Reflection groups of Lorentzian lattices.   *21 September 1999.*

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Contents.

1. Introduction.

The aim of this paper is to provide evidence for the following new principle: interesting reflection groups of Lorentzian lattices are controlled by certain modular forms with poles at cusps. We use this principle to explain many of the known examples of such reflection groups, and to find several new examples of reflection groups of Lorentzian lattices, including one whose fundamental domain has 960 faces.

We do not give a precise definition of what it means for a reflection group of a Lorentzian lattice to be interesting, mainly because there seem to be occasional counterexamples to almost any precise version of the principle. However the interesting groups should include the cases when the reflection group is cofinite, or more generally when the quotient of the full automorphism group by the reflection group contains a free abelian subgroup of finite index.

The main idea of this paper is roughly as follows (and is described in more detail in section 11). Suppose that L is an even level N Lorentzian lattice in $\mathbf{R}^{1,n}$. Then the idea is that if L has an interesting reflection group, there should be a modular form of weight $\frac{1-n}{2}$ and level N with certain rather mild singularities (called reflective singularities) at cusps. The singular theta correspondence of [B98a] associates a piecewise linear function on the

hyperbolic space of $\mathbf{R}^{1,n}$ to this modular form, and the singularities of this piecewise linear function should be reflection hyperplanes of the reflection group.

We do not have any sort of proof that interesting Lorentzian reflection groups always correspond to reflective modular forms (or even a good definition of what an interesting group is). However we can still use this rather vague correspondence to find new reflection groups, because it is usually easy to list examples of reflective forms using the results in the first 10 sections. In section 12 we use this to give many examples of reflection groups corresponding to reflective modular forms. In particular we show that most of the known examples in dimensions at least 5 can be found by systematically searching for reflective forms of small levels. (This does not seem to work so well for Lorentzian lattices of dimension 3 and possibly 4; Nikulin has found many examples that do not obviously correspond to reflective forms.) This gives some sort of structure to the collection of all nice Lorentzian reflection groups, which previously looked like a miscellaneous collection of unrelated examples found by half a dozen assorted methods. We also find many new examples of Lorentzian lattices with interesting reflection groups, including one whose fundamental domain has 960 faces (comfortably beating the previous record of 210 faces). The main problem with these examples is that there are almost too many of them: for small composite levels (4, 6, 8, 9) there are so many cases that we do not try to list them all but just give vague indications of how to list many of them. Somewhere round about level 25 the number of examples for each level decreases to a trickle, and above this level we only seem to get a few low dimensional lattices for each level. However there are probably still a few occasional examples when the level is a few hundred.

So reflective modular forms seem to be a useful practical method for finding interesting reflection groups of Lorentzian lattices. On the other hand there are occasional counterexamples to show that it does not always work. There are one or two high dimensional reflection groups which seem well behaved but do not appear to correspond to reflective modular forms, and on the other hand there are occasional Lorentzian lattices with non-zero reflective modular forms whose reflection groups are quite complicated. Moreover, when there is a reflective form, the reflection group of the Lorentzian lattice can have several different behaviors: for examples it might be cofinite, or it might have a free abelian subgroup of finite index, or it might have a space-like Weyl vector, or it might have none of these properties. So the relation between reflection group and reflective forms is useful in practice, but the theoretical side is still rather mysterious. The reader who wishes to tidy up the theory is warned that section 12 contains counterexamples to several plausible simplifying conjectures.

Sections 2 to 10 are mainly a summary of various assorted results that we need, most of which are minor variations of known results. The contents of these sections should mostly be clear from their titles.

Correction. A. Barnard pointed out a mistake in the formula for the Weyl vector in [B98a, theorem 10.4]. In the formula for ρ_z, the terms for $\lambda = 0$ were incorrectly omitted. So the condition $(\lambda, W) > 0$ in the sum should be deleted, and the factor of $\frac{1}{2}$ in front of the sum should be replaced by $\frac{1}{4}$.

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Notation and terminology.

~ A metaplectic double cover of a group.

$\sqrt{}$ The principal value of the square root, with $-\pi/2 < \arg(\sqrt{*}) \leq \pi/2$.

$'$ The dual of a lattice.

A A discriminant form.

A_n The A_n root lattice, or the elements of order n in a discriminant form A.

A^n The n'th powers of elements of A.

Γ A discrete subgroup of $\widetilde{SL}_2(\mathbf{R})$.

Δ The Dedekind delta function η^{24}.

D_n The D_n root lattice.

e_γ An element of a basis of $\mathbf{C}[L'/L]$.

$\mathbf{e}$ $\mathbf{e}(x) = \exp(2\pi i x)$, $\mathbf{e}_n(x) = \exp(2\pi i x/n)$.

E_k An Eisenstein series (see section 10), or the E_k root lattice.

η The Dedekind eta function.

θ_L A theta function of a lattice L.

g The genus of a subgroup of $SL_2(\mathbf{R})$.

$II_{m,n}$ The even unimodular lattice of dimension $m+n$ and signature $m-n$.

L An even lattice.

N The level of a modular form or discriminant form.

q^n $e^{2\pi i n \tau}$ or a discriminant form.

$\mathbf{Q}$ The rational numbers.

$\mathbf{R}$ The real numbers.

ρ_L A representation of $\widetilde{SL}_2(\mathbf{Z})$.

R_j The primitive elliptic element fixing the point j.

sign The signature of a lattice or discriminant form.

SL A special linear group.

τ A complex number with positive imaginary part, or the number of orbits of cusps.

Tr The trace of something.

$T_{a/c}$ The primitive parabolic element fixing the cusp a/c.

$\mathbf{Z}$ The integers.

χ A character. For χ_θ, χ_n, see section 5.

2. Modular forms.

In this section we recall the definition of a vector valued modular form and set up notation for the rest of the paper.

We define $\mathbf{e}(x)$ to be $\exp(2\pi i x)$, and we define $\mathbf{e}_n(x)$ to be $\exp(2\pi x/n)$

Recall that the group $SL_2(\mathbf{R})$ has a (metaplectic) double cover $\widetilde{SL}_2(\mathbf{R})$, whose elements can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$. The multiplication is defined so that the usual formulas for the transformation of modular forms work for half integer weights, which means that

$$(A, f(\cdot))(B, g(\cdot)) = (AB, f(B(\cdot))g(\cdot))$$

for $A, B \in SL_2(\mathbf{R})$ and f, g suitable functions on H. The group $\widetilde{SL}_2(\mathbf{Z})$ is the inverse image of $SL_2(\mathbf{Z}) \subset SL_2(\mathbf{R})$ in $\widetilde{SL}_2(\mathbf{R})$. The group $\widetilde{SL}_2(\mathbf{Z})$ is generated by S and T where $T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$ and $S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right)$, with $S^2 = (ST)^3 = Z$, $Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, i \right)$. The center is cyclic of order 4 and is generated by Z. The quotient by Z^2 is the group $SL_2(\mathbf{Z})$.

Suppose that Γ is a discrete subgroup of $\widetilde{SL}_2(\mathbf{R})$ that contains Z and is co-finite (this means that the quotient space has finite volume). Suppose that ρ is a representation of Γ on a finite dimensional complex vector space V_ρ. Choose $k \in \mathbf{Q}$. We define a modular form of weight k and type ρ to be a holomorphic function f on the upper half plane H with values in the vector space V_ρ such that

$$f \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k \rho \left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \right) f(\tau)$$

for elements $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right)$ of Γ. The expression $(c\tau + d)^k$ means of course $\sqrt{c\tau + d}^{2k}$ with the principal value of $\sqrt{\ }$. (We allow singularities at cusps.)

A modular form has a Fourier expansion at the cusp at infinity as follows. The Fourier coefficients $c_{n,\gamma} \in \mathbf{C}$ of f are defined by

$$f(\tau) = \sum_{n \in \mathbf{Q}} \sum_\gamma c_{n,\gamma} q^n e_\gamma$$

where q^n means $\mathbf{e}(n\tau)$ and where the sum runs over a basis e_γ of V_ρ consisting of eigenvectors of T. Note that n is not necessarily integral; more precisely, $c_{n,\gamma}$ is nonzero only if $n \equiv \lambda_\gamma \bmod 1$, where the eigenvalue of T on e_γ is $\mathbf{e}(\lambda_\gamma)$. We say that f is meromorphic at the cusp $i\infty$ if $c_{n,\gamma} = 0$ for $n << 0$, and we say f is meromorphic at the cusp a/c if $f((a\tau + b)/(c\tau + d))$ is meromorphic at $i\infty$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$. We say that f is holomorphic at cusps if the coefficients of the Fourier expansions at all cusps vanish for $n < 0$.

3. Discriminant forms and the Weil representation.

In this section we recall the definition of the Weil representation of a discriminant form, and prove some results about it that will be used in section 11.

We let L be a nonsingular even lattice of dimension $\dim(L)$ and signature $\text{sign}(L)$, with dual L'. The quotient L'/L is a finite group whose order is the absolute value of

the discriminant of the lattice L. We define a discriminant form A to be a finite abelian group with a $\mathbf{Q}/\mathbf{Z}$-valued quadratic form $\gamma \mapsto \gamma^2/2$. If L is an even lattice then L'/L is a discriminant form, with the quadratic form of L'/L given by the mod 1 reduction of $\gamma^2/2$, and conversely every discriminant form can be constructed in this way. (For the theory of the discriminant form of a lattice see [N].) This quadratic form on L'/L determines the signature mod 8 of L, by Milgram's formula

$$\sum_{\gamma \in L'/L} \mathbf{e}(\gamma^2/2) = \sqrt{|L'/L|}\mathbf{e}(\operatorname{sign}(L)/8).$$

We define the signature $\operatorname{sign}(A) \in \mathbf{Z}/8\mathbf{Z}$ of a discriminant form to be the signature mod 8 of any even lattice with that discriminant form. We let the elements e_γ for $\gamma \in L'/L$ be the standard basis of the group ring $\mathbf{C}[L'/L]$, so that $e_\gamma e_\delta = e_{\gamma+\delta}$.

A particularly important example ρ_A of a unitary representation of $\widetilde{SL}_2(\mathbf{Z})$, called the Weil representation of the discriminant form A, can be constructed as follows. The underlying space of ρ_A is the group ring $\mathbf{C}[A]$ of A, and the action is defined by

$$\rho_A(T)(e_\gamma) = \mathbf{e}((\gamma,\gamma)/2)e_\gamma$$

$$\rho_A(S)(e_\gamma) = \frac{\mathbf{e}(-\operatorname{sign}(A)/8)}{\sqrt{|A|}} \sum_{\delta \in A} \mathbf{e}(-(\gamma,\delta))e_\delta$$

where S and T are the standard generators of $\widetilde{SL}_2(\mathbf{Z})$. The representation ρ_A factors through the double cover $\widetilde{SL}_2(\mathbf{Z}/N\mathbf{Z})$ of the finite group $SL_2(\mathbf{Z}/N\mathbf{Z})$, where N is a positive integer such that $N\gamma^2/2$ is an integer for all $\gamma \in L'$. The smallest such integer N is called the level of A. In particular the representation ρ_A factors through a finite quotient of $\widetilde{SL}_2(\mathbf{Z})$. If L is an even lattice then we define ρ_L to be the representation $\rho_{L'/L}$.

We summarize some results about discriminant forms from [C-S, chapter 15, section 7]; for more details see [N] or [C-S]. We use a minor variation of the notation of [C-S] for discriminant forms. We recall that every discriminant form can be written as a sum of Jordan components (not uniquely if $p = 2$), and every Jordan component can be written as the sum of indecomposable Jordan components (usually not uniquely). The possible non-trivial Jordan components are as follows. We let $q > 1$ be a power of a prime p and n a positive integer and $t \in \mathbf{Z}/8\mathbf{Z}$. We define $antisquare$ by $antisquare(q^{\pm n}) = 0$ if q is a square or the exponent is $+n$, and $antisquare(q^{\pm n}) = 1$ if q is not a square and the exponent is $-n$. (See [C-S] page 370.)

For q odd the non-trivial Jordan components of exponent q are $q^{\pm n}$ for $n \geq 1$. The indecomposable components are $q^{\pm 1}$, generated by an element γ with $q\gamma = 0$, $\gamma^2 \equiv a/q$ mod 2 where a is an even integer with $\binom{a}{p} = \pm 1$. The component $q^{\pm n}$ is a sum of copies of q^{+1} and q^{-1}, with an even number of copies of q^{-1} if $\pm n = +n$ and an odd number if $\pm n = -n$. These components all have level q. The signature is given by $\operatorname{sign}(q^{\pm n}) = -n(q-1) + 4antisquare(q^{\pm n})$.

For q even the odd Jordan components of exponent q are $q_t^{\pm n}$. If $n = 1$ then $t \equiv \pm 1$ mod 8 if $\pm = +$ and $t \equiv \pm 3$ mod 8 if $\pm = -$. If $n = 2$ then $t \equiv 0, \pm 2$ mod 8 if $\pm = +$

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and $t \equiv 4, \pm 2 \bmod 8$ if $\pm = -$. For any n we have $t \equiv n \bmod 2$. The indecomposable components are $q_t^{\pm 1}$ for $\binom{t}{2} = \pm 1$ and are generated by an element γ with $q\gamma = 0$, $\gamma^2 \equiv t/q \bmod 2$. (Note that some of these are isomorphic to each other.) These components all have level $2q$. The signature is given by $\operatorname{sign}(q_t^{\pm n}) = t + 4antisquare(q^{\pm n})$.

For q even the non-trivial even Jordan components of exponent q are $q^{\pm 2n} = q_{II}^{\pm 2n}$. The indecomposable even Jordan components are $q^{\pm 2}$, which are generated by 2 elements γ and δ with $q\gamma = q\delta = 0$, $(\gamma, \delta) = 1/q$, $\gamma^2 \equiv \delta^2 \equiv 0 \bmod 2$ if $\pm = +$, $\gamma^2 \equiv \delta^2 \equiv 2/q \bmod 2$ if $\pm = -$. These components all have level q. The signature is given by $\operatorname{sign}(q^{\pm n}) = 4antisquare(q^{\pm n})$.

The sum of two Jordan components with the same prime power q can be worked out as follows: we add the ranks, multiply the signs in the exponent, and if any components have a subscript t we add together all subscripts t.

If A is a discriminant form, then we define A_n to be the elements of order n. We define A^n to be the n'th powers of elements of A, so that we have an exact sequence

$$0 \longrightarrow A_n \longrightarrow A \longrightarrow A^n \longrightarrow 0$$

and A^n is the orthogonal complement of A_n. We define A^{n*} to be the set of elements $\delta \in A$ such that $(\gamma, \delta) \equiv n\gamma^2/2 \bmod 1$ for all $\gamma \in A_n$, so that A^{n*} is a coset of A^n. We easily see that A^n is the same as A^{n*} if and only if the Jordan block of type 2^k (where $2^k || n$) is even. In any case, A^{n*} always contains an element δ with $2\delta = 0$.

Lemma 3.1. *Suppose that A is a discriminant form. Then*

$$\sum_{\gamma \in A} \mathbf{e}((\gamma, \delta) - n\gamma^2/2)$$

is 0 unless $\delta \in A^{n*}$ (in which case it has absolute value $\sqrt{|A||A_n|}$).

Proof. The square of the absolute value of this sum is

$$\sum_{\gamma_1, \gamma_2 \in A} \mathbf{e}((\gamma_1, \delta) - n\gamma_1^2/2 - (\gamma_2, \delta) + n\gamma_2^2/2)$$

$$= \sum_{\gamma_1, \gamma_2 \in A} \mathbf{e}((\gamma_1, \delta) - n\gamma_1^2/2 - n(\gamma_1, \gamma_2))$$

$$= |A| \sum_{\gamma_1 \in A_n} \mathbf{e}((\gamma_1, \delta) - n\gamma_1^2/2)$$

The map taking γ_1 to $\mathbf{e}((\gamma_1, \delta) - n\gamma_1^2/2)$ is a character of A_n, so this sum is 0 unless this is the trivial character, in other words unless $\delta \in A^{n*}$. This proves lemma 3.1.

