadic rationals are dense in $\mathbb{R}$. Finally, normalize by $\nu := \nu/2$. It is clear from the construction that for all σ, $\mu(\sigma) < 2\nu(\sigma)$. Hence, any test V for ν can be effectively transformed into a test W for μ covering the same reals by letting $W_n = V_{n+1}$.

(ii) $\Rightarrow$ (iii): Suppose ν is a continuous dyadic measure. By the construction in (i) $\Rightarrow$ (ii) we may also assume that $\nu(\sigma) > 0$ for all σ. Since ν is continuous, for every m there exists an l_m such that whenever $|\sigma| \geq l_m$, then $\nu(\sigma) \leq 2^{-m}$. Wlog we can assume that $l_m < l_{m+1}$. It is clear that l_m can be obtained recursively in ν. We define inductively a mapping $\phi : 2^{<\omega} \to 2^{<\omega}$ that will induce the desired homeomorphism. Let $\phi(\epsilon) = \epsilon$. Given τ, let

$$E_\tau = \{\sigma : |\sigma| = l_{|\tau|+1} \;\&\; \text{ is compatible with } \phi^{-1}(\tau)\}.$$

Find the least (with respect to the usual lexicographic ordering) $\sigma \in E_\tau$ such that

$$\sum_{\substack{\eta \leq \sigma \\ \eta \in E_\tau}} \nu(\eta) \geq \nu(\phi^{-1}(\tau))/2.$$

Map all strings $\eta \in E_\tau$, $\eta \leq \sigma$, to $\tau^\frown 0$, and map the remaining strings of E_τ to $\tau^\frown 1$. Let Φ the continuous mapping $2^\omega \to 2^\omega$ induced by ϕ. Obviously, Φ is recursive in ν. The continuity of ν ensures that Φ is total and one-one, while the lack of intervals of zero ν-measure yields that Φ is onto. Finally, the direct nature of the construction makes it easy to see that we can (recursively in ν) invert Φ.

It remains to show that $\Phi(x)$ is Martin-Löf random. Suppose not, then there exists an $\mathcal{L}$-test W that covers $\Phi(x)$. We claim that $V_n = \phi^{-1}(W_{n+1})$ is an ν-test that covers x. Since $\nu \leq_{\mathrm{T}} z$, V_n is uniformly enumerable in $\nu^{(n-1)}$. By construction we have that for any $\tau \in 2^{<\omega}$, $|\mu_\Phi(\tau) - \mathcal{L}(\tau)| \leq 2^{-|\tau|}$. Furthermore,

$$\sum_{\sigma \in V_n} \mu(\sigma) = \sum_{W_{n+1}} \mu_\Phi(\tau) \leq 2 \sum_{W_{n+1}} 2^{-|\tau|} \leq 2^{-n},$$

thus x is not ν-random, contradiction.

(iii) $\Rightarrow$ (iv): This is immediate.

(iv) $\Rightarrow$ (i): Suppose $x \equiv_{\mathrm{tt}} R$ via computable total functionals Φ, Ψ such that $\Phi(R) = x$ and $\Psi(x) = R$. Define a computable measure μ on the cylinders N_σ by letting μ equal the image measure $\mathcal{L}_\Phi$. If we discover that some element in the preimage of σ under Φ is incompatible with $\Psi(\sigma)$, we start distributing this contribution uniformly among all extensions of σ. This way we guarantee that μ is continuous. (Compare with the proof of Theorem 4.1.) x is random for μ, because we could lift any μ-test (V_n) that covers x to a $\mathcal{L}$-test (U_n) covering R, since we can effectively recognize the elements in preimage of some σ enumerated into V_n that are incompatible with Ψ, and which therefore do not have to be enumerated into U_n. The measure of the remaining elements in the preimage is bounded by $\mu(N_\sigma)$.

□

Woodin showed that outside the hyperarithmetical hierarchy the Posner-Robinson Theorem holds with truth-table equivalence.

Theorem 5.8 (Woodin [27])**.** *If $x \in 2^\omega$ is not hyperarithmetic, then there is a $z \in 2^\omega$ such that $x \equiv_{\mathrm{tt}(z)} z'$.*

It follows immediately that every member of NCR is hyperarithmetic.

Theorem 5.9. *If a real x is not hyperarithmetic, then there exists a continuos measure μ such that x is μ-random.*

Theorem 5.9 yields an interesting measure-theoretic characterization of Δ^1_1. The result can also be obtained via a game-theoretic argument using Borel determinacy, along with a generalization of the Posner-Robinson Theorem via Kumabe-Slaman forcing. This is part of a more general argument which shows that for all n, the set NCR_n of reals which are not n-random for some continuous measure is countable. Here n-random means that a test can resort to the $(n-1)$st jump of the measure. The countability result has an interesting metamathematical twist. This work be presented in a separate paper [22].

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