Vertex algebras, Kac-Moody algebras, and the Monster.

Proc Natl. Acad. Sci. USA Vol. 83, pp 3068-3071
Richard E. Borcherds,
Trinity College, Cambridge CB2 1TQ, England.

Communicated by Walter Feit, December 13, 1985
ABSTRACT It is known that the adjoint representation of any Kac-Moody algebra A can be identified with a subquotient of a certain Fock space representation constructed from the root lattice of A. I define a product on the whole of the Fock space the restricts to the Lie algebra product on this subquotient. This product (together with an infinite number of other products) is constructed using a generalization of vertex operators. I also construct an integral form for the universal enveloping algebra of any Kac-Moody algebra that can be used to define Kac-Moody groups over finite fields, some new irreducible integrable representations, and a sort of affinization of any Kac-Moody algebra. The "Moonshine representation of the Monster constructed by Frenkel and others also has products such as the ones constructed for Kac-Moody algebras, one of which extends the Griess product on the 196884-dimensional piece to the whole representation.

Section 1. Introduction.

Let A be any Kac-Moody algebra all of whose real roots have norm 2.(Everything here can be generalized to all Kac-Moody algebras but becomes a lot more complicated, so for simplicity I will mostly just describe this case.) A is defined by certain generators and relations depending on the Cartan matrix of A, and one of the most important problems about Kac-Moody algebras is to find a more explicit realization of A. This has been done only when A is finite dimensional or when A is affine, in which case A can be realized as a central extension of a twisted ring of Laurent series in some finite dimensional Lie algebra. Here we will construct a realization of an algebra that is usually slightly larger than A and is equal to A if A is finite dimensional or affine (in which case it is equivalent to the usual realization of A). For any even integral lattice R (for example, the root lattice of A), we will first construct a (well known) Fock space $V = V(R)$. Physicists have defined "vertex operators" for every element of R, which map V to the space $V\{z, z^{-1}\}$ of formal Laurent series in V, and the coefficients of these operators map V to V. I define a sort of generalized vertex operator for every element of V instead of just for elements of R. This operator is written as $:Q(u,z):(v)$ for u, v in V, and its coefficients are written as $u_n(v)$ for u, v in V and integers n. These products on V are not associative, commutative or skew commutative but satisfy several more complicated identities.

The product $u_0(v)$ is not a Lie algebra product on V, but it is a Lie algebra product on V/DV, where DV is the image of V under a certain derivation D. This Lie algebra V/DV contains the Kac-Moody algebra A as a subalgebra but is always far larger than A. To reduce V/DV to a smaller subalgebra, we will use the Virasoro algebra. This is spanned by the operators c_i and 1, where c is a certain element of V. The commutator of this algebra in V/DV also contains A and is not much larger than A; for example, we can calculate bounds on the dimensions of the root spaces of A from this that are sometimes the best possible.

If V is the infinite-dimensional representation of the monster constructed by Frenkel *et al.* (1) then V also has products $u_n(v)$ that satisfy several identities.

Section 2. Construction of the Fock space V.

In this section, we recall the construction of a certain Fock space V from an even lattice R and put several structures on V, such as a product, a derivation, and an inner product (see ref. 2).

For any even lattice R there is a central extension

$$0 \longrightarrow Z_2 \longrightarrow \hat{R} \longrightarrow R \longrightarrow 0$$

where Z_2 is a group of order 2 generated by an element ϵ and $\hat{R}$ has an element e^r for every element r of R, such that $e^r e^s = \epsilon^{(r,s)} e^s e^r$ and $e^r e^{-r} = \epsilon^{(r,r)}/2$. $\hat{R}$ is uniquely defined up to isomorphism by these conditions, and the automorphism group of $\hat{R}$ is an extension $Z_2^{\dim R} \cdot \mathrm{Aut}(R)$ (usually nonsplit). If R is the root lattice of a Kac-Moody algebra A, then $\mathrm{Aut}(R)$ is not usually a subgroup of $\mathrm{Aut}(A)$ in any natural way, but $\mathrm{Aut}(\hat{R})$ is, as we can prove by constructing A from $\hat{R}$

The Fock space V is a rational vector space given by the tensor product

$$Q(R) \otimes S(R(1)) \otimes S(R(2)) \ldots .$$

Here $Q(R)$ is the rational group algebra of $\hat{R}$ quotiented out by $\epsilon + 1$, so it has a basis of e^r for r in R and $e^r e^s = (-1)^{(r,s)} e^s e^r$. $R(i)$ is a copy of the rational vector space of R, and its elements are written $r(i)$ for r in R. $S(R(i))$ is the symmetric algebra on $R(i)$. A typical element of V might be $e^r s(1)^3 t(4)$ for r, s, t in R.