Lemma 3.2. *Suppose that $g \in \widetilde{SL}_2(\mathbf{Z})$ has image $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$. Then $\rho_A(e_0)$ is a linear combination of the elements e_γ for $\gamma \in A^{c*}$.*

Proof. It is sufficient to prove this when g is of the form $T^m ST^n S$ for some $m, n \in \mathbf{Z}$ with $(N, n) = (N, c)$, because g is a product of an element of this form by an element of $\tilde{\Gamma}_0(N)$ (where N is the level of A) and e_0 is an eigenvalue of $\tilde{\Gamma}_0(N)$.

We calculate the image of $1 = e_0 \in \mathbf{C}[A]$ under these elements of $\widetilde{SL}_2(\mathbf{Z})$. We know that

$$S(e_0) = \frac{\mathbf{e}(-\mathrm{sign}(A)/8)}{\sqrt{|A|}} \sum_{\gamma \in L'/L} e_\gamma$$

Applying T^n shows that

$$T^n S(1) = \frac{\mathbf{e}(-\mathrm{sign}(A)/8)}{\sqrt{|A|}} \sum_{\gamma \in A} \mathbf{e}(n(\gamma, \gamma)/2) e_\gamma.$$

Applying S again shows that

$$ST^n S(1) = \frac{\mathbf{e}(-\mathrm{sign}(A)/4)}{|A|} \sum_{\delta \in A} \sum_{\gamma \in A} \mathbf{e}((\gamma, \delta) + n(\gamma, \gamma)/2) e_\delta$$

Using lemma 3.1, we see that the coefficient of e_δ in this expression is 0 unless $\delta \in A^{c*}$. As all the elements e_δ are eigenvectors of T^m, the same is true for $T^m S T^n S(e_0)$. This proves lemma 3.2.

4. The singular theta correspondence.

We summarize some of the results from [B98a]. The main idea is that we can use modular forms with poles at cusps to construct some automorphic forms with singularities. In particular we can often use this to construct piecewise linear functions on hyperbolic space with singularities along the reflection hyperplanes of a reflection group, and this gives the connection between modular forms with singularities and nice hyperbolic reflection groups.

If L is a lattice then we define the Grassmannian $G(L)$ to be the set of maximal positive definite subspaces of $L \otimes \mathbf{R}$. It is a symmetric space acted on by the orthogonal group $O_L(\mathbf{R})$.

The Siegel theta function $\theta_{L+\gamma}$ of a coset $L + \gamma$ of L in L' is defined by

$$\theta_L(\tau; v^+) = \sum_{\lambda \in L+\gamma} \mathbf{e}(\tau \lambda_{v+}^2/2 + \bar{\tau} \lambda_{v-}^2/2)$$

for $\tau \in H$, $v^+ \in G(L)$. We will write Θ_L for the $\mathbf{C}[L'/L]$-valued function

$$\Theta_L(\tau; v) = \sum_{\gamma \in L'/L} e_\gamma \theta_{L+\gamma}(\tau; v).$$

Siegel's transformation formula for Θ_L under $\widetilde{SL}_2(\mathbf{Z})$ ([B98a, theorem 4.1]) is given by

$$\Theta_L\left(\frac{a\tau + b}{c\tau + d} v\right) = (c\tau + d)^{b^+/2} (c\bar{\tau} + d)^{b^-/2} \rho_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) \Theta_L(\tau; v).$$

We define $\Phi(v, F)$ by

$$\Phi(v, F) = \int_{SL_2 \backslash H} \bar{\Theta}_L(\tau; v) F(\tau) y^{b^+/2 - 2} dx dy$$

as in section 6 of [B98a]. By theorem 6.2 of [B98a], $\Phi(v, F)$ is an automorphic function of $v \in G(L)$ whose only singularities are on points of the form $\gamma^\perp$, for $\gamma \in L'$, $\gamma^2 < 0$, where there is a nonzero coefficient $c_{\gamma^2/2, \gamma}$ of F.

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Theorem 4.1. *Suppose L is an even lattice of signature $(2, b^-)$ and F is a modular form of weight $1 - b^-/2$ and representation ρ_L, which is holomorphic on H and meromorphic at cusps and whose coefficients $c_\lambda(m)$ are integers for $m \leq 0$. Then there is a meromorphic function $\Psi_L(Z_L, F)$ for $Z \in P$ with the following properties.*

1. *$\Psi_L(Z_L, F)$ is an automorphic form of weight $c_0(0)/2$ for the group $\mathrm{Aut}(L, F)$ with respect to some unitary character χ of $\mathrm{Aut}(L, F)$.*
2. *The only zeros or poles of Ψ_L lie on the rational quadratic divisors $\lambda^\perp$ for $\lambda \in L$, $\lambda^2 < 0$ and are zeros of order*

$$\sum_{\substack{0 < x \in \mathbf{R} \\ x\lambda \in L'}} c_{x\lambda}(x^2 \lambda^2/2).$$

   (or poles if this number is negative).
3. *Ψ_L is a holomorphic function if the orders of all zeros in item 2 above are nonnegative. If in addition L has dimension at least 5, or if L has dimension 4 and contains no 2 dimensional isotropic sublattice, then Ψ_L is a holomorphic automorphic form. If in addition $c_0(0) = b^- - 2$ then Ψ_L has singular weight so the only nonzero Fourier coefficients of Ψ_L correspond to vectors of K of norm 0.*

This follows from theorem 13.3 of [B98a].

If L is Lorentzian, in other words if $\mathrm{sign}(L) = 2 - \dim(L)$, then the set of all 1 dimensional positive definite subspaces of L is a copy of hyperbolic space of dimension $\dim(L) - 1$.

Theorem 4.2. *Suppose M is a Lorentzian lattice of dimension $1 + b^-$. Suppose that F is a modular form of type ρ_M and weight $(1/2 - b^-/2, 0)$ which is holomorphic on H and meromorphic at cusps and all of whose Fourier coefficients $c_\lambda(m)$ are real for $m < 0$. Finally suppose that if $c_\lambda(\lambda^2/2) \neq 0$ and $\lambda^2 < 0$ then reflection in $\lambda^\perp$ is in $\mathrm{Aut}(M, F, C)$. Then $\mathrm{Aut}(M, F, C)$ is the semidirect product of a reflection subgroup and a subgroup fixing the Weyl vector $\rho(M, W, F)$ of a Weyl chamber W.*

This is a special case of theorem 12.1 of [B98a].

Both theorems 4.1 and 4.2 depend on integrating the vector valued modular form against a vector valued theta function over a fundamental domain of $SL_2(\mathbf{Z})$. In this paper we usually start with a complex valued modular form for $\Gamma_0(N)$ rather than a vector valued form as used in theorems 4.1 and 4.2. There are two more or less equivalent ways to use these theorems on complex valued forms of level N. First, instead of integrating a vector valued form times the vector valued theta function of a lattice over a fundamental domain of $SL_2(\mathbf{Z})$, we can integrate a scalar valued modular form times the theta function of a lattice over a fundamental domain of $\Gamma_0(N)$. Alternatively we can first induce the complex valued modular form for $\Gamma_0(N)$ up to a vector valued modular form for $SL_2(\mathbf{Z})$, and then apply the theorems directly to part of this vector valued form. For these constructions to work, it is necessary and sufficient for the complex valued form to be a modular form for some character χ of $\tilde{\Gamma}_0(N)$, where the scalar valued theta function of the lattice is a modular form of character χ and level $\mathrm{sign}(L)/2$ for $\tilde{\Gamma}_0(N)$. Several sections of this paper describe how to find such modular forms. Note that the singularities of the automorphic

form associated to a level N modular form depend on all poles at all cusps of this form, not just the poles at $i\infty$.

Theorem 4.2 is very useful in practice for finding Lorentzian lattices with interesting reflection groups, because we just find lattices together with modular forms satisfying the conditions of the theorem. However there is a problem with using it for theoretical purposes: it seems hard to give useful general conditions under which the Weyl vector is nonzero or has positive norm. If the Weyl vector happens to be zero then of course theorem 4.2 does not say anything. In practical examples this does not matter because we can just check in each case to see whether the vector is zero (which does happen occasionally). Note the rather curious fact that in this paper we do not need to use the fact that theorem 4.2 has been proved (or even that it is true!) because we are only using it to suggest interesting places to look for lattices, and whenever we find a lattice using theorem 4.2 we still have to prove its properties directly because of the possibility that the Weyl vector is 0.

5. Theta functions.

In this section we work out the level and character of theta functions of even lattices. Most of the results are known, but there seems to be no convenient reference giving the results in the generality we require.

Lemma 5.1. *Suppose that N is a positive integer. If $4 \nmid N$ then two characters of $\Gamma_0(N)$ are the same provided that they have the same values on the elements such that $c > 0$, $d > 0$, and $d \equiv 1 \bmod 4$. If $4|N$ then two characters of $\tilde{\Gamma}_0(N)$ are the same provided that they have the same values on Z and on the elements such that $c > 0$, $d > 0$, and $d \equiv 1 \bmod 4$.*

Proof. It is sufficient to show that the images of the elements mentioned above generate $\Gamma_0(N)$. Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. We will show how to multiply it by powers of elements of the generating set above so that it becomes an element of the generating set, which will prove the lemma. We first note that T is in the group generated by the set above, because the generating set is closed under left multiplication by T and is nonempty. If d is even then c is odd so we can multiply it on the right by T so that d is odd, hence we can assume that d is odd.

Next we arrange that $d \equiv 1 \bmod 4$. If $4|N$ we multiply by Z if necessary so that $d \equiv 1 \bmod 4$. If $4 \nmid N$ we multiply on the right by $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$ if necessary to make c not divisible by 4, and then multiply on the right by a suitable power of T so that $d \equiv 1 \bmod 4$.

We now have to make c and d positive (without changing $d \bmod 4$). We multiply on the right by a suitable power of $\begin{pmatrix} 1 & 0 \\ 4N & 1 \end{pmatrix}$ to make c positive without changing d. Finally we multiply on the right by a suitable power of $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ to make d positive without changing c. The result is in the generating set, so this proves lemma 5.1.

We define the symbol $\left(\frac{c}{d}\right)$ for all pairs of coprime integers c and d as follows. The symbol is multiplicative in both c and d. If d is an odd prime it is just the usual Legendre

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symbol. If d is 2 it is 1 if $c \equiv \pm 1$ mod 8 and -1 otherwise. If $d = -1$ it is 1 if $c > 0$ and -1 if $c < 0$. Finally we define $\left(\begin{smallmatrix} 0 \\ \pm 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pm 1 \\ 0 \end{smallmatrix}\right) = 1$.

We now define some characters χ_n (for n a positive integer) and χ_θ of $\tilde{\Gamma}_0(N)$. We suppose that if p is an odd prime occurring an odd number of times in the prime factorization of n then it divides N. Also suppose that if 2 occurs an odd number of times in the prime factorization of n then 8 divides N. We define the character χ_n of $\Gamma_0(N)$ by

$$\chi_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} d \\ n \end{pmatrix}.$$

Lemma 5.2. *Let $\theta_{A_1} = \sum_{n \in \mathbf{Z}} q^{n^2}$ be the theta function of the A_1 lattice. There is a (unique) character χ_θ of the metaplectic double cover of $\Gamma_0(4)$ such that*

$$\theta_{A_1} \left(\frac{a\tau + b}{c\tau + d} \right) = \chi_\theta \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \sqrt{c\tau + d}\, \theta_{A_1}(\tau).$$

(In other words θ_{A_1} is a modular form for $\tilde{\Gamma}_0(4)$ of weight $1/2$ and character χ_θ.) The values are given by

$$\chi_\theta \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) = \begin{cases} \pm \begin{pmatrix} c \\ d \end{pmatrix} & \text{if } d \equiv 1 \text{ mod } 4 \\ \pm(-i)\begin{pmatrix} c \\ d \end{pmatrix} & \text{if } d \equiv 3 \text{ mod } 4 \end{cases}$$

In particular, $\chi_\theta(Z) = -i$.

Proof. This follows from the theorem on page 148 of [Ko].

Lemma 5.3. *Suppose $4|N$. Then the kernel of the character χ_θ of $\widetilde{\Gamma}_0(N)$ maps isomorphically onto $\Gamma_1(4) \cap \Gamma_0(N)$, and if we identify the kernel with this image then $\widetilde{\Gamma}_0(N)$ is the product $(\Gamma_1(4) \cap \Gamma_0(N)) \times \mathbf{Z}/4\mathbf{Z}$ (where $\mathbf{Z}/4\mathbf{Z}$ is its center, generated by Z). The lifting of $\Gamma_1(4) \cap \Gamma_0(N)$ to $\tilde{\Gamma}_0(N)$ is given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\sqrt{c\tau + d} \right)$$

Proof. This follows immediately from lemma 5.2 because χ_θ is a character whose values are ± 1 or $\pm i$, and $\chi_\theta(Z) = -i$, and $\Gamma_0(N)$ is the product of its center of order 2 (generated by Z) and the subgroup $\Gamma_1(4) \cap \Gamma_0(N)$. This proves lemma 5.3.

We define the group $\Gamma_0^2(N)$ to be the subgroup of $\Gamma_0(N)$ of elements whose diagonal entries are squares in $\mathbf{Z}/n\mathbf{Z}$. If $4 \nmid N$ then $\Gamma_0^2(N)$ is the intersection of the kernels of the characters χ_p for p an odd prime dividing N. If $4|N$ then $\Gamma_0^2(N)$ can be lifted to a subgroup of $\tilde{\Gamma}_0(N)$ as in lemma 5.3, and is the intersection of the kernels of the characters χ_θ and χ_p of $\tilde{\Gamma}_0(N)$ for p a prime dividing $N/4$.

Theorem 5.4. *Suppose that A is a discriminant form of level dividing N. If b and c are divisible by N then $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \in \widetilde{SL}_2(\mathbf{Z})$ acts on the Weil representation $\mathbf{C}[A]$ by*

$$g(e_\gamma) = \chi_A(g)e_{a\gamma}$$

where χ_A is the character of $\tilde{\Gamma}_0(N)$ given by

$$\chi_A = \begin{cases} \chi_\theta^{\text{sign}(A)+\left(\frac{-1}{|A|}\right)-1}\chi_{|A|2^{\text{sign}(A)}} & \text{if } 4|N \\ \chi_{|A|} & \text{if } 4\nmid N \end{cases}$$

Proof. First assume that A has even signature. Choose an even lattice in a positive definite space with discriminant form A. Then [E94, corollary 3.1] and the discussion on [E94, page 94] show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(e_\gamma) = \left(\frac{(-1)^{\text{sign}(A)/2}|A|}{d}\right)e_{a\gamma}$$

provided that d is odd and positive. By lemma 5.1 it is sufficient to check that this is equal to the value of χ_A when $g = Z$ and when $d \equiv 1 \bmod 4$, $d > 0$. But in the latter case $\left(\frac{-1}{d}\right) = 1$ and $\left(\frac{|A|}{d}\right) = \left(\frac{d}{|A|}\right)$, so this has the same character values as $\chi_{|A|}$. Also if $d \equiv 1 \bmod 4$ and $\text{sign}(A)$ is even then $\chi_{2^{\text{sign}(A)}}$ and $\chi_\theta^{\text{sign}(A)}$ and $\chi_\theta^{\left(\frac{-1}{A}\right)-1}$ are all 1 on the element g. Therefore the two characters coincide on elements with $d \equiv 1 \bmod 4$.

As $Z(e_\gamma) = (-i)^{\text{sign}(A)}e_{-\gamma}$ we see that $\chi_A(Z) = (-i)^{\text{sign}(A)}$ We now check that the characters are equal on the element Z. If $4\nmid N$ this follows from $\chi_{|A|}(Z) = \left(\frac{-1}{|A|}\right) = (-1)^{\text{sign}(A)/2}$. If $4|N$ this follows from $\chi_2(Z) = 1$, $\chi_\theta^{\text{sign}(A)}(Z) = (-i)^{\text{sign}(A)}$, and $\chi_{|A|}(Z) = \left(\frac{-1}{|A|}\right) = (-i)^{1-\left(\frac{-1}{|A|}\right)} = \chi_\theta^{1-\left(\frac{-1}{|A|}\right)}(Z)$. This proves theorem 5.4 when A has even signature.

We can do the case of odd signature very quickly by reducing it to the case of even signature as follows. If A has odd signature then $4|N$, and the discriminant form $A \oplus \langle 2 \rangle$ has determinant $2|A|$ (where $\langle 2 \rangle$ is the discriminant form of the A_1 lattice and has order 2). Theorem 5.4 for the element $\gamma \in A$ now follows from lemma 5.2 and theorem 5.4 applied to the element $\gamma + 0 \in A \oplus \langle 2 \rangle$. This proves theorem 5.4.

6. Eta quotients.

In this section we work out the levels and characters of some eta quotients. We will use these results in section 12 to construct examples of modular forms of given characters.

The function $\eta(t\tau)$ has a zero of order $(t,c)^2/24t$ at the cusp a/c.

Lemma 6.1. (Rademacher.) Recall that $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$ is the Dedekind eta function. Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ with $c > 0$. Then

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau+d}\right)\sqrt{c\tau+d}\,\eta(\tau)$$

where χ_η is a character of $\widetilde{SL}_2(\mathbf{Z})$ with values given as follows:

$$\chi_\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau+d}\right) = \begin{cases} \pm\left(\frac{d}{c}\right)\mathbf{e}_{24}(-3c+bd(1-c^2)+c(a+d)) & c \text{ odd}, c > 0 \\ \pm\left(\frac{-d}{-c}\right)\mathbf{e}_{24}(3c-6+bd(1-c^2)+c(a+d)) & c \text{ odd}, c < 0 \\ \pm\left(\frac{c}{d}\right)\mathbf{e}_{24}(3d-3+ac(1-d^2)+d(b-c)) & d \text{ odd}, c \geq 0 \\ \pm\left(\frac{-c}{d}\right)\mathbf{e}_{24}(-3d-9+ac(1-d^2)+d(b-c)) & d \text{ odd}, c < 0 \end{cases}$$

Proof. The cases with $c \geq 0$ follow from the theorem in [R, p. 163]. The cases with $c < 0$ follow easily from the cases with $c > 0$ and the fact that $\chi_\eta(Z) = -i$. This proves lemma 6.1.

Theorem 6.2. *Suppose that we are given a positive integer N, and integers r_δ for $\delta | N$ and $|A|$ with $|A|/\prod_{\delta|N} \delta^{r_\delta}$ a rational square. Suppose that $\frac{1}{24}\sum_{\delta|N} r_\delta \delta$ and $\frac{N}{24}\sum_{\delta|N} r_\delta/\delta$ are both integers. Then*

$$\prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$$

is a modular form for $\tilde{\Gamma}_0(N)$ of weight $k = \sum_\delta r_\delta/2$ and character equal to $\chi_{|A|}$ if $4 \nmid N$, and to $\chi_\theta^{2k+\binom{-1}{|A|}-1}\chi_{2^{2k}|A|}$ if $4|N$.

Proof. By lemma 5.1 it is enough to check that the characters are equal whenever $c > 0$ and $d \equiv 1 \bmod 4$, and also that they are equal on Z if $4|N$.

If $c > 0$ and $d \equiv 1 \bmod 4$ then by lemma 6.1 the character value is given by

$$\prod_{\delta|N} \left(\mathbf{e}_{24}(3d - 3 + a(c/\delta)(1 - d^2) + d(b\delta - c/\delta))\binom{c/\delta}{d} \right)^{r_\delta}$$

$$= \mathbf{e}_{24}\left(db\sum_{\delta|N}\delta r_\delta + (a - d - ad^2)c\sum_{\delta|N}r_\delta/\delta + 3(d-1)\sum_{\delta|N}r_\delta \right)\binom{c}{d}^{\sum_{\delta|N} r_\delta}\left(\prod_{\delta|N}\delta^{r_\delta}\atop d\right)$$

$$= i^{(d-1)k}\binom{c}{d}^{2k}\binom{|A|}{d}$$

$$= \binom{d}{2}^{2k}\binom{c}{d}^{2k}\binom{d}{|A|}$$

If $4 \nmid N$ then $2k$ is even so this is the value of the character $\chi_{|A|}$. If $4|N$ then this is the value of $\chi_\theta^{2k}\chi_{2^{2k}|A|}$, which is the same as the value of $\chi_\theta^{2k+\binom{-1}{|A|}-1}\chi_{2^{2k}|A|}$ because $\chi_\theta = \pm 1$ whenever $d \equiv 1 \bmod 4$. So both characters have the same value whenever $c > 0$ and $d \equiv 1 \bmod 4$.

Finally we have to check that both characters are equal on Z whenever $4|N$. This follows by the same argument used in theorem 5.4. This proves theorem 6.2.

Theorem 6.2 generalizes some theorems of Newman ([N57], [N59]), who did the case of weight 0 and trivial character.

7. Dimensions of spaces of modular forms.

In this section we recall the formulas for the dimensions of some spaces of modular or cusp forms associated to a representation ρ of a discrete co-finite subgroup of $\widetilde{SL}_2(\mathbf{R})$. For weight at least 2 the dimension is given by either the Riemann-Roch theorem or the Selberg trace formula. More generally if G is a group acting on A then it also acts on the spaces of cusp forms, and we calculate the character of these representations. These results are used in sections 9 and 12.

For weight 1/2 forms Serre and Stark described an explicit basis as follows.

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Theorem 7.1. *Suppose that χ is an even Dirichlet character mod N. Then a basis for the space of modular forms of weight $1/2$ and character $\chi_\theta \chi$ for $\Gamma_0(N)$ is given by the forms*

$$\sum_{n \in \mathbf{Z}} \psi(n) q^{tn^2}$$

where ψ is a primitive even character of conductor $r(\psi)$, t is a positive integer such that $4r(\psi)^2 t$ divides N, and $\chi(n) = \psi(n)\left(\frac{D}{n}\right)$ for all n coprime to N, where D is the discriminant of the quadratic field $\mathbf{Q}[\sqrt{t}]$. (Note that ψ is determined by t and χ.)

Proof. This is theorem A of [S-S, p.34].

The dimensions of spaces of holomorphic modular forms can all be worked out as follows. For weight less than 0 there are no non-zero forms, and weight 0 is trivial as these are just constants. For weight > 2 we can work out the dimension using the Selberg trace formula (see below) or the Riemann-Roch theorem, and with a bit more care this also works for weight 2 (there are extra correction terms coming from weight 0 forms in this case). For weight $1/2$ the Serre-Stark theorem gives an explicit basis, and this can be used to do the case of weight $3/2$ because the Selberg trace formula gives the difference of dimensions for weights k and $2 - k$. This leaves the case of weight 1, which seems to be the hardest case to do. In general weight 1 forms are closely related to odd 2-dimensional complex representations of the Galois group of $\mathbf{Q}$. Fortunately, for the low level cases we are interested in, the weight 1 forms are usually easy to construct explicitly using Eisenstein series and theta series of 2-dimensional lattices (mainly because the exotic Galois representations only occur for higher levels).

Now we use the Selberg trace formula to find the dimensions of spaces of forms of weight at least 2.

If X is a finite order automorphism of a finite dimensional complex vector space V with eigenvalues $\mathbf{e}(-\beta_j)$ for $1 \le j \le \dim(V)$ and $0 \le \beta_j < 1$, then we define $\delta_\infty(X)$ to be $\sum(1/2 - \beta_j)$, and we define $\delta_N(X)$ to be $\delta_\infty(X) - \dim(V)/2N$. More generally, if g is an endomorphism of V commuting with the action of G then we define $\delta_{\rho,\infty}(X, g)$ to be

$$\sum (1/2 - \beta_j) \mathrm{Tr}(g | V^{\mathbf{e}(\beta_j)X})$$

where the sum is over the distinct eigenvalues $\mathbf{e}(-\beta_j)$ of X, and we put $\delta_N(X, g) = \delta_\infty(X, g) - \mathrm{Tr}(g)/2N$.

Lemma 7.2. *If ρ is a representation of a group containing X on a finite dimensional complex vector space and $X^N = 1$ then*

$$\delta_N(X, g) = \frac{1}{N} \sum_{0 < j < N} \frac{\mathrm{Tr}_\rho(X^j g)}{1 - \mathbf{e}(j/N)}$$

$$\delta_\infty(X, g) = \frac{\mathrm{Tr}(g)}{2N} + \frac{1}{N} \sum_{0 < j < N} \frac{\mathrm{Tr}_\rho(X^j g)}{1 - \mathbf{e}(j/N)}$$

Proof. The trace of g on the subspace of ρ on which X^{-1} has eigenvalue $\mathbf{e}(k/N)$ is

$$\frac{1}{N} \sum_{j \bmod N} \mathrm{Tr}_\rho(gX^j) \mathbf{e}(jk/N).$$

Therefore

$$\delta_N(X) = \mathrm{Tr}_\rho(g)(1/2 - 1/2N) - \sum_{0 \le k < N} \frac{k}{N} \times \frac{1}{N} \sum_{j \bmod N} \mathrm{Tr}_\rho(gX^j)\mathbf{e}(jk/N)$$

$$= \mathrm{Tr}_\rho(g)(1/2 - 1/2N) - \frac{1}{N} \sum_{j \bmod N} \mathrm{Tr}_\rho(gX^j) \sum_{0 \le k < N} \frac{k}{N}\mathbf{e}(jk/N)$$

$$= \frac{1}{N} \sum_{0 < j < N} \frac{\mathrm{Tr}_\rho(gX^j)}{1 - \mathbf{e}(j/N)}.$$

This proves lemma 7.2.

We will write $ModForm(\Gamma, k, \rho)$ for the space of modular forms of weight k and representation ρ for Γ that are holomorphic at cusps.

Lemma 7.3. *Suppose that Γ is a discrete subgroup of $\widetilde{SL}_2(\mathbf{R})$ of co-finite volume and containing Z. Let ρ be a complex representation of Γ of finite dimension d on which Z acts multiplication by some constant. Choose $k \in \mathbf{Q}$ with $k > 2$. Then the dimension of the space of $ModForm(\Gamma, k, \rho)$ is equal to 0 unless Z acts as $\mathbf{e}(-k/2)$, in which case it is*

$$(k-1)\dim(\rho)\omega(F)/4\pi + \sum_{1 \le j \le \rho} \delta_{\rho,\nu_j}(\mathbf{e}(k/2\nu_j)R_j) + \sum_{1 \le j \le \tau} \delta_{\rho,\infty}(T_j)$$

where
$\omega(F)$ *is the hyperbolic area of the fundamental domain F, and is equal to*

$$2\pi\left(2g - 2 + \sum_{1 \le j \le \rho}(1 - 1/\nu_j) + \tau\right)$$

g *is the genus of the compactification of $\Gamma \backslash H$*
ρ *is the number of elliptic fixed points in a fundamental domain.*
ν_j *is the order of the j'th elliptic fixed point, so the subgroup of Γ fixing the j'th elliptic fixed point is cyclic, generated by R_j with $R_j^{\nu_j} = Z$.*
R_j *is the primitive elliptic element corresponding to j. An elliptic element is called primitive if it is conjugate to the "clockwise" element*

$$\left(\begin{pmatrix} \cos(\pi/\nu) & -\sin(\pi/\nu) \\ \sin(\pi/\nu) & \cos(\pi/\nu) \end{pmatrix}, \sqrt{\sin(\pi/\nu)\tau + \cos\pi/\nu}\right) \in \widetilde{SL}_2(\mathbf{R})$$

fixing i.
τ *is the number of orbits of cusps of Γ acting on H.*
T_j *is the unique element conjugate to T^{-1} under $\widetilde{SL}_2(\mathbf{R})$ such that the stabilizer of the j'th cusp is generated by T_j and Z.*

Proof. If Z acts as multiplication by some constant not equal to $\mathbf{e}(-k/2)$ then the transformation of modular forms under the element Z immediately shows that any

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modular form of weight k is 0. If Z acts as $\mathbf{e}(-k/2)$ then we get a "multiplier system" χ in the sense of [F, 1.3.4] from the representation ρ by putting $\chi\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) = \rho\left(\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \sqrt{c\tau + d}\right)\right)$ where we choose the value of the square root with $-\pi/2 < \arg(\sqrt{c\tau + d}) \le \pi/2$. (Note that the variable k used in [F] is half that used here.)

By [F, theorem 2.5.5] the dimension of the space of modular forms is given by

$$(k-1)d\omega(F)/4\pi + \sum_{1\le j\le\rho}\left(\frac{d}{2} - \frac{d}{2\nu_j} - \frac{\alpha_j}{\nu_j}\right) + \frac{d\tau}{2} - \sum_{1\le j\le\tau}\beta_j$$

where

$d = \dim(\rho)$

α_j is the sum of the numbers α_{jp} for $1 \le p \le d$, where the eigenvalues of R_j are $\mathbf{e}(-(k/2 + \alpha_{jp})/\nu_j)$ and $\alpha_{jp} \in \{0, 1, \ldots, \nu_j - 1\}$. ([F, p. 66–68])

β_j is the sum of the numbers β_{jp} $(1 \le p \le d)$ where the eigenvalues of T_j^{-1} are $\mathbf{e}(\beta_{jp})$ and $0 \le \beta_{jp} < 1$.

It is easy to check that

$$\frac{d}{2} - \frac{d}{2\nu_j} - \frac{\alpha_j}{\nu_j} = \delta_{\rho,\nu_j}(\mathbf{e}(k/2\nu_j)R_j)$$

and

$$\frac{d}{2} - \beta_j = \delta_{\rho,\infty}(T_j)$$

Putting everything together proves lemma 7.3.

Remark. If the weight is greater than 2 then the dimension of the space of all cusp forms is given by subtracting the dimension of the space of Eisenstein series, which is the sum over all cusps j of the of the dimension of the subspace of ρ fixed by T_j.

Corollary 7.4. *Suppose that ρ is a finite dimensional representation of Γ. Suppose that $k \in \mathbf{Q}$ and $k > 2$. Then*

$$\dim(ModForm(\Gamma, k, \rho)) = \frac{1}{4}\sum_{0\le j<4}\mathbf{e}(jk/2)\psi(Z^j)$$

where $\psi(g)$ is given by

$$\psi(g) = \frac{(k-1)\omega(F)}{4\pi}\mathrm{Tr}_\rho(g) + \sum_{1\le j\le\rho}\delta_{\rho,\nu_j}(\mathbf{e}(k/2\nu_j)R_j, g) + \sum_{1\le j\le\tau}\delta_{\rho,\infty}(T_j, g).$$

Proof. Break up ρ into the eigenspaces of Z and apply lemma 7.3 to each eigenspace. This proves corollary 7.4.

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Corollary 7.5. *Suppose that ρ is a finite dimensional representation of Γ acted on by a finite group G. Suppose that $k \in \mathbf{Q}$ and $k > 2$. Then the character of $ModForm(\Gamma, k, \rho)$, considered as a representation of G, is given by*

$$\mathrm{Tr}(g|ModForm(\Gamma, k, \rho)) = \frac{1}{4} \sum_{0 \leq j < 4} \mathbf{e}(jk/2)\psi(gZ^j)$$

where ψ given by the formula of corollary 7.4 and $g \in G$.

Proof. Note that the dimension of $ModForm(\Gamma, k, \sigma)$ is given by $\mathrm{Tr}(M|\sigma)$ for some element M in the group ring of G, whenever σ is a representation of Γ satisfying the conditions of corollary 7.4. Therefore

$$\begin{aligned}
&\mathrm{Tr}(g|ModForm(\Gamma, k, \rho)) \\
&= \sum_{\chi \in Irred(G)} \chi(g) \dim((ModForm(\Gamma, k, \rho) \otimes \bar{\chi})^G) \\
&= \sum_{\chi \in Irred(G)} \chi(g) \dim(ModForm(\Gamma, k, (\rho \otimes \bar{\chi})^G)) \\
&= \sum_{\chi \in Irred(G)} \chi(g) \mathrm{Tr}(M|(\rho \otimes \bar{\chi})^G) \\
&= \mathrm{Tr}(Mg|\rho).
\end{aligned}$$

(The sums are over the sets $Irred$ of irreducible representations of G, and $\bar{\chi}$ is the dual of the representation χ.) Therefore to find the trace of g, we just insert an extra factor of g whenever we have a trace in the formula for ψ. This proves corollary 7.5.

8. The geometry of $\Gamma_0(N)$.

We summarize some standard results about the cusps and elliptic points of the group $\Gamma_0(N)$. We need this information in order to use the formulas of section 7. For proofs see [Sh] or [Mi].