V has the following structures.

(i) V is an algebra as each of the pieces of the tensor product defining V is. V would be commutative except that e^r and e^s do not always commute.

(ii) There are linear maps D and deg from V to V such that $De^r = r(1)e^r$, $Dr(i) = ir(i+1)$, and D is a derivation. $\deg(e^r) = \frac{1}{2}(r,r)e^r$, $\deg r(i)v = r(i)(iv + \deg v)$. If $\deg u = iu$ we can say that u has degree i. We write $D^{(i)}$ for the operator $D^i/i!$.

(iii) V has a Cartan involution ω. ω acts on R by $\omega(e^r) = e^{-r}$, and this becomes an automorphism of V with $\omega(e^r) = e^{-r}$, $\omega(r(i)) = -r(i)$.

(iv) We define the operators $r(i)$ on V for r in R and integers i as follows: If $i > 0$ then $r(i)$ is multiplication by $r(i)$. If $i = 0$ then $r(i)e^s = (r,s)e^s$. If $i < 0$ then $r(i)e^s = 0$. $[r(i), s(j)] = j(r,s)$ if $i = -j$, 0 otherwise. (These properties characterize the operators $r(i)$.)

(v) V has a unique inner product $(,)$ such that the operator $r(i)$ is the adjoint of $r(-i)$, and $(e^r, e^s) = 1$ if $r = s$, 0 otherwise.

(vi) The integral form V_Z of V is defined to be the smallest subring of V containing all the e^r and closed under $D^{(i)}$ for $i \geq 0$. This integral form is compatible with all the structures above; i.e., it is preserved by the Cartan involution ω and the operators $r(i)$, and the inner product is integral on it. It is generated as a ring by e^r, $r(1)$, $(r(2) + r(1)^2)/2$,

$(2r(3)+3r(2)r(1)+r(1)^3)/6\ldots$, which are Schur polynomials in $r(1)$, $r(2)/2$, $r(3)/3\ldots$. If W is the sublattice of V_Z of elements of "R grading" 0 and degree i, then the determinant of W is an integral power of the determinant of R and, in particular, if R is unimodular then so is W. (V is graded by the lattice R by letting e^r have degree r and letting $r(i)$ have degree 0.)

Section 3. Vertex Operators.

For each u in V we will define a map u from V to the ring of formal Laurent series $V\{z, z^{-1}\}$. If u is of the form e^r then these operators are just vertex operators, and if u is a product of $r(i)$s then these operators have been constructed by Frenkel (3).

We can define $Q(r, z)$ to be the formal expression

$$\sum_{i\neq 0} r(i)z^i/i + r(0)\log(z) + r$$

and define $Q(r(i), z)$ for $i \geq 1$ to be $(d/dz)^i Q(r, z)/(i-1)!$. If $u = e^r \prod_i r_i(n_i)$ is an element of V then we define $Q(u, z)$ to be the formal expression

$$e^{Q(r,z)} \prod_i Q(r_i(n_i), z).$$

This is not an operator from V to $V\{z, z^{-1}\}$ as it does not converge, but we can make it into an operator by "normal ordering" it. This means that in each term of the formal expression $Q(u, z)$ we rearrange all terms e^r and $r(i)$ so that the "creation operators" e^r and $r(i)$ ($i \geq 1$) occur to the left of all "annihilation operators" $r(i)$ ($i \leq 0$). Note that all annihilation operators commute with each other, and so do all creation operators except for e^r and e^s. The normal ordering of $Q(u, z)$ is denoted by $:Q(u, z):$, and this is a well-defined operator from V to $V\{z, z^{-1}\}$. We define $u_n(v)$ for u, v in V and integers n by

$$u_n(v) = \text{the coefficient of } z^{-n-1} \text{ in } :Q(u, z):(v).$$

If u and v are in the integral form of V then so is $u_n(v)$. If u and v have degrees i, j then $u_n(v)$ has degree $i + j - n - 1$. The operator $r(i)$ is equal to $r(1)_{-i}$.