The group $\Gamma_0(N)$ has index $N \prod_{p|N}(1 + 1/p)$ (where the product is over all primes dividing N).

The equivalence class of the cusp a/c of $\Gamma_0(N)$ is determined by the invariants (c, N) (a divisor of N) and $(c/(c, N))^{-1}a$ (an element of $(\mathbf{Z}/(c, N/c))^*$). A complete set of representatives for the cusps is given by a/c for $c|N$, $c > 0$, $0 < a \leq (c, N/c)$, $(a, c) = 1$. The cusp a/c has width $N/(c^2, N)$.

In the rest of this section we work out the values of the characters χ_θ and χ_p on the elements R_j and $T_{a/c}$ of section 7 associated to elliptic points or cusps of $\Gamma_0(N)$.

Lemma 8.1. *If $(a, c) = 1$ then the element $T_{a/c}$ of $\widetilde{\Gamma}_0(N)$ is given by*

$$T_{a/c} = \left(\begin{pmatrix} 1 + act & -a^2t \\ c^2t & 1 - act \end{pmatrix}, \sqrt{c^2t\tau + 1 - act} \right)$$

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where $t = N/(c^2, N)$.

Proof. We can assume $c > 0$ as the case $c < 0$ follows from this, and the case $c = 0$ is trivial to check. The element $T_{a/c}$ is the conjugate of some element of the form $\left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, 1 \right)$ for $t > 0$, so is equal to

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \sqrt{-c\tau + a} \right)$$

where t is the smallest positive integer such that this element is in $\widetilde{\Gamma}_0(N)$, and b and d in $\mathbf{Z}$ are chosen so that $ad - bc = 1$. If we evaluate this element (keeping careful track of the values of the square roots) we find it is equal to the expression in the lemma. This is in $\widetilde{\Gamma}_0(N)$ if and only if $N | c^2 t$, so $t = N/(c^2, N)$. This proves lemma 8.1.

Lemma 8.2. *If p is an odd prime dividing N then $\chi_p(T_{a/c}) = 1$.*

Proof. By lemma 8.1 we see that $\chi_p(T_{a/c})$ is 1 provided that $1 - act$ is a square mod p. But this is always true because p must divide either c or $t = N/(c^2, N)$ if it divides N. This proves lemma 8.2.

Lemma 8.3. *Suppose that $8|N$ and $(a, c) = 1$. Then $\chi_2(T_{a/c})$ is -1 in the 3 cases $2||c$ and $8||N$, or $4||c$ and $8||N$, or $4||c$ and $16||N$, and is 1 otherwise.*

Proof. By lemma 8.1 we have

$$\chi_2(T_{a/c}) = \chi_2 \left(\begin{pmatrix} 1 + act & -a^2 t \\ c^2 t & 1 - act \end{pmatrix}, \sqrt{c^2 t\tau + 1 - act} \right) = \left(\frac{1 - act}{2} \right),$$

so $\chi_2(T_{a/c})$ is 1 if $1 - act \equiv 1 \bmod 8$ and is -1 if $1 - act \equiv 5 \bmod 8$ (where $t = N/(c^2, N)$). Note that $act \equiv 1 \bmod 4$ as $8|N$. We do a case by case check to show that act is 4 mod 8 in the 3 cases listed above, and is 0 mod 8 otherwise.

1. If c is odd then $8|t = N/(c^2, N)$, so we can assume that c is even (and hence a is odd).
2. If $8|c$ then $act \equiv 0 \bmod 8$, so we can assume that $2||c$ or $4||c$.
3. If $2||c$ and $16|N$ then $4|t$ so $act \equiv 0 \bmod 8$.
4. If $2||c$ and $8||N$ then $2||t$ so $act \equiv 4 \bmod 8$.
5. If $4||c$ and $32|N$ then $2|t$ so $act \equiv 0 \bmod 8$.
6. If $4||c$ and $32 \nmid N$ then t is odd so $act \equiv 4 \bmod 8$.

This proves lemma 8.3.

Lemma 8.4. *Suppose that $4|N$. If $2||c$ and $4||N$ then $\chi_\theta(T_{a/c}) = -i^t$, with $t = N/(c^2, N)$. If $2||c$ and $8||N$ then $\chi_\theta(T_{a/c}) = -1$. Otherwise $\chi_\theta(T_{a/c}) = 1$*

Proof. By lemmas 8.1 and 5.2, we see that if $4|act$ then

$$\chi_\theta(T_{a/c}) = \chi_\theta \left(\begin{pmatrix} 1 + act & -a^2 t \\ c^2 t & 1 - act \end{pmatrix}, \sqrt{c^2 t\tau + 1 - act} \right) = \left(\frac{c^2 t}{1 - act} \right) = \left(\frac{1 - act}{t} \right).$$

This is equal to 1 unless $2||t$ and $2||c$, in which case it is -1 and $8||N$. So lemma 8.4 is true whenever $4|act$.

If c is odd then $4|t$ so $4|act$, and if $4|c$ then $4|act$, so we can assume that $2||c$. If $8|N$ and $2||c$ then $2|t$ so $4|act$. So we can also assume that $4||N$, which implies that a and t are odd and $act \equiv 2 \bmod 4$. Then by lemmas 8.1 and 5.2,

$$\chi_\theta(T_{a/c}) = -i\left(\frac{t}{1-act}\right) = -i(-1)^{(t-1)(1-act-1)/4}\left(\frac{1-act}{t}\right) = -i(-1)^{(t-1)/2} = -i^t.$$

This proves lemma 8.4.

Finally we work out the values of characters on elliptic elements. Note that if the characters χ_θ or χ_2 are non-trivial, then $4|N$ so there are no elliptic elements. So we only have to do the characters χ_p on elliptic elements for p an odd prime.

Lemma 8.5. *Suppose that $\Gamma_0(N)$ has an elliptic fixed point of order 2 fixed by the primitive elliptic element $R \in \Gamma_0(N)$ as in lemma 7.3, and let p be an odd prime dividing N (so that $p \equiv 1 \bmod 4$). Then $\chi_p(R) = (-1)^{(p-1)/4}$.*

Proof. We know that $R^2 = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $R = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ for some a, b, c, and $\chi_p(R) = \left(\frac{a}{p}\right)$. As $-a^2 - bc = 1$ and $N|c$ and $p|N$ we see that $a^2 \equiv -1 \bmod p$, so that a is an element of order exactly 4 in $(\mathbf{Z}/p\mathbf{Z})^*$, and hence is in the (unique) index 2 subgroup of $(\mathbf{Z}/p\mathbf{Z})^*$ if and only if $p \equiv 1 \bmod 8$. But $\chi_p(a) = 1$ if and only if a is in the index 2 subgroup of $(\mathbf{Z}/p\mathbf{Z})^*$. This proves lemma 8.5.

Lemma 8.6. *Suppose that $\Gamma_0(N)$ has an elliptic fixed point of order 3 fixed by the primitive elliptic element $R \in \Gamma_0(N)$ as in lemma 7.3, and let p be an odd prime dividing N. Then $\chi_p(R) = (-1)^{(p-1)/2}$.*

Proof. The character χ_p has order 2, so

$$\chi_p(R) = \chi_p(R^3) = \chi_p(Z) = \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}.$$

This proves lemma 8.6.

9. An application of Serre duality.

In this section we summarize some results from [B99]. We will later often need to find modular forms with given singularities at cusps. In this section we show how it is sometimes possible to prove the existence of modular forms with given singularities without writing them down explicitly. The main idea is that Serre duality shows that the space of obstructions to finding a modular form with given singularities at cusps is a space of cusp forms whose dimension can usually be worked out explicitly. If this space of obstructions has smaller dimension than the space of potential solutions, then at least one solution must exist.

If κ is a cusp of Γ, let q_κ be a uniformizing parameter at κ on $\Gamma\backslash H$. For a representation ρ on V_ρ, let V_ρ^* denote the dual. Let

$$PowSer_\kappa(\Gamma, \rho) = \mathbf{C}[[q_\kappa]] \otimes V_\rho$$

be the space of formal power series in q_κ with coefficients in V_ρ, let

$$Laur_\kappa(\Gamma, \rho) = \mathbf{C}[[q_\kappa]][q_\kappa^{-1}] \otimes V_\rho$$

be the space of formal Laurent series in q_κ with coefficients in V_ρ, and let

$$Sing_\kappa(\Gamma, \rho) = \frac{Laur_\kappa(\Gamma, \rho)}{PowSer_\kappa(\Gamma, \rho)}$$

be the space of possible singularities of V_ρ valued Laurent series at κ. The two spaces $PowSer_\kappa(\Gamma, \rho^*)$ and $Sing_\kappa(\Gamma, \rho)$ are paired into $\mathbf{C}$ by taking the residue

$$\langle f, \phi \rangle = \mathrm{Res}\big(f\phi\, q_\kappa^{-1}\, dq_\kappa\big),$$

for $f \in PowSer_\kappa(\Gamma, \rho^*)$ and $\phi \in Sing_\kappa(\Gamma, \rho)$. Here the product of f and ϕ is defined using the pairing of V_ρ and V_ρ^*.

Then the spaces

$$Sing(\Gamma, \rho) = \oplus_\kappa Sing_\kappa(\Gamma, \rho)$$

and

$$PowSer(\Gamma, \rho^*) = \oplus_\kappa PowSer_\kappa(\Gamma, \rho^*),$$

where κ runs over the Γ-inequivalent cusps, are paired by the sum of the local pairings at the cusps.

There are maps

$$\lambda : CuspForm(\Gamma, k, \rho^*) \longrightarrow PowSer(\Gamma, \rho^*)$$

and

$$\lambda : SingModForm(\Gamma, 2 - k, \rho) \longrightarrow Sing(\Gamma, \rho),$$

defined in the obvious way by taking the Fourier expansions of their nonpositive part at the various cusps.

We define the space $Obstruct(\Gamma, k, \rho)$ of obstructions to finding a modular form of type ρ and weight k which is holomorphic on H and has given meromorphic singularities at the cusps to be the space

$$Obstruct(\Gamma, k, \rho) = \frac{Sing(\Gamma, \rho)}{\lambda(SingModForm(\Gamma, k, \rho))}.$$

Lemma 9.1. *Suppose that ρ is a representation of Γ factoring through some finite quotient of Γ and $k \in \mathbf{Q}$ with $Z = \mathbf{e}(-k/2)$ on ρ. Then*

$$Obstruct(\Gamma, 2 - k, \rho)$$

is dual to

$$CuspForm(\Gamma, k, \rho^*)$$

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(and both spaces are finite dimensional).

Proof. This can be proved in the same way as theorem 3.1 of [B99] (which is really a special case of Serre duality). The only real difference is that in [B99] the space $Sing$ is different because it also includes information about the constant terms of functions. This has the effect of replacing the space of holomorphic modular forms in [B99] by a space $CuspForm(\Gamma, k, \rho^*)$ of cusp forms. This proves lemma 9.1.

Example 9.2. In one of the examples later on, we need to know that there is a non-zero weight -7 form of character χ_3 for $\Gamma_0(3)$ whose singularities are a pole of order at most 1 at $i\infty$ and a pole of order at most 3 at 0, and such that the coefficient of q_3^2 at the cusp 0 vanishes. The space of possible singularities is spanned by q^{-1} at $i\infty$ and by q_3^{-1} and q_3^{-3} at 0, so it is 3 dimensional. The space of obstructions is the space of weight $2 - -7 = 9$ cusp forms of character χ_3 for $\Gamma_0(3)$, which has dimension 2. This is less than 3, so the space of forms with the singularities above is at least one dimensional, so a nonzero form exists. (For an explicit formula for it see section 12.)

Warning. This method only gives a lower bound for dimensions of spaces of forms with given singularities. In practice this lower bound is often the exact dimension, but there are occasional examples where the lower bound is 0 but nevertheless there is a nonzero form.

10. Eisenstein series.

We summarize some standard results about Eisenstein series that we will use in section 12.

Lemma 10.1. *Assume that $k \geq 2$ is even. Put*

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} q^n \sum_{d|n} d^{k-1}.$$

If $k > 2$ then E_k is a modular form of weight k for $\Gamma_0(1)$. If $\sum_{d|N} a_d/d = 0$ then $\sum_{d|N} a_d E_2(\tau)$ is a modular form of weight 2 for $\Gamma_0(N)$.

Proof. These are just the usual Eisenstein series for $SL_2(\mathbf{Z})$.

Lemma 10.2. *Assume that $k \geq 2$ is integral and let χ be a non-principal Dirichlet character mod N with $\chi(-1) = (-1)^k$. Then*

$$E_k(\tau, \chi) = \sum_{n \geq 1} q^n \sum_{d|n} d^{k-1} \chi(n/d)$$

is a modular form of weight k and character χ for $\Gamma_0(N)$.

Proof. See [Mi, theorems 7.1.3 and 7.2.12 and lemma 7.1.1].

Lemma 10.3. *Let χ be a non-principal Dirichlet character mod N with $\chi(-1) = -1$. Then*

$$E_1(\tau, \chi) = 1 + \frac{2}{L(0, \chi)} \sum_{n \geq 1} q^n \sum_{d|n} \chi(n/d),$$

where

$$L(0, \chi) = -B_{1,\chi} = - \sum_{0 < n < N} \chi(n)n/N,$$

is a modular form of weight k and character χ for $\Gamma_0(N)$.

Proof. See [Mi, theorem 7.2.13 and lemma 7.1.1].

Example 10.4. The Eisenstein series $E_1(\tau, \chi_3)$ is a weight 1 modular form for $\Gamma_0(3)$ of character χ_3 whose power series is

$$E_1(\tau, \chi_3) = 1 + 6 \sum_{n>0} q^n \sum_{d|n} \chi_3(n/d) = 1 + 6q + 6q^3 + 6q^4 + O(q^7).$$

This form is also the theta function of the A_2 lattice. (It is used in Wiles' proof of Fermat's last theorem to show that any weight 1 form is congruent mod 3 to a weight 2 form: just multiply by $E_1(\tau, \chi_3)$!)

11. Reflective forms.

Suppose that L is a lattice of level dividing N. We define a singularity at a cusp a/c of $\Gamma_0(N)$ to be **reflective** if it is a linear combination of terms $q^{-1/n}$ for n a positive integer, such that every norm $-2/n$ element of $(L'/L)^{c*}$ has order dividing n. Note that if a norm $-2/n$ element of L'/L has order dividing n then every lift of it to a norm $-2/n$ element of L' is a root of L'. We say that a modular form is **reflective** for L if it is a meromorphic modular form of weight $\text{sign}(L)/2$, level N, character χ_L, and its only singularities are reflective singularities at cusps.

This definition is the result of a lot of experimentation to find a definition that is easy to use and that also seems to give most of the "interesting" lattices. There are many possible variations of it, some of which we now briefly describe. First of all, we could use vector valued modular forms rather than scalar valued modular forms. The main problem with this is practical inconvenience: it is not that easy (though possible) to work with modular forms taking values in a space of dimension (say) 2^9. Moreover examples suggest that the most interesting vector valued modular forms are often invariant under $\text{Aut}(A)$. This suggests using invariant vector valued modular forms instead, but examples suggest that these are very closely related to scalar valued modular forms of level N, so we may as well use these. The "allowable" singularities can also be varied. For example, we could also allow singularities of the form q^{-n} such that A has no vectors of norm $-2n$. One problem with this is that it turns out there are then too many modular forms with these singularities if the p rank of A is 1 for some prime p. Another possibility is to allow singularities of the automorphic form that correspond to characteristic vectors as well as roots. The reason for this is that this would include the reflection groups of the lattices $I_{1,20}$, $I_{1,21}$, $I_{1,22}$, $I_{1,23}$ that were described in [B87].

The main theme of this paper is that lattices of negative signature that have a non-zero reflective modular form tend to be "interesting". The meaning of "interesting" depends on the dimension of the maximal positive definite sublattice. For example, negative definite lattices might be similar to the Leech lattice, Lorentzian lattices should have interesting reflection groups, and lattices in $\mathbf{R}^{2,n}$ should have interesting automorphic forms associated with them. The main reason for the definition of reflective forms is the following lemma.

Lemma 11.1. *Suppose that f is a reflective modular form for the lattice L. Then all singularities of the automorphic form of f on $G(L)$ are orthogonal to negative norm roots of L.*

Proof. By theorem 6.2 of [B98a], the singularities of Φ are orthogonal to vectors $\gamma \in L'$ such that $\gamma^2 < 0$ and the vector valued form

$$\sum_{g \in \widetilde{SL}_2(\mathbf{Z})/\Gamma_0(N)} g(f)\rho_A(g)(1)$$

of f has a singularity of type $q^{\gamma^2/2}e_\gamma$. By lemma 3.2, if the vector valued form has such a singularity, then f has a singularity $q^{\gamma^2/2}$ at some cusp a/c with the image of $\gamma \in A^{c*}$. By the definition of a reflective form this implies that γ is a root of L'. This proves lemma 11.1.

Warning. If a level N modular form f is nonzero it is possible that the corresponding vector valued modular form is zero. And even if the vector valued modular form has nontrivial singularities at cusps, it is possible that the corresponding automorphic form is zero. Although we normally expect a singularity of a vector valued form to give a singularity of the corresponding automorphic form, it is possible that all such singularities correspond to vectors of an empty set, or it is possible that all singularities happen to cancel each other out. (This is related to the well known problem that the image of a non-zero form under the theta correspondence can be zero.)

For lattices of some given level N it is usually easy to determine the reflective singularities, though the answer sometimes involves many different cases. The following lemma gives a simple condition for a singularity to be reflective, which in practice covers many cases.

Lemma 11.2. *Suppose a/c is a cusp of $\Gamma_0(N)$ and L is an even lattice of level equal to N. Then a pole of order 1 at a/c is reflective if (c, N) is a Hall divisor of N.*

Proof. The cusp a/c has width $h = N/(c^2, N) = N/(c, N)$. To show that $q_h^{-1} = q^{-1/h}$ is a reflective singularity at a/c it is enough to show that all elements $\alpha \in A^{c*}$ with $(\alpha, \alpha) \equiv -2/h \bmod 2$ satisfy $h\alpha = 0$. But this is obvious because $A^{c*} = A^c$ is the set of elements of A of order dividing h because (N, c) is a Hall divisor of N. This proves lemma 11.2.

12. Examples.

In this section we give some examples of lattices with reflective modular forms. For a given level the method for finding such forms is as follows:

1. Work out the ring of modular forms for $\Gamma_0^2(N)$, paying particular attention to the forms with zeros only at cusps.
2. Work out the possible reflective singularities for each possible discriminant form of level N.
3. Try to find modular forms with reflective singularities for each discriminant form.
4. Try to find lattices corresponding to these modular forms.

5. For each lattice with a non-zero reflective modular form, see if it is connected with any interesting reflection groups, automorphic forms, moduli spaces, or Lie algebras.

The examples probably include most interesting cases for small prime level, but become less and less complete as the level gets larger because the number of cases to consider becomes rather large. What usually seems to happen is that for each level there is some "critical" signature, with the property that almost all lattices up to that signature have non-zero reflective modular forms, but beyond that signature there are only a few isolated examples, usually with p-rank at most 2 for some prime p. For example, for level $N = 1$ the critical signature is -24, corresponding to the Leech lattice, the lattice $II_{1,25}$ whose reflection group was described by Conway, and so on, while for level $N = 2$ the critical signature is -16 corresponding to the Barnes-Wall lattice and so on.

There are several other methods for finding examples of lattices with reflective modular forms. First, the inverses of eta quotients of elements of Conway's group of automorphisms of the Leech lattice are often reflective modular forms for various lattices. Second, Hauptmoduls of genus zero subgroups of $SL_2(\mathbf{R})$ are often reflective modular forms for lattices of signature 0; see the case $N = 17$ below for some examples of this. More generally, several other modular functions of genus 0 subgroups with poles of order 1 at some cusps are reflective modular forms for some lattices. (The restriction to genus 0 subgroups is just an observation: most cases seem to be related to genus 0 subgroups. I do not know of a good theoretical reason for this.) Third, Y. Martin [M] gave a list of many eta quotients with multiplicative coefficients, and again many of these seem to be the inverses of reflective modular forms. (Note that there are many multiplicative eta quotients not on Y. Martin's list that also appear as reflective modular forms, because Y. Martin restricted himself to forms of integral weight whose conjugate under the Fricke involution was also multiplicative.) Y. Martin's list contains many examples high level, up to level 576, which suggests that there should be many reflective modular forms beyond the examples in this section.

Many of the calculations with modular forms in this section were done using the PARI calculator [BBCO].

In most of the examples below, the tables have the following meaning. "Group" describes the relevant subgroup of $SL_2(\mathbf{Z})$, "index" gives its index in $SL_2(\mathbf{Z})$, ν_2, ν_3, and ν_∞ are the numbers of elliptic points of orders 2 and 3 and the numbers of cusps of a fundamental domain, and "genus" gives the genus of the corresponding compact Riemann surface. The second table in each section is a table of the cusps, with one line per cusp. The column "cusps" gives a representative cusp, "width" is the width of the cusp, "characters" lists the non-trivial values of characters on the normalized generator of the subgroup fixing a cusp, "η" lists an eta function with a zero of order "zero" at this cusp and no other zero and with weight "weight" and character "character".

The Hilbert function is the rational function of x whose coefficients give the dimensions of the spaces of modular forms of various weights. Sometimes we put in extra variables u_p and u_θ, which describe the dimensions of spaces with various characters.

We sometimes write q_n for $\mathbf{e}(\tau/n)$.

$N = 1$.

All the results we get for this case are well known, but this case is still useful as a

warming up exercise.

The discriminant form A has order 1, and the group $\tilde{\Gamma}_0(1)$ is just $\widetilde{SL}_2(\mathbf{Z})$. The character χ is always 1.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(1)$	1	1	1	1	0

cusps	width	η	zero	weight
$i\infty$	1	1^{24}	1	12

The ring of modular forms is a polynomial ring with generators given by the Eisenstein series E_4 and E_6 of weights 4 and 6, and the Hilbert function is $1/(1-x^4)(1-x^6)$. The critical weight is $k = 12$, and the critical form is $\Delta(\tau) = \eta(\tau)^{24}$. By looking at the forms $E_4(\tau)^n/\Delta(\tau)$ we see that every even lattice L with $N = 1$ and $\text{sign}(L) \geq -24$ has a nonzero reflective modular form. The signature must be 0 mod 8.

Next we find some possible reflective forms. By lemma 11.2, poles of order at most 1 at the cusp are reflective singularities. The modular forms with poles of order at most 1 at the cusp are exactly those of the form (holomorphic modular form)$/\Delta$. So there are no examples of weight less than -12, and the ones of weights -12, -8, and -4 are multiples of $1/\Delta$, E_4/Δ, and E_4^2/Δ.

We now look at some of these cases in more detail.

For signature -24 we take f to be $1/\Delta(\tau) = q^{-1} + 24 + \cdots$. We get an automorphic form for the lattice $II_{2,26}$ of singular weight $24/2 = 12$. The corresponding Lie algebra is the fake monster Lie algebra. Its Weyl group is Conway's reflection group of the lattice $II_{1,25}$, which has a norm 0 Weyl vector. The critical lattice of this Weyl vector is of course the Leech lattice.

The lattice $II_{4,28}$ has a reflective form, and is the underlying integral lattice of Allcock's largest quaternionic reflection group [A]. The lattices $II_{n,16+n}$ for various values of $n > 2$ appear in the moduli space of K3 surfaces, possibly with some extra structure such as a B-field. The existence of a reflective form for these lattices appears to be significant in the corresponding moduli spaces as it is usually necessary to discard points of the Grassmannian that are orthogonal to norm -2 vectors.

For signature -8 and -16 we find the lattices $II_{1,9}$ and $II_{1,17}$ whose (arithmetic) reflection groups were first described by Vinberg [V75].

For signature 0 we take f to be $j(\tau) - 744 = q^{-1} + 196884q + \cdots$. We get an automorphic function for the lattice $II_{2,2}$, which is more or less $j(\sigma) - j(\tau)$ in suitable coordinates. The corresponding Lie algebra is the monster Lie algebra. The reflection group is not very interesting as it is of order 2 (and is the Weyl group of the monster Lie algebra).

$N = 2$.

The discriminant form A has order 2^{2n} for some non-negative integer n. The character χ_A is always trivial as A always has square order and signature divisible by 4.

The group $\Gamma_0(2) = \Gamma_0^2(2)$ has 2 cusps which can be taken as $i\infty$ (of width 1) and 0 (of width 2). It has one elliptic point of order 2, which can be take as the point $(1+i)/2$, fixed by $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(2)$	3	1	0	2	0

cusps	width	η	zero	weight
0	2	$1^{16}2^{-8}$	1	4
$i\infty$	1	$1^{-8}2^{16}$	1	4

The ring of modular forms for $\Gamma_0^2(2) = \Gamma_0(2)$ is a polynomial ring on the generators $-E_2(\tau) + 2E_2(2\tau) = \theta_{D_4}(\tau) = 1 + 24q + 24q^2 + O(q^3)$ of weight 2 with a zero at the elliptic point, and $E_4(\tau)$ of weight 4. The Hilbert function is $1/(1-x^2)(1-x^4)$.

All poles of order at most 1 at cusps are reflective by lemma 11.2 as $N = 2$ is square-free. If $A = II(2^{-2})$ then A has no non-zero elements of norm 1 mod 2, so a pole of order 2 at the cusp 0 is also reflective. (There are also other possible reflective singularities for lattices of high 2-rank.)

By looking at the form $\Delta_{2+}(\tau)^{-1}$, with order 1 poles at all cusps, we see that all level 2 even lattices of signature at least -16 have reflective modular forms. The Lorentzian lattices $II_{1,17}(2^{+8})$ and $II_{1,17}(2^{+10})$ have norm 0 Weyl vectors; their reflection groups are not arithmetic, but are similar to the case of $II_{1,25}$. The remaining Lorentzian lattices of dimension at most 18 have positive norm Weyl vectors, so their reflection groups are arithmetic. The lattice $II_{1,17}(2^{+2})$ is the even sublattice of an odd unimodular lattice; see [V75]. The arithmetic reflection group of the lattice $II_{1,17}(2^{+4})$ appeared recently in Kondo and Keum's work [K-K] as the Picard lattice of the Kummer surface of a generic product of elliptic curves, and can be obtained as the orthogonal complement of a D_4^2 in $II_{1,25}$.

The reflection group of $II_{1,17}(2^{+6})$ seems to be the highest dimension of a "new" example of an arithmetic reflection group in this paper. This lattice has an unusually complicated fundamental domain, with 896+64 sides. It can be described as follows. Let K be the 16 dimensional even lattice in the genus $II_{0,16}(2^{+6})$ that has root system A_1^{16}. It can be constructed by applying construction A of [C-S, chapter 5] to the first order Reed-Muller code of length 16 with 32 elements (see [C-S, p. 129], which uses construction B applied to this code to construct the Barnes-Wall lattice). Then $L = II_{1,17}(2^{+6})$ can be constructed as $K \oplus II_{1,1}$. The fundamental domain of the reflection group R of L has 2 norm 0 vectors z, z' of type K (together with many other norm 0 vectors of other lattices). The generalized Weyl vector is given by $\rho = (z + z')/2$. The fundamental domain domain has 64 walls corresponding to norm -2 roots of L and 896 walls corresponding to norm -4 roots of L (or equivalently to norm -1 roots of L'). The norm -2 simple roots split into 2 groups of 32. The first group of 32 have inner product 0 with z and -1 with z' and correspond to the 16 coordinate vectors of K and their negatives. The other 32 have inner product -1 with z and 0 with z' and correspond to the 32 elements of the Reed-Muller code. (There is of course an automorphism of the fundamental domain exchanging z and z' and the two groups of 32 norm -2 simple roots.) The 896 norm -1 simple roots of L' all have inner product 1 with both z and z' and correspond to the elements of the dual of the Reed-Muller code whose weight is 2 mod 4. The Reed-Muller code has Hamming weight enumerator $x^{16} + 30x^8y^8 + y^{16}$, so by the MacWilliams identity ([C-S, p.78]) the dual code has weight enumerator

$$\frac{(x+y)^{16} + 30(x+y)^8(x-y)^8 + (x-y)^{16}}{32}$$
$$= x^{16} + 140y^4x^{12} + 448y^6x^{10} + 870y^8x^8 + 448y^{10}x^6 + 140y^{12}x^4 + y^{16}$$

and therefore there are 448+448=896 elements of length 2 mod 4. Alternatively, L can be constructed as the orthogonal complement of a certain A_1^8 in the Dynkin diagram of $II_{1,25}$. (Note that the Dynkin diagram of $II_{1,25}$ contains more than one orbit of subsets isomorphic to A_1^8. The orbit we use has the special property that it is not contained in a $A_1^7 A_2$ sub-diagram.) The 64 norm -2 roots correspond to the 64 ways to extend the A_1^8 to an A_1^9, and the 896 norm -1 simple roots correspond to the 896 ways of extending it to an $A_3 A_1^6$ diagram. These $A_3 A_1^6$ diagrams are the same as those used by Kondo in [K] to describe the automorphism group of a generic Jacobian Kummer surface. The automorphism group of the Dynkin diagram of L has a normal subgroup of order 2^{10} and the quotient is the alternating group A_8. Unfortunately $II_{1,17}(2^{+6})$ cannot be the Picard lattice of a K3 surface: S. Kondo pointed out to me that its 2-rank (6) is larger than its codimension (4) in $II_{3,19}$.

The lattices $II_{n,n+20}(2^{-2})$ also have a reflective modular form $\theta_{D_4}(\tau)/\Delta(\tau)$. In particular the Lorentzian lattice $II_{1,21}(2^{-2})$ has an arithmetic reflection group; see [B87, p.149] for a description of its fundamental domain. Esselmann [E] showed that this is essentially the only example of a cofinite reflection group of a Lorentzian lattice of dimension at least 21. (Of course we can find trivial variations of this example by taking the Atkin-Lehner conjugate $II_{1,21}(2^{-20})$, or by multiplying all norms by a constant.)

We can find some automorphic forms of singular weight, corresponding to lattices L and reflective modular forms f as follows.

1. $L = II_{2,18}(2^{+10})$, $f = \eta_{1^{-8}2^{-8}}$. This is the denominator function of a generalized Kac-Moody algebra of rank 18. This example is related to the element $2A$ of $\mathrm{Aut}(\Lambda)$, of cycle shape $1^8 2^8$. There are 24 lattices in the genus $II_{0,16}(2^{+8})$ by [S-V], one of which is the Barnes-Wall lattice, and the others all have root systems of rank 16.

2. $L = II_{2,10}(2^{+2})$, $f = 2^4 \eta_{1^{-16}2^8}$. This example is related to the element $-2A$ of $\mathrm{Aut}(\Lambda)$.

3. $L = II_{2,10}(2^{+10})$, $f = \eta_{1^8 2^{-16}}$

The first case is closely related to the reflection group of the lattice $II_{1,17}(2^{+8})$, whose fundamental domain has a nonzero norm 0 vector fixed by its automorphism group, as in the lattice $II_{1,25}$. For more about this lattice and its reflection group see [B90].

The last two cases are really the same, since the lattices are Atkin-Lehner conjugates of each other, and the automorphic forms we get are more or less the same. This automorphic form is the denominator function of two generalized Kac-Moody superalgebras, and is also closely related to the moduli space of Enriques surfaces. See [B98a, example 13.7], [B96] and [Sch] for more details.

If R is the reflection group of the lattice $II_{1,17}(2^{+2})$ generated by the reflections of norm -2 vectors and D is its fundamental domain, then $\mathrm{Aut}(D)$ has a finite index subgroup isomorphic to $\mathbf{Z}$ and fixes a nonzero norm 0 vector z. However there seems to be no reflective modular form for $\Gamma_0(2)$ corresponding to this reflection group.

The remaining level 2 cases of signatures -4, -8 and -12 are left to the reader; they all give known arithmetic reflection groups, often associated to unimodular lattices as in [V75].

$N = 3.$

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(3)$	4	0	1	2	0

cusps	width	η	zero	weight
0	3	$1^9 3^{-3}$	1	3
$i\infty$	1	$1^{-3} 3^9$	1	3

The character χ_3 is trivial for forms of even weight and nontrivial for forms of odd weight. The forms of integer weights and arbitrary character are the same as the forms for $\Gamma_1(3)$ and trivial character. Note that $\Gamma_0(3)$ is the product of $\Gamma_1(3)$ and its center of order 2 generated by Z.