Section 4. Vertex Algebras.

We will list some identities satisfied by the operators u_n and show how to construct Lie algebras from them. u, v, and w denote elements of V, and 1 is the unit of V.

For any even lattice R the operators u_n on V satisfy the following relations.

(i) $u_n(w) = 0$ for n sufficiently large (depending on u, w). This ensures convergence of the following formulae.

(ii) $1_n(w) = 0$ if $n \neq -1$, w if $n = -1$.

(iii) $u_n(1) = D^{(-n-1)}(u)$.

(iv) $u_n(v) = \sum_{i\geq 0}(-1)^{i+n+1} D^{(i)}(v_{n+i}(u))$.

(v) $(u_m(v))_n(w) = \sum_{i\geq 0}(-1)^i \binom{m}{i}(u_{m-i}(v_{n+i}(w)) - (-1)^m v_{m+n-i}(u_i(w)))$.

(The binomial coefficient $\binom{m}{i}$ is equal to $m(m-1)\ldots(m-i+1)/i!$ if $i \geq 0$ and 0 otherwise.)

We will call any module with linear operators $D^{(i)}(u)$ and bilinear operators $u_n(v)$ satisfying relations *i*-*v* above a vertex algebra, so V is a vertex algebra. (When we work with Kac-Moody algebras that so not have all real roots of norm 2, we can also construct a space V and operators u_n; however n is not always integral, u lies in a subspace of V depending on n, and u_n acts on a space that is different from V.)

Another example of a vertex algebra is given by taking any ring with a derivation [i.e., maps $D^{(i)}$ with $D^{(i)} = 0$ for $i < 0$, 1 for $i = 0$,

$$D^{(i)}D^{(j)} = \binom{i+j}{i} D^{(i+j)}$$

$$D^{(i)}(uv) = \sum_j D^{(j)}(u)D^{(i-j)}(v)]$$

and defining $u_n(v)$ to be $D^{(-n-1)}(u)v$. This satisfies conditions *i*-*iii* and *v* and satisfies condition *iv* if and only if the ring is commutative. It also satisfies $u_n(v) = 0$ if $n \geq 0$, and conversely and vertex algebra satisfying this comes from a unique ring with derivation. Hence vertex algebras are a generalization of commutative rings with derivations.

A module over a vertex algebra V is a module W with operators u_n on W satisfying relations *i*-*v* above for u, v in V, w in W. In particular V is a V module. (Warning—if V comes from a ring with derivation then vertex algebra modules over V are not the same as ring modules over V.)

If V is any vertex algebra then V/DV is a Lie algebra, where DV is the sum of all the spaces $D^{(i)}(V)$ for $i \geq 1$ and where the Lie algebra product is $[u, v] = u_0(v)$. Note that $u_0(v)$ is not antisymmetric on V. Any V module W becomes a module for the Lie algebra V/DV by letting v in V/DV act as v_0 on W. (If v is in DV then v_0 is 0.) In particular V is a V/DV module and is usually a nonsplit extension of the adjoint representation of V/DV. The operators $D^{(i)}$ and the products $u_n(v)$ on V are invariant under the action of V/DV. (V/DV can be extended to a larger Lie algebra $V[z, z^{-1}]/DV[z, z^{-1}]$ of operators on V that is spanned by all the operators u_n, but this algebra does not leave the products $u_n(v)$ invariant; see Section 8.)

The free vertex algebra on some set of generators does not exist because of relation *i*. However if for each pair of generators u, v we fix an integer $n(u, v)$ and include the relations $u_i(v) = 0$ for $i \geq n(u, v)$ then there is a universal vertex algebra with these generators and relations. It can be constructed as a subalgebra of the vertex algebra $V(R)$ for a certain lattice R depending on the $n(u, v)$s, and in particular any relation between the operators u_n that holds for all the vertex algebras constructed from lattices can be deduced from relations *i*-*v*.

Section 5. The Virasoro Algebra.

We will construct a representation of the Virasoro algebra on V using some operators c_n, which are the Segal operators, and use this to reduce the space V/DV.

We assume that R is nonsingular, and we let c be the element $\frac{1}{2}\sum_i r_i(1)r_i'(1)$ of V, where r_i runs over some base of R and r_i' is the dual base. We write L_i for the operator c_{i+1}, and we find that the L_i have the following properties:

$$L_{-1} = D, \qquad L_0 = \deg$$
$$[L_i, L_j] = (i-j)L_{i+j} + (i^3 - i)\dim(R)\delta_{i,-j}/12$$
L_{-i} is the adjoint of L_i.