The ring of modular forms for $\Gamma_0^2(3) = \Gamma_1(3)$ is a polynomial ring generated by $\theta_{A_2}(\tau) = E_1(\tau,\chi_3) = 1+6q+6q^3+6q^4+O(q^7)$ of weight 1 and $E_3(\tau,\chi_3) = \eta(\tau)^{-3}\eta(3\tau)^9 = q+3q^2+9q^3+13q^4+24q^5+O(q^6)$ of weight 3. The Hilbert function is $1/(1-u_3x)(1-u_3x^3)$.

Next we find some reflective singularities. At the cusp $i\infty$ the singularity q^{-1} is reflective. At the cusp 0 the singularity q_3^{-1} is reflective. If the discriminant form A is $II(3^{\pm 1})$ or $II(3^{-2})$ then A has no nonzero elements of norm 0 mod 2, so the singularities q_3^{-1}, q_3^{-3} are reflective at the cusp 0.

The forms $\Theta_{A_2}(\tau)^n/\Delta_{3+}$ show that all level 3 even lattices of signature at least -12 have non-zero reflective modular forms.

There are also some other examples of reflective modular forms for lattices of small 3-rank. If we take the signature to be -18 and take A to be $II(3^{+1})$ then the form $\Theta_{E_6}(\tau)/\Delta(\tau)$ is reflective. This can be used to show that the reflection group of the lattice $II_{1,19}(3^{+1})$ is arithmetic. This reflection group was first found by Vinberg [V85] in his investigations of the "most algebraic" K3 surfaces. The lattice can also be constructed as the orthogonal complement of an E_6 in $II_{1,25}$, and this gives another proof that the reflection group is arithmetic [B87].

Next take the signature to be -14 and take A to be $II(3^{-1})$. We let f be the form

$$\frac{E_1(\tau,\chi_3)^5 - 270 E_1(\tau,\chi_3)^2 \eta(\tau)^{-3}\eta(3\tau)^9}{\Delta(\tau)} = q^{-1} - 216 - 9126q + O(q^2).$$

The constant 270 is chosen so that the coefficient of $q^{-2/3} = q_3^{-2}$ of $f(-1/\tau) = -i3^{-5/2}\tau^3(-9q^{-1} + 810q^{-1/3} + 1944 + 53136q^{2/3} + O(q))$ vanishes. So the automorphic forms with singularities constructed from f have all their singularities orthogonal to roots. However something unexpected now happens: the $\mathbf{C}[A]$ valued modular form induced from f is identically zero! So the piecewise linear automorphic forms constructed from f as in theorem 4.2 have no singularities and are also zero.

In spite of this the reflection group of $L = II_{1,15}(3^{-1})$ still has a nonzero vector (of norm 0) in the fundamental domain fixed by the automorphism group of the fundamental domain. To see this, we represent L as the orthogonal complement of an A_2 in $II_{1,17}$. Then the quotient of $\mathrm{Aut}^+(L)$ by the reflection group can be worked out using theorem 2.7 of [B98b] and turns out to be an infinite dihedral group, which has an index 2 subgroup isomorphic to $\mathbf{Z}$. Next we can classify the primitive norm 0 vectors z of L, and we find that there are just two orbits, with the lattice $z^\perp/z$ having root systems $E_8 E_6$ and D_{13}. Fix ρ to be a primitive norm 0 vector corresponding to a lattice with root system D_{13}. As D_{13} has rank one less than the corresponding lattice, there is a group $\mathbf{Z}$ of automorphisms of the fundamental domain fixing z. This group $\mathbf{Z}$ has finite index in the full automorphism group of the fundamental domain, so the full automorphism group of the fundamental

domain must fix z. However z is not quite a Weyl vector, as it has zero inner product with some simple roots (forming an affine D_{13} Dynkin diagram) and has inner product 1 with the others.

There are 10 lattices in the genus $II_{0,12}(3^{+6})$ [S-V]. One is the Coxeter-Todd lattice with no roots, and the others all have root systems of rank 12.

There are also some automorphic forms of singular weight corresponding to the following lattices and reflective forms:

1 $II_{2,14}(3^{-8})$, $\eta_{1^{-6}3^{-6}}$.
2 $II_{2,8}(3^{+7})$, $\eta_{1^3 3^{-9}}$.
3 $II_{2,8}(3^{+3})$, $3^2\eta_{1^{-9}3^3}$.

The lattice $II_{2,8}(3^{+5})$ appears in [A-C-T], where it is the underlying integral lattice of a unimodular Eisenstein lattice. The automorphic forms for this case have been studied in great detail by Freitag in [A-F], [F99]. In particular there is one of weight 12 (coming from the function $27\eta_{1^{-9}3^3}$) whose restriction to complex hyperbolic space $\mathbf{C}H^4$ vanishes (to order 3) exactly along the reflection hyperplanes of a certain complex reflection group related to the moduli space of cubic surfaces. So its cube root is an automorphic form of weight 4 with order 1 zeros along all the reflection hyperplanes [A].

We have seen above that sometimes the Weyl vector of a reflective form unexpected vanishes because all the singularities just happen to cancel out. Another way that the Weyl vector can unexpectedly vanish is if the vectors corresponding to the singularities happen not to exist (usually when the p-rank of A is small). For example, for the lattice $L = II_{2,8}(3^{-1})$, the automorphic form is constant even though the vector valued modular form has non-trivial singularities. The singularities of the vector valued modular form imply that automorphic form has zeros corresponding to all norm $4/3$ vectors of L', but L' happens to have no such vectors so the automorphic form is constant.

$N = 4$.

The group $\Gamma_0(4)$ has 3 cusps which can be taken as $i\infty$ (of width 1) and 0 (of width 4) and $1/2$ (of width 1). It has no elliptic points and has genus 0.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(4)$	6	0	0	3	0

cusps	width	characters	η	zero	weight	character
1=0	4		$1^8 2^{-4}$	1	2	
1/2	1	$\chi_\theta = -i$	$1^{-2} 2^5 4^{-2}$	1/4	1/2	χ_θ
$1/4 = i\infty$	1		$2^{-4} 4^8$	1	2	

The double cover of $\Gamma_0(4)$ is the product of its center of order 4 (generated by Z) and a subgroup that can be identified with $\Gamma_1(4)$.

The ring of modular forms of integral or half integral weight for $\Gamma_1(4)$ is a polynomial ring generated by $\theta_{A_1}(\tau) = 1 + 2q + 2q^4 + O(q^9)$ of weight $1/2$ and $\eta(\tau)^8\eta(2\tau)^{-4} = 1 - 8q + \cdots$ of weight 2. The Hilbert function is $1/(1 - u_\theta x^{1/2})(1 - x^2)$. The ideals of cusp forms vanishing at $i\infty$, 0, or $1/2$ are generated by $\eta(2\tau)^{-4}\eta(4\tau)^8$, $\eta(\tau)^8\eta(2\tau)^{-4}$, $\eta(\tau)^{-2}\eta(2\tau)^5\eta(4\tau)^{-2}$. Note that the last function has a zero of order $1/4$ at $1/2$. The ideal of cusp forms of even weight is generated by $\Delta_{4+}(\tau) = \eta(2\tau)^{12}$ of weight 6.

If L is a unimodular positive definite lattice then $\theta_L(2\tau)$ is a modular form for $\Gamma_1(4)$.

Next we find some reflective singularities. To reduce the number of cases to consider we will assume that $A = L'/L$ has exponent 2, so that A is $II(2_t^{+n})$ for some t and n. As usual, poles of order 1 are reflective singularities at the cusps $i\infty$ and 0. At the cusp $1/2$, poles of order $1/4$ and $1/2$ are reflective, because all elements of A^{2*} have order 1 or 2. If the parity vector of A does not have norm 0 mod 2 then a pole of order 1 at $1/2$ is also reflective. Finally, at the cusp 0 poles of order 1 or 2 are reflective, and if A has no non-zero vectors of norm 0 mod 2 then poles of order 4 are reflective.

The form $\eta(\tau)^{-12}\eta(2\tau)^{-2}\eta(4\tau)^4$ shows that all level 4 exponent 2 lattices of signature -14 have non-zero reflective modular forms. By multiplying this form by $\theta_{A_1}(\tau)^n$ for $n \geq 1$ we see that the level 4 lattices of signature at least -14 have non-zero reflective modular forms.

We can find many examples of eta quotients that are eigenforms of Hecke operators by finding eta quotients with poles of order at most 1 at all cusps. This gives 15 non-constant examples as follows: $\eta_{1^{-2}2^54^{-2}}$, $\eta_{1^{-4}2^{10}4^{-4}}$, $\eta_{1^{-8}2^{20}4^{-8}}$, $\eta_{1^82^{-4}}$, $\eta_{1^62^14^{-2}}$, $\eta_{1^42^64^{-4}}$, $\eta_{2^{16}4^{-8}}$, $\eta_{2^{-4}4^8}$, $\eta_{1^{-2}2^14^6}$, $\eta_{1^{-4}2^64^4}$, $\eta_{1^{-8}2^{16}}$, $\eta_{1^{-8}2^84^{-8}}$, $\eta_{1^{-6}2^34^{-6}}$, $\eta_{1^{-4}2^{-2}4^{-4}}$, $\eta_{2^{-12}}$. The inverses of these forms are often reflective forms for various lattices. Note that the forms $\eta_{1^{-6}2^{15}4^{-6}}$, $\eta_{1^22^{11}4^{-6}}$, $\eta_{1^{-6}2^{11}4^2}$, $\eta_{1^22^74^2}$ are eigenfunctions of Hecke operators, but as they have a zero of order $3/4$ at $1/2$ their inverses do not usually give reflective automorphic forms (except for rather special discriminant forms). The form $\eta_{1^22^74^2}$ is the highest weight eta product I know of that is an eigenform and has non-integral weight.

Most of the time lattices of positive signature with reflective forms do not seem to be interesting, but there are some exceptions. For example, there is a reflective form for the lattice $II_{2,1}(2_1^{+1})$. The corresponding automorphic form is essentially E_6, which is the denominator function of a generalized Kac-Moody algebras. See [B95, section 15, example 2] for more details.

The lattices $II_{1,19}(2_6^{+2})$ $II_{1,15}(2_2^{+2})$ $II_{1,11}(2_6^{+2})$ have reflective forms of type θ_{D_n}/Δ. They are the even sublattices of odd unimodular lattices, and have cofinite reflection groups, as was first found by Vinberg ([V75]).

The function θ_{E_7}/Δ is a reflective form for the lattices $II_{n,n+17}(2_{-1}^{+1})$. In particular we find the Nikulin's example of the Lorentzian lattice $II_{1,18}(2_{-1}^{+1})$ whose reflection group is arithmetic. This example can also be constructed as the orthogonal complement of an E_7 in $II_{1,25}$.

Yoshikawa [Y] used the automorphic forms coming from the modular forms $\eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\theta_{A_1}(\tau)^k$ to construct automorphic products (for odd unimodular lattices). These automorphic products are the squares of discriminant forms of various moduli spaces of "generalized Enriques surfaces", and can also be constructed using analytic torsion.

$N = 5$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(5)$	6	2	0	2	0

cusps	width	η	zero	weight
0	5	1^55^{-1}	1	2
$i\infty$	1	$1^{-1}5^5$	1	2

The ring of modular forms for $\Gamma_0^2(5)$ is not a polynomial ring, but is generated by the

3-dimensional space of weight 2 forms, which is spanned by $\eta(\tau)^5\eta(5\tau)^{-1}$, $\eta(\tau)^{-1}\eta(5\tau)^5$ (of nontrivial character) and $E_2(\tau) - 5E_2(5\tau)$ (of trivial character). The Hilbert function is $(1 + x^2)/(1 - u_5x^2)^2$.

Remark. The ring of all modular forms of integral weight for $\Gamma_1(5)$ is a polynomial ring generated by the weight 1 Eisenstein series $1 + (3 + i)(q + (1 - i)q^2 + (1 + i)q^3 - iq^4 + q^5 + O(q^6))$ and its complex conjugate. These correspond to the two complex conjugate order 4 characters of $\mathbf{Z}/4\mathbf{Z}$, and each of them has a simple zero at one of the elliptic points and no other zeros. The subring of forms of even weight is the ring of modular forms for $\Gamma_0^2(5)$.

Even lattices of level 5 all have signature divisible by 4. The form $\Delta_5(\tau)^{-1}$ shows that all level 5 even lattices of signature -8 and even 5-rank have non-zero reflective modular forms. (So does the lattice of 5-rank 1; see below.) If we multiply this form by products of powers of $E_2(\tau) - 5E_2(5\tau)$ and $\eta(\tau)^5\eta(5\tau)^{-1}$ we also see that all even level 5 lattices of signature at least -4 have non-zero reflective modular forms.

For the discriminant forms $A = II(5^{\pm 1})$ or $II(5^{+2})$ the only norm 0 element is 0, so q_5^{-1} and q_5^{-5} are all reflective singularities at the cusp 0. If we take A to be $II(5^{-1})$ and take the signature to be -8 then there is a reflective form. This gives a Lorentzian lattice $II_{1,9}(5^{-1})$ with a reflection group of finite index. If we take L to be $II_{1,17}(5^{-1})$ then there is a reflective automorphic form. (This is slightly surprising as the space of forms with a pole of order at most 1 at $i\infty$ and a pole of order at most 5 at 0 is 2 dimensional, so we would normally expect there to be no such forms satisfying the 2 conditions that the coefficients of q_5^{-2} and q_5^{-3} both vanish. However it turns out that these two conditions are not independent; in fact the modular form we get has "complex multiplication", (see [Ri]) meaning that the coefficient of q_5^n is 0 whenever $n \equiv 2, 3 \bmod 5$.) In spite of the existence of a non-zero reflective modular form, the reflection group of $II_{1,17}(5^{-1})$ is not cofinite, and does not even have virtually free abelian index. (In particular, this lattice is a counterexample to several otherwise plausible conjectures about Lorentzian lattices with non-zero reflective modular forms.) As a substitute for this, the lattice is very closely related to Bugaenko's largest example of a co-compact hyperbolic reflection group. In fact $II_{1,17}(5^{-1})$ can be made into a lattice over $\mathbf{Z}[\phi]$, and Bugaenko [B] showed that the corresponding hyperbolic reflection group was co-compact. The relationship between Bugaenko's reflection group and the reflective form is rather mysterious. The lattice has 5 orbits of primitive norm 0 vectors, corresponding to the 5 elements of the genus $II_{0,16}(5^{-1})$, which have root systems A_2A_{14}, E_7A_9, E_6D_9, $E_8E_75A_1$, $D_{14}A_15A_1$.

It is possible to produce some examples of co-compact hyperbolic reflection group from level 5 lattices as follows.

Lemma 12.1. *Suppose that L is an even Lorentzian lattice of level 5, and suppose that there is a self adjoint endomorphism ϕ of L such that $\phi^2 = \phi + 1$. Let H^ϕ be the hyperbolic space of the Lorentzian eigenspace $(L \otimes \mathbf{R})^\phi$. Then the subgroup of the reflection of L acting on H^ϕ is a hyperbolic reflection group of H^ϕ. If W is cofinite then W^ϕ is co-compact.*

Proof. Let H be the hyperbolic space of L, and H^ϕ the subspace of it fixed by ϕ. The main point is that the intersection of any reflection hyperplane of W with H^ϕ is a reflection hyperplane of the group W^ϕ acting on H^ϕ. To see this, recall that a reflection of W is the reflection of a norm -2 vector of L or a norm $-2/5$ vector of L'. First suppose

first that v is a norm -2 vector of L. As $v^\perp$ intersects H^ϕ, $v^{\phi\perp}$ and v must generate a negative definite space. This easily implies that v and $v^\perp$ span a lattice isomorphic to A_1^2, and the product of two reflections of this lattice is the automorphism -1 which commutes with ϕ. This is a reflection of W^ϕ acting on H^ϕ whose reflection hyperplane is $v^\perp \cap H^\phi$. The argument when v is a norm $-2/5$ vector of L' is similar.

It now follows that W^ϕ is a reflection group acting on H^ϕ whose fundamental domain is the intersection of H^ϕ with a fundamental domain of W acting on H.