In particular the operators L_i and 1 span a copy of the Virasoro algebra. If R is a (possibly singular) lattice contained in a nonsingular lattice S, then the operators L_i for $i \geq -1$ on the vertex algebra of S restrict to operators on the vertex algebra of R that do not depend on the lattice S containing R. In particular if $i \geq -1$ then the operator L_i can be defined on the vertex algebra V of R even when R is singular. We define the physical subspace P^i to be the elements v of V with $L_n(v) = iv$ if $n = 0$, 0 if $n \geq 1$. If v is in P^1 then the operator v_0 commutes with the Virasoro algebra so it preserves all the spaces P^i. If v in P^1 is equal to Du for some u in V then u is in P^0, so P^1/DP^0 is a Lie algebra acting on V and commuting with the Virasoro algebra. More generally if u is in P^1 then

$$[L_j, u_k] = ((j+1)(i-1) - k)u_{j+k}.$$

If R contains the root lattice of the Kac-Moody algebra A' (possibly quotiented out by some null lattice) then A' can be mapped to P^1/DP^0 by

$$e_i = e^{r_i}$$
$$f_i = -e^{-r_i}$$
$$h_i = r_i(1).$$

Here r_i are the simple roots of A, and e_i, f_i, and h_i are the usual generators for the derived algebra A' of A. It is easy to check that these elements are in P^1 and satisfy the relations for A', so we obtain a representation of A'. If the root lattice of A' is singular, we can either quotient out by the kernel of the bilinear form on it, in which case we will not obtain a faithful representation of A', or embed it in a nonsingular lattice R, in which case some of the elements $r(1)$ will be outer derivations of A'.

If r is any nonzero vector of R the then dimension of the r subspace of P^1 or P^0 is $p_{d-1}(1-(r,r)/2)$ or $p_{d-1}(-(r,r)/2)$, where d is the dimension of R and p_{d-1} is the number of partitions into $d-1$ colors. Hence the dimension of the r subspace of P^1/DP^0 is equal to $p_{d-1}(1-(r,r)/2) - p_{d-1}(-(r,r)/2)$, and this is an upper bound for the multiplicity of roots of A (provided A is connected and not affine so that the kernel of the map from A' to P^1/DP^0 is in the center of A').

Example: If R is the 18-dimensional even unimodular Lorentzian lattice $II_{17,1}$ and A is the Kac-Moody algebra whose Dynkin diagram is that of R, then A has roots of norm 2, 0, -2, and -4 whose multiplicity is equal to the upper bound given above. However there are several Kac-Moody algebras for which numerical evidence suggests the better upper bound $p_{d-2}(1-(r,r)/2)$ for the multiplicities of roots (3). Frenkel used the no-ghost theorem to prove this stronger upper bound when R is 26 dimensional and Lorentzian. In this case P^1/DP^0 has a normal subalgebra such that the quotient by this subalgebra is a simple algebra with root spaces of dimension $p_{d-2}(1-(r,r)/2)$ for $r \neq 0$.

If the lattice R is odd we can use it to construct a "super vertex algebra" V acted on by a super Virasoro algebra spanned by elements 1, L_i, and $G_{i+1/2}$. The space

$G_{1/2}P^{1/2}/DP^0$ is then a Lie algebra. (Not a proper superalgebra!) For example if R is $I_{9,1}$ then $G_{1/2}P^{1/2}/DP^0$ has a normal subalgebra such that the quotient by this subalgebra is a simple Lie algebra with root spaces of dimensions equal to the coefficient of $x^{(1-(r,r))/2}$ in $\prod_{i\geq 1}(1-x^{i-1/2})^8(1-x^i)^{-8}$. This simple algebra contains the Kac-Moody algebra that has a simple root r' for every vector r of the lattice E_8, with $(r',s') = 1-\frac{1}{2}(r-s,r-s)$. (This is similar to the "monster Lie algebra", which has a simple root r' for every vector r of the Leech lattice, with $(r',s') = 2-\frac{1}{2}(r-s,r-s)$.)

The operator L_1 can be used to describe the adjoint of u_n: if u has degree i then the adjoint of u_n is

$$(-1)^i \sum_j L_1^j(\omega(u))_{2i-j-n-2}/j!$$

In particular if u is in P^1 then the adjoint of u_0 is $-\omega(u)_0$, so the adjoint of e_i is f_i and the adjoint of h_i is $-h_i$.

Section 6. The Representations $L(r)$.

If r is any element of the weight lattice R' of R we construct an irreducible A module whose largest weight is r, and these representations generalize the highest weight and adjoint representations of A.

We first assume that r is in R. We take the space P^i with $i = (r,r)/2$. This has a maximal graded submodule not containing e^r, and if we quotient out by this we get an irreducible module that we denote by $L(r)$. $L(r)$ has the following properties.

(i) $L(r)$ is irreducible.

(ii) The weight r has multiplicity 1, and if $(s,s) > (r,r)$ then s has multiplicity 0. This implies that $L(r)$ is integrable, so in particular if s and t are conjugate under the Weyl group they have the same multiplicity.

(iii) $L(r)$ has a nonzero contravariant bilinear form, which is unique up to multiplication by a constant. (This is not necessarily positive definite unless r is a highest or lowest weight vector.)

(iv) All weights of $L(r)$ have finite multiplicities. (I do not know of any formula for the multiplicities except in the cases below.)

(v) If r is a highest or lowest weight vector then $L(r)$ is the corresponding highest or lowest weight module, and if r is a real root of A then $L(r)$ is a quotient of the adjoint representation (and equal to A' modulo its center if this is simple).

(vi) If r and s are conjugate under the Weyl group then $L(r) = L(s)$. (The converse is not true; for example r and s could be two real roots of A in different orbits of $\mathrm{Aut}(R)$.)

If r is an element of R' not in R then the construction above does not work because e^r is not in V, so we construct the space V_r by replacing $Q(R)$ in the tensor product defining V by $Q(R+r)$. All the operators u_n for u in V act on V_r, and we can construct $L(r)$ as a subquotient of V_r as above.

Problem: Is $L(r)$ the only A module satisfying conditions *i* and *ii* above? (It is if r is a highest or lowest weight vector, and in this case condition *ii* implies condition *i*.)

Section 7. Integral forms for Kac-Moody algebras.

We constructed an integral form V_Z for V in Section 3. Here we will use this to find an integral form for the universal enveloping algebra for the Kac-Moody algebra A.

For each r in the weight lattice R' we define the integral form $L_Z(r)$ of $L(r)$ to be the elements of $L(r)$ represented by elements in the integral form of V_r. (If r is not in R then the integral form of V_r is $e^r V_Z$.) Similarly the integral form A_Z of A is the set of elements of A represented by elements of V_Z. This acts on all the $L_Z(r)$ because u_n preserves the integral form of V_r Finally we define the integral form U_Z of the universal enveloping algebra U of A to be the subalgebra of U preserving all the $L_Z(r)$s. Calculation shows that U_Z contains $(e_i)^n/n!$ and $(f_i)^n/n!$ for all integers $n \geq 0$ where the es and fs are the generators for A. We can therefore use U_Z to define Kac-Moody groups over finite fields (or over any commutative ring) in the same way that Chevalley groups are defined, by using the automorphisms $\exp(te_i)$ and $\exp(tf_i)$ of $L_Z(r) \otimes F$ for t in the finite field F.

The element c of V is not usually in V_Z, but V_Z can be extended to a larger integral form containing $2c$ and containing c if $\dim(R)$ is even. In any case the operators L_i for $i \geq -1$ and $L_{\pm 1}^n/n!$ preserve V_Z. (Warning—these operators do not preserve the integral form of V_r for r not in R.)

Tits has also constructed an integral form for Kac-Moody algebras (4).

Section 8. Affinization.

If A is a Kac-Moody algebra with root lattice R we can construct a sort of affinization of A, which when A is finite dimensional is just the affine algebra of A When A is finite dimensional the affinization is also a Kac-Moody algebra, but this is not true in general.

To construct the affinization we form the lattice R_1 that is the sum of R and a one-dimensional lattice generated by s with $(s,s)=0$ and let V_1, P_1^i be the Fock space and physical spaces of R_1. Then we define the affinization $\bar{A}$ of A to be the subalgebra of P_1^1/DP_1^0 generated by the elements $e^{ns \pm r_i}$ where n runs through the integers, and the rs are the simple roots of A.