Finally if W is cofinite then all norm 0 vectors in the fundamental domain of W are rational and therefore cannot be fixed by ϕ, so the fundamental domain of W^ϕ has no norm 0 vectors in it and is therefore compact. This proves lemma 12.1.

Unfortunately, this lemma does not give the largest examples found by Bugaenko.

If we take A to be $II(5^{+1})$ and take the signature to be -12 then q_5^{-4} is a reflective singularity at 0 as A has no nonzero elements of norm $-4/5 \bmod 2$, and q_5^{-5} is reflective as any norm 0 element of A is 0. So A has a reflective modular form of weight -6, level 5, and character χ_5 whose singularity at $i\infty$ is a multiple of q^{-1} and whose singularity at 0 is a linear combination of q_5^{-1} and q_5^{-4} and q_5^{-5}. (There is a 2 dimensional space of such forms.) This gives a Lorentzian lattice $L = II_{1,13}(5^{+1})$ with $\mathrm{Aut}^+(L)/R$ infinite dihedral. It is the orthogonal complement of an A_4 in $II_{1,17}$. This case is similar to $II_{1,15}(3^{-1})$.

The (level 1) form E_6/Δ has a singularity at 0 of the form $q^{-1} = q_5^{-5}$ so it is a reflective modular form for the lattice $L = II_{1,13}(5^{-2})$. The corresponding vector valued modular form is 0. The lattice L is a module over $\mathbf{Z}[\phi]$. It may be the lattice of the orthogonal complement of an $I_2(5)$ in Bugaenko's lattice, which would imply that it has a co-compact reflection group.

As in the case $N = 3$ we also get a few examples of automorphic forms of singular weight coming from the reflective forms $\eta_{1^{-4}5^{-4}}$, $\eta_{1^{-1}5^5}$, and $\eta_{1^5 5^{-1}}$. There are 5 lattices in the genus $II_{0,8}(5^{+4})$ corresponding to the case $II_{2,10}$ by [S-V]. One has no roots and by [S-H p. 744] the roots systems of the other 4 are $A_1^4 5 A_1^4$, $A_2^2 5 A_2^2$, $A_4 5 A_4$, $D_4 5 D_4$.

Problem: does the lattice $II_{2,6}(5^{+3})$ correspond to some nice moduli space, in the same way that the corresponding lattices $II_{2,10}(2^{+2})$ and $II_{2,8}(3^{+5})$ for levels 2 and 3 correspond to the moduli spaces of Enriques surfaces or cubic surfaces?

$N = 6$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(6)$	12	0	0	4	0

cusps	width	η	zero	weight
1=0	6	$1^6 2^{-3} 3^{-2} 6^1$	1	1
1/2	3	$1^{-3} 2^6 3^1 6^{-2}$	1	1
1/3	2	$1^{-2} 2^1 3^6 6^{-3}$	1	1
1/6 = $i\infty$	1	$1^1 2^{-2} 3^{-3} 6^6$	1	1

The group $\Gamma_0(6)$ is the product of $\Gamma_1(6)$ and its center of order 2 generated by Z. The forms of trivial character have even weight, and those of nontrivial character have odd weight.

The ring of modular forms of integral weight for $\Gamma_0(6)$ is a polynomial ring generated by $E_1(\tau, \chi_3)$ and $E_1(2\tau, \chi_3)$. The Hilbert function is $1/(1 - u_3 x)^2$. The ideal of cusp forms is generated by $\Delta_{6+}(\tau) = \eta(\tau)^2 \eta(2\tau)^2 \eta(3\tau)^2 \eta(6\tau)^2$ of weight 4.

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We can find eta quotients with an given integral order poles at the cusps. In particular we find the following 15 non-constant holomorphic eta quotients with zeros of order at most 1 at all cusps: $\eta_{1^62^{-3}3^{-2}6^1}$, $\eta_{1^{-3}2^63^16^{-2}}$, $\eta_{1^{-2}2^13^66^{-3}}$, $\eta_{1^12^{-2}3^{-3}6^6}$, $\eta_{1^32^33^{-1}6^{-1}}$, $\eta_{1^{-1}2^{-1}3^36^3}$, $\eta_{1^42^{-2}3^46^{-2}}$, $\eta_{1^{-2}2^43^{-2}6^4}$, $\eta_{1^72^{-5}3^{-5}6^7}$, $\eta_{1^{-5}2^73^76^{-5}}$, $\eta_{1^12^43^56^{-4}}$, $\eta_{1^42^13^{-4}6^5}$, $\eta_{1^52^{-4}3^16^4}$, $\eta_{1^{-4}2^53^46^1}$, $\eta_{1^22^23^26^2}$. Their inverses give numerous examples of reflective forms for various lattices. For example, $\eta_{1^{-2}2^{-2}3^{-2}6^{-2}}$ is a reflective form for all even level 6 lattices of signature -8, and by multiplying by a power of $E_1(\tau, \chi_3)$ we get reflective forms whenever the signature is at least -8. The other eta quotients give many examples where roots of certain norms are excluded.

We can also find many examples of signature less than -8 if we restrict the 2-rank or 3-rank to be at most 2. One example of such a lattice with an arithmetic reflection group is the orthogonal complement $II_{1,15}(2^{-2}3^{-1})$ of a $D_4 E_6$ root system in $II_{1,25}$.

$N = 7$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(7)$	8	0	2	2	0

cusps	width	η	zero	weight
0	7	1^77^{-1}	2	3
$i\infty$	1	$1^{-1}7^7$	2	3

The ring of modular forms of integral weight for $\Gamma_0^2(7)$ is generated by $E_1(\tau, \chi_7)$ (of weight 1 which vanishes at both elliptic points), $\Delta_{7+}(\tau)$ (of weight 3 which vanishes at both cusps), and the two weight 3 Eisenstein series. The Hilbert function is $(1+u_7 x^3)/(1-u_7 x)(1-u_7 x^3)$. The ideal of cusp forms is generated by $\Delta_{7+}(\tau) = \eta(\tau)^3\eta(7\tau)^3$ of weight 3. Note that the ideal of forms vanishing at $i\infty$ is not principal. The function $\eta(\tau)^4\eta(7\tau)^{-4}$ is a Hauptmodul for $\Gamma_0(7)$.

Remark. The ring of modular forms for $\Gamma_1(7)$ of integral weight has a simpler structure: it is generated by the three weight 1 forms $E_1(\tau, \chi_7)$, $E_1(\tau, \chi)$, $E_1(\tau, \bar{\chi})$, where χ is a character of $\mathbf{Z}/7\mathbf{Z}$ of order 6 and $\bar{\chi}$ is its complex conjugate. The ideal of relations between these generators is generated by $E_1(\tau, \chi_7)^2 - E_1(\tau, \chi)E_1(\tau, \bar{\chi})$. We can even embed this into a polynomial ring of modular forms: all the zeros of the forms $E_1(\tau, \chi)$ and $E_1(\tau, \bar{\chi})$ have order 2, so their square roots are also modular forms (of half integral weight for a strange character of $\Gamma_0(7)$), and they generate a polynomial ring whose elements of integral weight are the modular forms for $\Gamma_1(7)$.

Any even lattice of level 7 and signature at least -6 has a reflective form of the form $E_1(\tau, \chi_7)^n \eta_{1^{-3}7^{-3}}$.

The automorphic form associated to $\eta_{1^{-3}7^{-3}}$ and the lattice $II_{2,8}(7^{+5})$ has singular weight and is the denominator function of a generalized Kac-Moody algebra. The reflection group of $II_{1,7}(7^{-3})$ has a norm 0 Weyl vector. The corresponding genus $II_{0,6}(7^{-3})$ has 3 elements [S-H, proposition 3.4a], and by [S-H table 1] there is one with no roots (corresponding to the norm 0 Weyl vector), one with root system A_37A_3 and one with root system $A_1^37A_1^3$.

For the discriminant form $II(7^{-1})$ the singularities q_7^{-1} and q_7^{-7} are reflective. The lattice $II_{1,11}(7^{-1})$ has a reflective form, and is the orthogonal complement of an A_6 in $II_{1,17}$. The quotient $\text{Aut}^+(L)/R$ is infinite dihedral, and fixes a norm 0 vector corresponding to a lattice in the genus $II_{0,10}(7^{-1})$ with root system D_9. There is a second lattice in

this genus, isomorphic to the sum of E_8 and a 2 dimensional definite lattice of determinant 7, so its root system is $E_8 A_1 7A_1$.

$N = 8$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(8)$	12	0	0	4	0

cusps	width	characters		η	zero	weight	character
1=0	8			$1^4 2^{-2}$	1	1	χ_θ^2
1/2	2	$\chi_2 = \chi_\theta = -1$		$1^{-2} 2^5 4^{-2}$	1/2	1/2	χ_θ
1/4	1	χ_2	$= -1$	$2^{-2} 4^5 8^{-2}$	1/2	1/2	$\chi_\theta \chi_2$
$1/8 = i\infty$	1			$4^{-2} 8^4$	1	1	χ_θ^2

The double cover of $\Gamma_0(8)$ is the product of its center of order 4 (generated by Z) and a subgroup that can be identified with its image $\Gamma_0(8) \cap \Gamma_1(4)$. The ring of modular forms of integral or half integral weight for $\Gamma_0^2(8) = \Gamma_1(8)$ is a polynomial ring generated by $\eta(2\tau)^{-2} \eta(4\tau)^5 \eta(8\tau)^{-2}$ (of weight 1/2 and character $\chi_2 \chi_\theta$) and $\eta(\tau)^{-2} \eta(2\tau)^5 \eta(4\tau)^{-2}$ (of weight 1/2 and character χ_θ). The Hilbert function is $1/(1 - u_\theta u_2 x^{1/2})(1 - u_\theta x^{1/2})$.

There are many level 8 discriminant forms and many possible reflective singularities. Together they give a bewildering number of examples of level 8 lattices with reflective forms; they are probably best left to a computer to classify. As examples we will just mention $II_{1,18}(4_7^{+1})$, $II_{1,16}(4_1^{+1})$, $II_{1,14}(4_3^{-1})$, $II_{1,12}(4_5^{-1})$. These are the even sublattices of some of the odd unimodular lattices with cofinite reflection groups found by Vinberg and Kaplinskaja [V-K].

$N = 9$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(9)$	12	0	0	4	0

cusps	width	η	zero	weight
1=0	9	$1^3 3^{-1}$	1	1
1/3, 2/3	1	$1^{-3} 3^{10} 9^{-3}$	1,1	2
$1/9 = i\infty$	1	$3^{-1} 9^3$	1	1

The group $\Gamma_0(9)$ is the product of its center of order 2 and the group $\Gamma_0^2(9)$, so a form has nontrivial character if and only if it has odd weight. The ring of modular forms of integral weight for $\Gamma_0^2(9)$ is a polynomial ring generated by $\eta(\tau)^3 \eta(3\tau)^{-1}$ and $\eta(9\tau)^3 \eta(3\tau)^{-1}$. The Hilbert function is $1/(1 - x)^2$.

For the sake of completeness we also describe generators for the ring of all integral weight modular forms for $\Gamma_1(9)$. This is a 3 dimensional free module over the ring of modular forms for $\Gamma_0^2(9)$, with a basis consisting of 1 and the two weight 1 Eisenstein series for the two order 6 characters of $\mathbf{Z}/6\mathbf{Z}$. Each of these Eisenstein series has zeros of order 1/3 and 2/3 at the cusps 1/3 and 2/3 (not necessarily in that order). Note that the character of a modular form can be read off from the parity of its weight and the fractional part of the order of the zero at 1/3. We can embed this ring in a polynomial ring, generated by the cube roots of the two weight 1 modular forms with poles of order 1 at 1/3 or 2/3.

The form $\eta(\tau)^{-3} \eta(3\tau)^2 \eta(9\tau)^{-3}$ is a reflective form for all even level 9 lattices of signature -4, and by multiplying it by a suitable power of (say) $\theta_{A_2}(\tau)$ we get reflective forms whenever the signature is at least -4.

A few examples of modular forms that might correspond to automorphic forms of singular weight on some lattices are $\eta_{3^{-8}}$, $\eta_{1^{-3}3^1}$, $\eta_{3^19^{-3}}$, and $\eta_{1^{-3}3^29^{-3}}$.

$N = 10$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(10)$	18	2	0	4	0

cusps	width	η	zero	weight
$1=0$	10	$1^{10}2^{-5}5^{-2}10^1$	3	2
$1/2$	5	$1^{-5}2^{10}5^{1}10^{-2}$	3	2
$1/5$	2	$1^{-2}2^{1}5^{10}10^{-5}$	3	2
$1/10 = i\infty$	1	$1^{1}2^{-2}5^{-5}10^{10}$	3	2

The ring of modular forms of integral weight for $\Gamma_0^2(10)$ is generated by the 7 dimensional space of forms of weight 2, and the ideal of relations between them is generated by 15 quadratic relations. The Hilbert function is $(1 + 5x^2)/(1 - x)^2$.

Remark. We can embed the ring of modular forms for $\Gamma_0^2(10)$ in a polynomial ring as follows. The ring of modular forms of integral weight for $\Gamma_1(10)$ is generated by the 4 dimensional space of forms of weight 1. We can find two of these forms that have zeros of order 3 at the two elliptic points of order 2. Their cube roots are modular forms of weight $1/3$ for characters of order 3 of $\Gamma_1(10)$, and generate a polynomial ring in 2 variables. The space of modular forms for $\Gamma_0^2(10)$ can be identified with the polynomials of degree divisible by 6.

The cube root of the product of any three of the forms above with zeros only at one cusp is a form with a zero of order 1 at 3 of the 4 cusps, and these 4 forms are a basis of the space of weight 2 forms with non-trivial character. Their inverses are the functions $\eta_{1^{-1}2^{-2}5^{-3}10^2}$, $\eta_{1^{-2}2^{-1}5^210^{-3}}$, $\eta_{1^{-3}2^25^{-1}10^{-2}}$, $\eta_{1^22^{-3}5^{-2}10^{-1}}$.

Any one of these 4 functions (of non-trivial character and weight -2) shows that any even level 10 lattice of signature -4 and odd 5-rank has a reflective form. There is also a weight -2 form of trivial character whose poles and zeros are a pole of order 1 at each cusp and order 1 zeros at the two elliptic points. (Construction: take a linear combination of the weight 2 Eisenstein series with trivial character that vanishes at 2 cusps (this automatically vanishes at the two elliptic points), then divide it by an eta product with order 2 zeros at these cusps and order 1 zeros at the other two cusps.) This is a reflective form for the even level 10 lattice of signature -4 with even 5-rank.

So every even level 10 lattice of signature ≥ -4 has a reflective form.

$N = 11$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(11)$	12	0	0	2	1

cusps	width	η	zero	weight
0	11	$1^{11}11^{-1}$	5	5
$i\infty$	1	$1^{-1}11^{11}$	5	5

The ring of modular forms of integral weight for $\Gamma_0^2(11)$ is generated by $E_1(\tau, \chi_{11})$ (of weight 1 which vanishes at both elliptic points), $\Delta_{11+}(\tau)$ (of weight 2 which vanishes at both cusps), and the two weight 3 Eisenstein series. The Hilbert function is $(1 + x^3)/(1 - x)(1 - x^2)$. The ideal of cusp forms is generated by $\Delta_{11+}(\tau) = \eta(\tau)^2\eta(11\tau)^2$ of weight 2.

The ideal of forms vanishing at $i\infty$ is not principal. The function $\eta(\tau)^{12}\eta(11\tau)^{-12}$ has a pole of order 5 at $i\infty$ and a zero of order 5 at 0 and is a modular function for $\Gamma_0(11)$, showing that 0 is a torsion point of order 5 on the modular elliptic curve of $\Gamma_0(11)$. (In fact this point generates the subgroup of rational points on this elliptic curve.)

The forms $E_1(\tau,\chi_{11})^n\eta_{1^{-2}11^{-2}}$ show that all even lattices of level 11 and signature at least -4 are reflective. The lattice $II_{1,7}(11^{-1})$ has a reflection group of infinite dihedral index in its automorphism group. The corresponding genus $II_{0,6}(11^{-1})$ contains just one lattice, which has root system D_5.

$N = 12.$

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(12)$	24	0	0	6	0

cusps	width	characters		η	zero	$weight$
1=0	12			$1^6 2^{-3} 3^{-2} 6^1$	2	1
1/2	3	χ_θ	$=\ i$	$1^{-6} 2^{15} 3^2 4^{-6} 6^{-5} 12^2$	2	1
1/3	4			$1^{-2} 2^1 3^6 6^{-3}$	2	1
1/4	3			$2^{-3} 4^6 6^1 12^{-2}$	2	1
1/6	1	χ_θ	$=\ -i$	$1^2 2^{-5} 3^{-6} 4^2 6^{15} 12^{-6}$	2	1
$1/12 = i\infty$	1			$2^{14} 4^{-2} 6^{-3} 12^6$	2	1

The ring of modular forms is generated by the forms $\theta(\tau)$, $\theta(3\tau)$, $E_1(\tau,\chi_3)$, and $E_1(2\tau,\chi_3)$ and the Hilbert function is $(1 + x^{1/2} + 2x)/(1 - x^{1/2})(1 - x)$.

There are quite a lot of modular forms whose zeros are all zeros of order at most 1 at cusps: we can find forms with zeros of order $1/2$ at $1/2$ and $1/6$ and an odd number of zeros at the other cusps, or forms whose zeros at $1/2$ and $1/6$ have orders $(0,0)$, $(1/4, 3/4)$, $(3/4, 1/4)$, or $(1,1)$ and that have an even number of zeros at the other cusps. The maximum weight of these forms is 3, attained by the form $\eta(2\tau)^3\eta(6\tau)^3$ with a zero of order 1 at every cusp. The inverses of these forms are reflective forms for many lattices of signature up to -6.

$N = 13.$

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(13)$	14	2	2	2	0

cusps	width	η	zero	weight
0	13	$1^{13} 13^{-1}$	7	6
$i\infty$	1	$1^{-1} 13^{13}$	7	6

The Hilbert function is $(1 + 2x^2 + 6x^4 + 5x^6)/(1 - x^2)(1 - x^6)$. The space of weight 2 forms for $\Gamma_0^2(13)$ is 3 dimensional, spanned by $E_2(\tau) - 13E_3(13\tau)$, $E_2(\tau,\chi_{13})$, and the cusp form $E_1(\tau,\chi)^2 - E_2(\tau,\bar{\chi})^2$ where χ is an order 4 character of $(\mathbf{Z}/13\mathbf{Z})^*$. The ring of modular forms for $\Gamma_1(13)$ is generated by the 6 dimensional space of forms of weight 1, which has a basis of the 6 forms $E_1(\tau,\chi)$ as χ runs through the 6 odd characters of $(\mathbf{Z}/13\mathbf{Z})^*$. Each of these weight 1 Eisenstein series has a zero at an elliptic point of order 2 and 2 zeros at elliptic points of order 3 and no other zeros.

The function $\eta_{1^2 13^{-2}}$ is a Hauptmodul for $\Gamma_0(13)$. There is also a Hauptmodul for $\Gamma_0(13)+$. These give automorphic forms for the lattice $II_{2,2}(13^{+2})$, which are the denominator functions for generalized Kac-Moody algebras related to elements of order 13 in the monster group.

There are no modular forms of negative weight with poles of order at most 1 at the cusps, as can be seen from the relation (number of zeros) = weight×index/12 = weight×7/6 and the fact that the weight is even and there are only 2 cusps. The space of cusp forms of weight 4 and character χ_{13} has dimension 2, and as this is the space of obstructions to finding a form of weight -2 and character χ_{13} with given singularities, we see that there is a nonzero form of weight -2 and character χ_{13} whose singularities are a pole of order 1 at $i\infty$ and a singularity at 0 with terms involving only q^{-1} and q^{-3}. This is a reflective form for the lattices $II_{n,4+n}(13^{+1})$.

The lattice $L = II_{1,5}(13^{+1})$ is one of the lattices with $\mathrm{Aut}(L)/R(L)$ infinite dihedral. This is the orthogonal complement of an A_{12} in $II_{1,17}$. There is a unique lattice in the genus $II_{0,4}(13^{+1})$, and it has root system is D_3. This can be seen from the fact that all such lattices are the orthogonal complement of an A_{12} in an even 16 dimensional self dual negative definite lattice.

$N = 14$.

The group $\Gamma_0(14)$ is the product of $\Gamma_0^2(14)$ and its center of order 2 generated by Z.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(14)$	24	0	0	4	1

cusps	width	η	zero	weight
1=0	14	$1^{14}2^{-7}7^{-2}14^1$	6	3
1/2	7	$1^{-7}2^{14}7^114^{-2}$	6	3
1/7	2	$1^{-2}2^17^{14}14^{-7}$	6	3
1/14 = $i\infty$	1	$1^12^{-7}7^{-7}14^{14}$	6	3

The ring of modular forms of integral weight for $\Gamma_0^2(14)$ is generated by $E_1(\tau, \chi_7)$, $E_1(2\tau, \chi_7)$, and $\Delta_{14+}(\tau)$. The Hilbert function is $(1+x^2)/(1-x)^2$ The ideal of cusp forms is generated by $\Delta_{14+}(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$ of weight 2. We can construct some automorphic forms on lattices of signature -2 or -4 of singular weight from the modular forms $\eta_{1^{-2}2^17^{-2}14^1}$, $\eta_{1^12^{-2}7^114^{-2}}$, and $\eta_{1^{-1}2^{-1}7^{-1}14^{-1}}$.

$N = 15$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(15)$	24	0	0	4	1

cusps	width	η	zero	weight
1=0	15	$1^{15}3^{-5}5^{-3}15^1$	8	4
1/3	5	$1^{-5}3^{15}5^115^{-3}$	8	4
1/5	3	$1^{-3}3^15^{15}15^{-5}$	8	4
1/15 = $i\infty$	1	$1^13^{-3}5^{-5}15^{15}$	8	4

This case seems very similar to the case $N = 14$. The ring of modular forms of integral weight for $\Gamma_0^2(15)$ is generated by $E_1(\tau, \chi_3)$, $E_1(5\tau, \chi_3)$, and $\Delta_{15+}(\tau)$. The Hilbert function is $(1+x^2)/(1-x)^2$.

The ideal of cusp forms is generated by $\Delta_{15+}(\tau) = \eta(\tau)\eta(5\tau)\eta(3\tau)\eta(15\tau)$ of weight 2. We can construct some automorphic forms on lattices of signature -2 or -4 of singular weight from the modular forms $\eta_{1^{-2}3^15^115^{-2}}$, $\eta_{1^13^{-2}5^{-2}15^1}$, and $\eta_{1^{-1}3^{-1}5^{-1}15^{-1}}$.

$N = 16$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(16)$	24	0	0	6	0

cusps	width	characters	η	zero	weight	character
1=0	16		$1^2 2^{-1}$	1	1/2	χ_θ
1/2	4		$1^{-2} 2^5 4^{-2}$	1	1/2	χ_θ
1/4, 3/4	1	$\chi_2 = -1$	$2^{-2} 4^5 8^{-2}$	1/2,1/2	1/2	$\chi_2 \chi_\theta$
1/8	1		$4^{-2} 8^5 16^{-2}$	1	1/2	χ_θ
$1/16 = i\infty$	1		$8^{-1} 16^2$	1	1/2	χ_θ

Note that the isomorphism class of a cusp a/c is no longer always determined by $(c, 16)$.

The ring of modular forms of integral or half integral weight for $\Gamma_0^2(16)$ is generated by the 3-dimensional space of forms of weight 1/2, which is spanned by the 5 forms listed above. The Hilbert function is $(1 + \chi_2\chi_\theta x^{1/2})/(1 - \chi_\theta x^{1/2})^2$. As in the case $N = 8$ there seem to be rather a lot of examples.

$N = 17$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(17)$	18	2	0	2	1

All modular forms for $\Gamma_0^2(17)$ have even weight. The Hilbert function is $(1 + (1 + 2u_{17})x^2 + (3 + 4u_{17})x^4 + x^6)/(1 - x^2)(1 - x^4)$. The group $\Gamma_0(17)+$ has genus 0, and its Hauptmodul $q^{-1} + 7q + 14q^2 + O(q^3)$ has poles of order 1 at all cusps.

The Hauptmodul with poles of order 1 at all cusps is a reflective modular form for lattices $II_{n,n}(17^{\pm 2m})$. For example, we get automorphic forms for the lattice $II_{2,2}(17^{+2})$. The automorphic form for $II_{2,2}(17^{+2})$ is the denominator function of a generalized Kac-Moody algebra associated with an element of order 17 of the monster group. We get similar statements if we replace 17 by any of the other primes $p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59$, or 71 such that $\Gamma_0(p)+$ has genus 0.

$N = 18$.

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(18)$	36	0	0	8	0

cusps	width	η	zero	weight
1=0	18	$1^6 2^{-3} 3^{-2} 6^1$	3	1
1/2	9	$1^{-3} 2^6 3^1 6^{-2}$	3	1
1/3, 2/3	2	$1^{-6} 2^3 3^{20} 6^{-10} 9^{-6} 18^3$	3,3	2
1/6, 5/6	1	$1^3 2^{-6} 3^{-10} 6^{20} 9^3 18^{-6}$	3,3	2
1/9	2	$3^{-2} 6^1 9^6 18^{-3}$	3	1
$1/18 = i\infty$	1	$3^1 6^{-2} 9^{-3} 18^6$	3	1

The Hilbert function is $(1 + 2u_3 x)/(1 - u_3 x)^2$, and the ring of modular forms is generated by the 4-dimensional space of forms of weight 1, which is spanned by $E_1(\tau, \chi_3)$, $E_1(2\tau, \chi_3)$, $E_1(3\tau, \chi_3)$, and $E_1(6\tau, \chi_3)$.

There are many eta quotients with poles of order 1 at all cusps: for any set of 3 or 6 cusps (with multiplicities) such that cusps with the same denominator have the same multiplicity there is an eta quotient of weight 1 or 2 with these zeros. This gives 12 such eta quotients of weight 1 and 8 of weight 2. There are 4 of weight 1 with no poles at the

cusps 1/3, 2/3, 1/6, or 5/6, and the inverses of these are reflective forms for even level 18 lattices of signature -2.

$N = 20.$

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(20)$	36	0	0	6	1

cusps	width	characters	η	zero	weight
1=0	20		$1^{10}2^{-5}5^{-2}10^1$	6	2
1/2	5	$\chi_\theta = -i$	$1^{-10}2^{25}4^{-10}5^210^{-5}20^2$	6	2
1/4	5		$2^{-5}4^{10}10^120^{-2}$	6	2
1/5	4		$1^{-2}2^15^{10}10^{-5}$	6	2
1/10	1	$\chi_\theta = -i$	$1^22^{-5}4^25^{-10}10^{25}20^{-10}$	6	2
$1/20 = i\infty$	1		$2^14^{-2}10^{-5}20^{10}$	6	2

The space of weight 1/2 forms is spanned by $\eta_{1^{-2}2^54^{-2}}$ (zero of order 5/4 at 1/2 and order 1/4 at 1/10) and $\eta_{5^{-2}10^520^{-2}}$ (zero of order 1/4 at 1/2 and order 5/4 at 1/10). The space of cusp forms of weight 2 and trivial character is spanned by $\eta_{2^210^2}$; it has zeros of order 1 at all cusps.

The form $1^12^14^{-1}5^{-1}10^120^1$ has zeros of order 1/2 at 1/2 and 1/10 and zeros of order 1 at 1 and 1/20. The form $1^{-1}2^14^15^110^120^{-1}$ has zeros of order 1/2 at 1/2 and 1/10 and zeros of order 1 at 1/4 and 1/5. Their inverses are reflective forms for even lattices of level 20, signature -2, and even 5-rank.

$N = 23.$

Group	index	ν_2	ν_3	ν_∞	genus
$\Gamma_0(23)$	24	0	0	2	2

cusps	width	η	zero	weight
0	23	$1^{23}23^{-1}$	22	11
$i\infty$	1	$1^{-1}23^{23}$	22	11

The ring of modular forms of integral weight for $\Gamma_0^2(23)$ is generated by $\Delta_{23+}(\tau) = \eta(\tau)\eta(23\tau)$, and $E_1(\tau, \chi_{23})$, and one of the two weight 3 Eisenstein series. The Hilbert function is $(1 + x^3)/(1 - x)^2$.

The ideal of cusp forms is generated by $\Delta_{23+}(\tau) = \eta(\tau)\eta(23\tau)$ of weight 1. This is the lowest level for which there is a cusp form of weight 1.

Remark. The modular function $\eta(\tau)^{12}\eta(23\tau)^{-12}$ has a pole of order 11 at $i\infty$ and a zero of order 11 at 0, and shows that the cusp 0 gives a torsion point of order 11 on the modular abelian surface of $\Gamma_0(23)$. See [Sh, p. 197] for more about this.

The forms $E_1(\tau, \chi_{23})^n/\Delta_{23+}(\tau)$ show that all level 23 lattices of signature at least -2 have reflective forms.

The automorphic form of the lattice $II_{2,4}(23^{+3})$ and the function $1/\Delta_{23+}$ has singular weight and is the denominator function of a generalized Kac-Moody algebra. This generalized Kac-Moody algebra contains the Feingold-Frenkel rank 3 Kac-Moody algebra as a subalgebra, and can be used to explain why the root multiplicities of the Feingold-Frenkel algebra are often given by values of the partition function. See [Ni] for details.

$N = 28.$

The eta product $1^1 2^{-1} 4^1 7^1 14^{-1} 28^1$ has weight 1, character χ_7, and its zeros are order 1 zeros at the cusps 1/1, 1/4, 1/7, and 1/28. (It is nonzero at the cusps 1/2 and 1/14.) So its inverse is a reflective form for any level 28 even lattice of signature -2 and odd 7-rank.

$N = 30.$

Here are some weight 1 eta quotients with order 1 zeros at 6 of the 8 cusps. $2^1 3^1 5^1 6^{-1} 10^{-1} 30^1$ (nonzero at 1/3, 1/5), $1^1 3^{-1} 5^{-1} 6^1 10^1 15^1$ (nonzero at 1/6, 1/10), $1^{-1} 2^1 3^1 5^1 15^{-1} 30^1$ (nonzero at 1/2, 1/30), $1^1 2^{-1} 6^1 10^1 15^1 30^{-1}$ (nonzero at 1/1, 1/15). These are all modular forms for the character χ_{15}. So any even lattice of level 30 and signature -2 and odd 5-rank has a reflective form. (The 3-rank is automatically odd for any such lattice.) The maximal order of an automorphism of the Leech lattice with fixed points is 30, and 3 of the eta quotients above occur as generalized cycle shapes of such order 30 automorphisms.

There is no special reason for stopping at $N = 30$: there are hints that there might be examples for N up to a few hundred.

13. Open problems.

We list a few suggestions for further research.

Problem 13.1. Find some analogue of reflective forms for other sorts of hyperbolic reflection group. In particular explain why the complex hyperbolic reflection groups found by Allcock [A] have underlying integral lattices with non-zero reflective modular forms. Is this true of all complex hyperbolic reflection groups (except perhaps in small dimensions)? Is there a relation between the co-compact hyperbolic reflection groups found by Bugaenko [B] and lattices with reflective forms?

Problem 13.2. Find some sort of converse theorem that implies that all "interesting" lattices of some sort have non-zero reflective forms. Bruinier [Br] has recently proved some related converse theorems, showing that certain sorts of automorphic infinite products always come from modular forms with singularities.

Problem 13.3. Are there any other reflection groups of high dimensional lattices with co-finite volume other than those listed in section 12?

Problem 13.4. Which of the rank 3 hyperbolic lattices classified by Nikulin [N97] have reflective forms? Note that in part III of Nikulin's papers there are many examples where the Weyl vector has negative norm.

Problem 13.5. Classify all holomorphic eta quotients whose zeros at cusps are all of order at most 1. This would be suitable for a computer.

Problem 13.6. Write a computer program to classify the lattices such that the space of possible reflective singularities is greater than the dimension of the space of cusp forms that give obstructions to the existence of such a singularity. This would give a large class of examples of lattices with reflective forms, and the examples in section 12 suggest that this would include most of them.

Problem 13.7. The group $\Gamma_0(24^2)$ has 48 cusps, all conjugate under its normalizer. The form $1/\eta(24\tau)$ has poles of order 1 at all cusps. Are there any lattices for which it is a reflective form?

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