$\bar{A}$ is an extension $N.A[z,z^{-1}]$ of an algebra N with an infinite descending central series by the algebra of Laurent polynomials in A. When A is finite dimensional and simple, N is one dimensional and we recover the usual affinization of A. If R_2 is the lattice that is the sum of R and a lattice generated by s, t with $(s,s)=(t,t)=0$, $(s,t)=1$, then $\bar{A}$ has many representations that can be constructed as subquotients of the subspaces P_2^i of V_2.

There is a second way to construct the affinization of A. If V is any vertex algebra we make $W = V[z,z^{-1}]$ into a vertex algebra by defining

$$uz_i^m(vz^n) := \sum_j \binom{m}{j} u_{i+j}(v) z^{m+n-j},$$

and we make V into a W module by defining

$$uz_i^m(v) := u_{i+m}(v).$$

(This is a special case of the tensor product of two vertex algebras acting on the tensor product of two of their modules; in this case the vertex algebras are V and the vertex algebra of the ring $Z[z,z^{-1}]$ with derivation $D^{(i)}z^j = \binom{j}{i} z^{j-i}$, and their modules are V

and a one-dimensional module generated by an element 1 with $z_i^j(1) = 1$ if $i + j = -1$, 0 otherwise. Note that this one-dimensional module is not a module for the ring $Z[z, z^{-1}]$.) The affinization of A is then a subalgebra of the Lie algebra W/DW, and this Lie algebra acts on V. In particular we obtain a formula for the commutator of two operators u_m and v_n on V:

$$[u_m, v_n] = \sum_j \binom{m}{j} u_j(v)_{m+n-j}.$$

If V is constructed from a lattice R and u is in $W = V[z, z^{-1}]$ then $Du = 0$ if and only if u is a multiple of 1, and the operator u_0 on V is 0 if and only if $u = Dv$ for some v in W. If u is in V then this holds if and only if $u = Dv + a$ for some $v \in V$ and some constant a.

V is usually irreducible under the action of $\bar{A}$.

Section 9. The monster.

Frenkel *et al.* (1) constructed an infinite-dimensional graded representation V of the monster F. This representation can be given the structure of a vertex algebra that is invariant under F and is similar to the vertex algebras constructed from positive definite lattices (or more precisely to the subspace of the complexification of such algebras fixed by the Cartan involution ω—i.e., their "compact forms"). In particular it has an element c such that the operators $L_i = c_{i+1}$ give a representation of the Virasoro algebra and it has a positive definite inner product such that the adjoint of u_n is given by the formula in *Section 5* (with $\omega(u) = u$). One important difference between this algebra and the ones coming from lattices is that the piece of degree 1 is 0 dimensional. We will call this vertex algebra V the Monster vertex algebra.

Any vertex algebra V with these properties (except that it does not have to have an action of the monster F on it) also has the following two properties for u and v in the degree 2 piece of V.

(i) $u_1(v) = v_1(u)$. When V is the Monster vertex algebra this is essentially the Griess product (5). Also u_1 is self adjoint.

(ii) Norton's inequality (see ref. 6): $x = (u_1(u), v_1(v)) - (u_1(v), u_1(v))$ is nonnegative and zero if and only if the operators u_1 and v_1 commute. In fact x is the norm of $w = u_0(v) - \frac{1}{2}D(u_1(v))$, and the operator w_2 is the commutator of u_1 and v_1.

There are a large number of other vertex algebras with these properties.

In particular the Griess product can be extended to the whole of V in a natural way, and there are also an infinite number of other products $u_n(v)$ on V invariant under the action of the Monster on V.

The product $u_1(v)$ is not symmetric on the whole of V. If we want symmetric products we can define the product $\times_n$ for any integer n by

$$u \times_n v = \sum_{i \geq 0} \frac{(-1)^i}{i+1} D^{(i)}(u_{n+1+i}(v))$$

and these are symmetric or antisymmetric depending on whether n is even or odd. If $n = 0$ this is equal to the Griess product $u_1(v)$ on the degree 2 piece of V, and $D(u \times_0 v) =$

$u_0(v)+v_0(u)$. These products have these properties for any vertex algebra over the rational numbers but do not seem to be as natural as the products $u_n(v)$.

This research was supported by Trinity College, Cambridge, England and by the Natural Sciences and Engineering Research Council of Canada.

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